Bull. Korean Math. Soc. **57** (2020), No. 1, pp. 127–131
https://doi.org/10.4134/BKMS.b190097
pISSN: 1015-8634 / eISSN: 2234-3016

TWO REMARKS ON THE GAME OF COPS AND ROBBERS

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Abstract. We discuss two unrelated topics regarding *Cops and Robbers*, a well-known pursuit-evasion game played on a simple graph. First, we address a recent question of Breen et al. and prove the PSPACE-completeness of the *cop throttling number*, that is, the minimal possible sum of the number k of cops and the number $\mathrm{capt}(k)$ of moves that the robber can survive against k cops under the optimal play of both sides. Secondly, we revisit a *teleporting* version of the game due to Wagner; we disprove one of his conjectures and suggest a new related research problem.

A pursuit-evasion game of *Cops and Robbers* was introduced forty years ago [1,18,20] and attracts a notable amount of attention in recent publications. This game is played on a simple graph G by two sides, a set of cops and a robber. In the beginning of the game, each of the cops takes some vertex of G as his initial position, and then the robber chooses an initial location of his. When the starting position of the game is determined, the players begin moving alternately with cops moving first. On their turn, each of the cops should either move to a vertex adjacent to his current position or stay in place, and so should do the robber on his move. Two or more cops are allowed to share the same vertex in the initial position and on any move. The cops' goal is to make one of them move to the vertex occupied by the robber; the robber wins if he can permanently avoid being captured. The *cop number* of a graph G, denoted $c(G)$, is the smallest cardinality of the set of cops needed to win the game. The *capture time* $\mathrm{capt}(k, G)$ is the number of moves that a team of k cops have to make before capturing the robber, provided that both players follow the optimal strategies [3–5].

1. The complexity of cop-throttling number

Breen et al. [5] introduced another invariant $\mathrm{th}_c(G)$, which they call *cop-throttling number* and define as

$$\mathrm{th}_c(G) = \min_{k \geq c(G)} \left(k + capt(k, G)\right).$$

Received January 27, 2019; Revised October 22, 2019; Accepted November 19, 2019.
2010 *Mathematics Subject Classification.* 05C57, 49N75.
Key words and phrases. Graph theory, cops and robbers.

They investigate this quantity and compare its behaviour with that of the domination number, diameter, radius, girth of a graph and other invariants. Also, the authors of [5] ask a question on the algorithmic complexity of the cop-throttling number, which we address in this section.

Problem 1.1 (Question 4.2 in [5])**.** What is the complexity of computing $\mathrm{th}_c(G)$?

Theorem 1.2. *Computing* $\mathrm{th}_c(G)$ *is a PSPACE-complete problem.*

The class of PSPACE-complete problems is a natural home for those two-player games which have a polynomial bound on the number of possible moves – like the $n \times n$ versions of Othello [13], Gomoku [19], and chess or checkers with the '50' in the 50-move drawing rule replaced by some fixed non-constant polynomial in n (see [10, 11]). *Cops and Robbers* is EXPTIME-complete in general [14], but in order to compute $\mathrm{th}_c(G)$ we can restrict ourselves with games of length at most $|G| + 1$, which is an obvious upper bound on the cop-throttling number [5]. Therefore, the problem in Question 1.1 could be expected to be PSPACE-complete; indeed we can deduce this result from the following theorem.

Theorem 1.3 (See [16])**.** *Given a simple graph G and $k \in \mathbb{N}$, it is PSPACE-hard to determine whether or not $c(G) \leq k$. This problem remains PSPACE-hard even when the yes-instances are promised*[1] *to satisfy $capt(G, k) < C|G|$ for some absolute constant $C \in \mathbb{N}$.*

Proof. The first sentence is the main result of [16]. The instances used in the reduction either satisfy $c(G) > k$ or allow k cops to capture the robber in a linear number of moves (see also Lemma 2.2 in [15] and the remark in the last paragraph of Section 1 in [16]), which implies the second claim. $\square$

Theorem 1.4. *Given a simple graph H and $t \in \mathbb{N}$, it is PSPACE-hard to decide whether or not* $\mathrm{th}_c(H) < t$.

Proof. We construct a polynomial reduction from the problem as in Theorem 1.3 as follows; we define $H(G)$ as the union of $C|G|$ disjoint copies of G and set $t = (k + 1)C|G|$. If (G, k) is a yes-instance, then we place the k cops in every copy of G in the vertices corresponding to the strategy that allows them to capture the robber in less than $C|G|$ moves provided that the robber chooses a vertex in that copy as his initial position. Since the copies of G are disjoint, the cops of the copy containing the initial location of the robber can indeed realize this strategy, so the throttling number of $H(G)$ is less than $kC|G| + C|G| = t$ when (G, k) is a yes-instance of the problem in Theorem 1.3. Conversely, if (G, k) is a no-instance, then we need at least t cops to capture

[1] By saying that an input of some computational problem is *promised* to satisfy some condition, we mean that the algorithms solving that problem should work properly on the inputs satisfying the condition and return anything or even not halt otherwise.

the robber – in fact, any initial placing of less than t cops on $H(G)$ will leave one of the copies attended by at most k cops, so the robber can choose a vertex in that copy and play his winning strategy there. $\square$

The proved result does actually imply Theorem 1.2 because the membership in PSPACE follows from standard techniques (and was proved, for instance, in Theorem 3.7 of [9]). Let us comment on another problem on the algorithmic complexity of the cop throttling number asked in [5].

Problem 1.5 (Question 4.3 in [5])**.** What is the complexity of determining whether $\mathrm{th}_c(G) \leq h$ for fixed h?

As pointed out to the author by an anonymous referee of *Discrete Mathematics*, the answer to this question comes from the existing results. The authors of [8] (see the proof of Theorem 1 in Subsection 2.2) construct a fixed-parameter tractable reduction from the dominating set problem to the cop number, that is, a function that, given a graph with dominating number γ as an input, returns a graph whose cop number is the same number γ and halts in time not exceeding $f(\gamma)L^\theta$, where f is a function depending only on γ, θ is an absolute constant, and L is the bit length of the input. A specific property of their reduction was the fact that the γ cops can capture the robber in one move, which means that the throttling number of the output graph of the reduction equals $\gamma + 1$. It follows that the throttling (and cop) numbers of a given graph are what is called $W[2]$-hard in the fixed parameter hierarchy, that is, are believed not to be fixed parameter tractable [6, 7]. On the other hand, the standard backtracking algorithm [12] allows one, for any fixed k, to decide whether a given graph G satisfies $c(G) \leq k$, and if this is the case, to compute $capt(G, k)$ in polynomial time. In particular, we can decide whether or not $\mathrm{th}_c(G) \leq h$ by an algorithm which halts in time polynomial for every fixed value of h.

2. A version with teleporting cops

Wagner [21] proposes a version of *Cops and Robbers* in which one of the players is allowed to ignore the structure of the graph. Namely, he allows the cops to move to arbitrary vertices of G except the position of the robber; the cops win if one of them occupies a vertex adjacent to the robber's position after the robber's turn. The smallest number of cops needed to catch the robber in this way is called the *teleporting cop number* in [21], and we denote it by $tc(G)$. Let us look at Conjecture 8 in [21].

Conjecture 2.1. *For any diameter-two graph G, we have $c(G) = tc(G)$.*

Theorem 2.2. *For any $c \in \mathbb{N}$, there exists a diameter-two graph H such that $tc(H) = 1$ and $c(H) \geq c$.*

Proof. We start with a diameter-two graph G satisfying $c(