Bull. Korean Math. Soc. **57** (2020), No. 1, pp. 133–138
https://doi.org/10.4134/BKMS.b190126
pISSN: 1015-8634 / eISSN: 2234-3016

HYPERBOLICALLY CLOSE TO $Q_p^\#$-SEQUENCES

Rauno Aulaskari, Shamil Makhmutov, and Jouni Rättyä

Abstract. It is shown that each sequence lying sufficiently close in the hyperbolic sense to a $Q_p^\#$-sequence for a meromorphic function f in the unit disc is also a $Q_p^\#$-sequence for f.

1. Introduction and results

Let $\mathcal{M}(\mathbb{D})$ denote the class of meromorphic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$. Green's function in $\mathbb{D}$ with logarithmic singularity at $a \in \mathbb{D}$ is $g(z,a) = -\log|\varphi_a(z)|$, where $\varphi_a(z) = (a-z)/(1-\overline{a}z)$ for all $z \in \mathbb{D}$. The function φ_a is the Möbius transformation of $\mathbb{D}$ which interchanges the point $a \in \mathbb{D}$ and the origin, and it is its own inverse. For $0 < p < \infty$, the class $Q_p^\#$ consists of $f \in \mathcal{M}(\mathbb{D})$ such that

$$\|f\|_{Q_p^\#}^2 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} f^\#(z)^2 g^p(z,a)\, dA(z) < \infty,$$

where $f^\#(z) = |f'(z)|/(1+|f(z)|^2)$ is the spherical derivative of f at z and $dA(z) = r dr\, d\theta$ for $z = re^{i\theta}$ denotes the element of Lebesgue area measure on $\mathbb{D}$. It is known that $Q_1^\#$ coincides with the class UBC of functions in $\mathcal{M}(\mathbb{D})$ of uniformly bounded Nevanlinna characteristic in $\mathbb{D}$ [5], and, for each $p > 1$, $Q_p^\#$ is the same as the class $\mathcal{N}$ of meromorphic normal functions [2], defined by the condition

$$\|f\|_{\mathcal{N}} = \sup_{z\in\mathbb{D}} f^\#(z)(1-|z|^2) < \infty.$$

Received February 1, 2019; Revised September 2, 2019; Accepted November 6, 2019.
2010 *Mathematics Subject Classification.* 30D45, 30D50.
Key words and phrases. Normal function, $\mathcal{N}$-sequence, spherical derivative, $Q_p^\#$-class, $Q_p^\#$-sequence.
This research was supported in part by Magnus Ehrnrooth Foundation, Ministerio de Economía y Competitivivad, Spain, projects MTM2014-52865-P and MTM2015-69323-REDT; La Junta de Andalucía, project FQM210; Academy of Finland project no. 268009.
This project was started by the first two authors and was finished by the latter two few years after our friend and colleague Rauno Aulaskari passed away in December 2014.

©2020 Korean Mathematical Society

Let $f \in \mathcal{M}(\mathbb{D})$. According to [1, Definition 1], a sequence $\{a_n\}_{n=1}^\infty$ in $\mathbb{D}$ is a $q_\mathcal{N}$-sequence for f if

$$\lim_{n\to\infty} f^\#(a_n)(1-|a_n|^2) = \infty.$$

Further, we say that $\{a_n\}_{n=1}^\infty$ is an $\mathcal{N}$-sequence for f if

$$\limsup_{n\to\infty} f^\#(a_n)(1-|a_n|^2) = \infty.$$

Then $\{a_n\}_{n=1}^\infty$ is an $\mathcal{N}$-sequence for f if and only if one of its subsequences is a $q_\mathcal{N}$-sequence for f. Similarly, if $0 < p < \infty$, then $\{a_n\}_{n=1}^\infty$ is a q_p-sequence for $f \in \mathcal{M}(\mathbb{D})$ according to [1, Definition 2] if

$$\lim_{n\to\infty} \int_\mathbb{D} f^\#(z)^2 g^p(z,a_n) dA(z) = \infty,$$

and it is a $Q_p^\#$-sequence for f if

$$\limsup_{n\to\infty} \int_\mathbb{D} f^\#(z)^2 g^p(z,a_n)\, dA(z) = \infty.$$

If $\{a_n\}_{n=1}^\infty$ is a $q_\mathcal{N}$-sequence for $f \in \mathcal{M}(\mathbb{D})$, then each sequence $\{b_n\}_{n=1}^\infty$ in $\mathbb{D}$ for which $\sigma(a_n, b_n) \to 0$, as $n \to \infty$, is a q_p-sequence for f for all $0 < p < \infty$ by [1, Theorem 1]. Here $\sigma(z,w) = |\varphi_z(w)|$ is the pseudohyperbolic distance between two points z and w in $\mathbb{D}$. Hence each $\{b_n\}_{n=1}^\infty$ satisfying $\sigma(a_n,b_n) \to 0$, as $n \to \infty$, is a $Q_p^\#$-sequence for $f \in \mathcal{M}(\mathbb{D})$ if $\{a_n\}_{n=1}^\infty$ is an $\mathcal{N}$-sequence for $f \in \mathcal{M}(\mathbb{D})$.

If $\{a_n\}_{n=1}^\infty$ is a $Q_p^\#$-sequence for $f \in \mathcal{M}(\mathbb{D})$, then each sequence $\{b_n\}_{n=1}^\infty$ in $\mathbb{D}$ for which $\sigma(a_n, b_n) \to 0$, as $n \to \infty$, is also a $Q_p^\#$-sequence for f by the proof of [1, Theorem 5]. The following theorem improves this result.

Theorem 1.1. *Let $0 < p < \infty$ and $f \in \mathcal{M}(\mathbb{D})$, and let $\{a_n\}_{n=1}^\infty$ be a $Q_p^\#$-sequence for f. Then there exists $\delta = \delta(f,p) \in (0,1)$ such that each sequence $\{b_n\}_{n=1}^\infty$ in $\mathbb{D}$ satisfying $\sigma(a_n,b_n) \le \delta$ for all $n \in \mathbb{N}$ is a $Q_p^\#$-sequence for f.*

By using Theorem 1.1 we obtain the following improvement of [1, Theorem 6].

Corollary 1.2. *Let $0 < p < p' < \infty$ and $f \in \mathcal{M}(\mathbb{D})$, and let $\{a_n\}_{n=1}^\infty$ be a $Q_p^\#$-sequence for f. If*

$$\limsup_{n\to\infty} \int_{\mathbb{D}\setminus\Delta(a_n,r)} f^\#(z)^2(1-|\varphi_{a_n}(z)|^2)^p\, dA(z) < \infty \tag{1.1}$$

for some $r \in (0,1)$, then there exists $\delta = \delta(f,p') \in (0,1)$ such that each sequence $\{b_n\}_{n=1}^\infty$ in $\mathbb{D}$ satisfying $\sigma(a_n,b_n) \le \delta$ for all $n \in \mathbb{N}$ is a $Q_{p'}^\#$-sequence for f.

In the forthcoming sections we will prove our results in the order of appearance.

2. Auxiliary result

To proof Theorem 1.1 we need the following auxiliary result.

Lemma 2.1. *Let $0 < p < \infty$ and $f \in \mathcal{M}(\mathbb{D})$, and let $\{a_n\}_{n=1}^{\infty}$ be an $\mathcal{N}$-sequence for f. Then there exists $\delta = \delta(f,p) \in (0,1)$ such that each sequence $\{b_n\}_{n=1}^{\infty}$ satisfying $\sigma(a_n, b_n) \le \delta$ for all $n \in \mathbb{N}$ is a $Q_p^{\#}$-sequence for f.*

Proof. Assume on the contrary to the assertion that for each $\delta \in (0,1)$ there exists a sequence $\{b_n\}_{n=1}^{\infty}$ in $\mathbb{D}$ such that $\sigma(a_n, b_n) \le \delta$ for all $n \in \mathbb{N}$, but

$$K = K(f,p,\{b_n\}) = \sup_{n\in\mathbb{N}} \int_{\mathbb{D}} f^{\#}(z)^2 g^p(z, b_n)\, dA(z) < \infty. \tag{2.1}$$

Let $\Delta(z,r) = \{\zeta \in \mathbb{D} : \sigma(z,\zeta) < r\}$ denote the pseudohyperbolic disc with center $z \in \mathbb{D}$ and radius $r \in (0,1)$, and let $D(z,R) = \{\zeta \in \mathbb{C} : |z-\zeta| < R\}$ be the Euclidean disc with center $z \in \mathbb{C}$ and radius $R > 0$. By the hypothesis, we can pick up a subsequence $\{a_n^1\}_{n=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ which is a $q_{\mathcal{N}}$-sequence for f. Let $\{b_n^1\}_{n=1}^{\infty}$ be the corresponding subsequence of $\{b_n\}_{n=1}^{\infty}$. Then (2.1) yields

$$\begin{aligned} K &\ge \int_{\mathbb{D}} f^{\#}(z)^2 g^p(z, b_n^1)\, dA(z) \ge \int_{\Delta(b_n^1, r)} f^{\#}(z)^2 g^p(z, b_n^1)\, dA(z) \\ &\ge \left(\log \frac{1}{r}\right)^p \int_{D(0,r)} f_n^{\#}(z)^2\, dA(z), \quad n \in \mathbb{N}, \quad 0 < r < 1, \end{aligned} \tag{2.2}$$

where $f_n = f \circ \varphi_{b_n^1}$ for all $n \in \mathbb{N}$. Choose $r = r(K,p) \in (0,1)$ sufficiently small such that $(-\log r)^p > K/\pi$. Then [3, Theorem 6] shows that $\{f_n : n \in \mathbb{N}\}$ is a normal family in $D(0,r)$. Therefore there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which converges uniformly on compact subsets of $D(0,r)$ to a meromorphic function h on $D(0,r)$ or to ∞. In the latter case, (2.2) yields

$$\pi > \lim_{k\to\infty} \int_{\overline{D(0,r/2)}} f_{n_k}^{\#}(z)^2\, dA(z) = \infty,$$

which is a contradiction, and thus the assertion follows. If the limit h is meromorphic on $D(0,r)$, then $f_{n_k}^{\#}$ converges uniformly to $h^{\#}$ on $\overline{D(0,r/2)}$ as $k \to \infty$. Fix now $\delta = \delta(r) > 0$ sufficiently small such that $\Delta(\varphi_{b_n}(a_n), \delta) \subset D(0,r/2)$ for all $n \in \mathbb{N}$. This is possible because $\sigma(a_n, b_n) \le \delta$ for all $n \in \mathbb{N}$. Then, by the uniform convergence on $\overline{D(0,r/2)}$,

$$\begin{aligned} &\lim_{k\to\infty} \int_{\Delta\left(a_{n_k}^1, \delta\right)} f^{\#}(z)^2 g^p\left(z, a_{n_k}^1\right)\, dA(z) \\ = &\lim_{k\to\infty} \int_{\Delta\left(\varphi_{b_{n_k}^1}(a_{n_k}^1), \delta\right)} f_{n_k}^{\#}(\zeta)^2 g^p\left(\zeta, \varphi_{b_{n_k}^1}\left(a_{n_k}^1\right)\right)\, dA(\zeta) \\ \le &\lim_{k\to\infty} \int_{D(0,r/2)} f_{n_k}^{\#}(\zeta)^2 g^p\left(\zeta, \varphi_{b_{n_k}^1}\left(a_{n_k}^1\right)\right) dA(\zeta) \end{aligned}$$

$$\leq \sup_{c\in D(0,\delta)} \int_{D(0,r/2)} h^{\#}(\zeta)^2 g^p(\zeta,c)\, dA(\zeta) < \infty,$$

because h is meromorphic in $D(0,r)$. Since $\{a_n^1\}_{n=1}^{\infty}$ is a $q_{\mathcal{N}}$-sequence for $f \in \mathcal{M}(\mathbb{D})$, so is its subsequence $\{a_{n_k}^1\}_{k=1}^{\infty}$. To complete the proof, it suffices to show that for each $q_{\mathcal{N}}$-sequence $\{c_n\}_{n=1}^{\infty}$ for $f \in \mathcal{M}(\mathbb{D})$ there exists a subsequence $\{c_{n_j}\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} \int_{\Delta\left(c_{n_j},\rho\right)} f^{\#}(z)^2 g^p\left(z, c_{n_j}\right)\, dA(z) = \infty \tag{2.3}$$

for each $\rho \in (0,1)$ and $0 < p < \infty$. This for $\rho = \delta$ contradicts what just have been proved, and thus gives the assertion. To prove (2.3), we employ the method used in the proof of [1, Theorem 4]. Assume on the contrary that there exist $\rho \in (0,1)$ and $0 < p < \infty$ such that

$$C = C(f,p,\{c_n\}) = \sup_{n\in\mathbb{N}} \int_{\Delta(c_n,\rho)} f^{\#}(z)^2 g^p\left(z,c_n\right)\, dA(z) < \infty.$$

Choose $r = r(f,p,\{c_n\}) \in (0,\rho)$ sufficiently small such that $2(-\log r)^{-p} C \leq \pi$. Then a change of variable gives

$$C \geq \int_{\Delta(c_n,r)} f^{\#}(z)^2 g^p\left(z,c_n\right)\, dA(z) \geq \left(\log\frac{1}{r}\right)^p \int_{D(0,r)} (f\circ\varphi_{c_n})^{\#}(z)^2\, dA(z),$$

and hence $g_n = f\circ\varphi_{c_n}$ satisfies $g_n^{\#}(0) = f^{\#}(c_n)(1-|c_n|^2) \leq 1/r$ by Dufresnoy's theorem [4, p. 83]. This is a contradiction, and hence the claimed subsequence exists. □

Note that the argument used in the end of the proof gives an easy way to see that each $\mathcal{N}$-sequence for $f \in \mathcal{M}(\mathbb{D})$ is a $Q_p^{\#}$-sequence for f.

3. Proof of Theorem 1.1

Let $\{a_n\}_{n=1}^{\infty}$ be a $Q_p^{\#}$-sequence for f, and let $\delta \in (0,1)$ be that of Lemma 2.1. Then either

$$\limsup_{n\to\infty} \int_{\mathbb{D}\setminus\Delta(a_n,\delta)} f^{\#}(z)^2 g^p(z,a_n)\, dA(z) = \infty \tag{3.1}$$

or

$$\limsup_{n\to\infty} \int_{\Delta(a_n,\delta)} f^{\#}(z)^2 g^p(z,a_n)\, dA(z) = \infty. \tag{3.2}$$

Assume first (3.1). The inequalities $1 - t \leq -\log t \leq \frac{1}{t}(1-t)$, valid for all $0 < t \leq 1$, imply

$$g(z,a_n) \leq \frac{1}{\delta}\left(1 - |\varphi_{a_n}(z)|\right) \leq \frac{C_1}{\delta}\left(1 - |\varphi_{b_n}(z)|\right) \leq \frac{C_1}{\delta} g(z,b_n),$$

where $C_1 = C_1(\delta) > 0$ is a constant, for all $z \in \mathbb{D} \setminus \Delta(a_n, \delta)$ and $b_n \in \overline{\Delta(a_n, \delta)}$. It follows that

$$\begin{aligned}
&\int_{\mathbb{D}\setminus\Delta(a_n,\delta)} f^\#(z)^2 g^p(z, a_n)\, dA(z) \\
&\le \left(\frac{C_1}{\delta}\right)^p \int_{\mathbb{D}\setminus\Delta(a_n,\delta)} f^\#(z)^2 g^p(z, b_n)\, dA(z) \\
&\le \left(\frac{C_1}{\delta}\right)^p \int_{\mathbb{D}} f^\#(z)^2 g^p(z, b_n)\, dA(z), \quad b_n \in \overline{\Delta(a_n, \delta)},
\end{aligned}$$

and hence every $\{b_n\}_{n=1}^\infty$ satisfying $\sigma(a_n, b_n) \le \delta$ for all $n \in \mathbb{N}$ is a $Q_p^\#$-sequence for f.

Assume now (3.2). Then there either exists a $q_{\mathcal{N}}$-sequence $\{c_{n_k}\}_{k=1}^\infty$ satisfying $\sigma(a_{n_k}, c_{n_k}) \le \delta/2$ for all $k \in \mathbb{N}$ or a constant $C_2 > 0$ such that $f^\#(z)(1 - |z|^2) \le C_2$ for all $z \in \Delta(a_n, \delta/2)$ and all $n \in \mathbb{N}$. In the first case, $\{b_{n_k}\}_{k=1}^\infty$ is a $Q_p^\#$-sequence for f by Theorem 2.1 if $\sigma(a_{n_k}, b_{n_k}) \le \delta/2$ for all $k \in \mathbb{N}$ because then $\sigma(c_{n_k}, b_{n_k}) \le \sigma(c_{n_k}, a_{n_k}) + \sigma(a_{n_k}, b_{n_k}) \le \delta/2 + \delta/2 = \delta$. In the latter case, by a change of variable,

$$\begin{aligned}
\int_{\Delta(a_n,\delta/2)} f^\#(z)^2 g^p(z, a_n)\, dA(z) &\le C_2^2 \int_{D(0,\delta/2)} \frac{\left(\log \frac{1}{|\zeta|}\right)^p}{(1 - |\zeta|^2)}\, dA(\zeta) \\
&< \infty, \quad n \in \mathbb{N}.
\end{aligned} \tag{3.3}$$

If now $\sigma(a_n, b_n) \le \frac{1-\delta}{2}$ for all $n \in \mathbb{N}$, then $\Delta(a_n, \delta) \subset \Delta(b_n, \frac{1+\delta}{2})$ because $\sigma(z, b_n) \le \sigma(z, a_n) + \sigma(a_n, b_n) \le \delta + \frac{1-\delta}{2} = \frac{1+\delta}{2}$ for all $z \in \Delta(a_n, \delta)$. This observation together with (3.2) and (3.3) shows that

$$\begin{aligned}
&\int_{\Delta(b_n,\frac{1+\delta}{2})\setminus\Delta(a_n,\delta/2)} f^\#(z)^2 g^p(z, b_n)\, dA(z) \\
&\ge \left(\log \frac{2}{1+\delta}\right)^p \int_{\Delta(a_n,\delta)\setminus\Delta(a_n,\delta/2)} f^\#(z)^2\, dA(z) \\
&\ge \left(\frac{\log \frac{2}{1+\delta}}{\log \frac{2}{\delta}}\right)^p \int_{\Delta(a_n,\delta)\setminus\Delta(a_n,\delta/2)} f^\#(z)^2 g^p(z, a_n)\, dA(z)\ .
\end{aligned}$$

Therefore

$$\limsup_{n\to\infty} \int_{\Delta(b_n,\frac{1+\delta}{2})\setminus\Delta(a_n,\delta/2)} f^\#(z)^2 g^p(z, b_n)\, dA(z) = \infty\ ,$$

and thus $\{b_n\}_{n=1}^\infty$ is a $Q_p^\#$-sequence for f. □

4. Proof of Corollary 1.2

The hypothesis (1.1) implies

$$\limsup_{n\to\infty} \int_{\Delta(a_n,r)} f^\#(z)^2 g^p(z, a_n)\, dA(z) = \infty.$$

If $r \leq \frac{1}{e}$, then $g^p(z,a_n) \leq g^{p'}(z,a_n)$ for all $z \in \Delta(a_n,r)$, and hence $\{a_n\}_{n=1}^{\infty}$ is a $Q_{p'}^{\#}$-sequence for f. The assertion now follows by Theorem 1.1. If $r > \frac{1}{e}$, then either

$$\limsup_{n\to\infty} \int_{\Delta(a_n,\frac{1}{e})} f^{\#}(z)^2 g^p(z,a_n)\, dA(z) = \infty$$

or

$$\limsup_{n\to\infty} \int_{\Delta(a_n,r)\setminus\Delta(a_n,\frac{1}{e})} f^{\#}(z)^2 g^p(z,a_n)\, dA(z) = \infty.$$

In the former case we may proceed as before, meanwhile in the latter case we have

$$g^p(z,a_n) \leq \frac{1}{|\varphi_{a_n}(z)|^p}\,(1-|\varphi_{a_n}(z)|)^p \leq \frac{e^p}{(1-r)^{p'-p}} g^{p'}(z,a_n)$$

for all $z \in \Delta(a_n,r) \setminus \Delta\left(a_n,\frac{1}{e}\right)$. Again the assertion follows by Theorem 1.1. □

References

[1] R. Aulaskari, S. Makhmutov, and H. Wulan, *On q_p sequences*, in Finite or infinite dimensional complex analysis and applications, 117–125, Adv. Complex Anal. Appl. **2**, Kluwer Acad. Publ., Dordrecht, 2004.

[2] O. Lehto and K. I. Virtanen, *Boundary behaviour and normal meromorphic functions*, Acta Math. **97** (1957), 47–65. https://doi.org/10.1007/BF02392392

[3] Sh. A. Makhmutov, *Classes of meromorphic functions characterized by a spherical derivative*, Dokl. Akad. Nauk SSSR **287** (1986), no. 4, 789–794.

[4] J. L. Schiff, *Normal Families*, Universitext, Springer-Verlag, New York, 1993. https://doi.org/10.1007/978-1-4612-0907-2

[5] S. Yamashita, *Functions of uniformly bounded characteristic*, Ann. Acad. Sci. Fenn. Ser. A I Math. **7** (1982), no. 2, 349–367. https://doi.org/10.5186/aasfm.1982.0733

Rauno Aulaskari
Department of Physics and Mathematics
University of Eastern Finland
P.O.Box 111, 80101 Joensuu, Finland

Shamil Makhmutov
Department of Mathematics
College of Science
Sultan Qaboos University
P.O. Box 36, PC 123 Al Khodh, Muscat, Sultanate of Oman
Email address: makhm@squ.edu.om

Jouni Rättyä
Department of Physics and Mathematics
University of Eastern Finland
P.O.Box 111, 80101 Joensuu, Finland
Email address: jouni.rattya@uef.fi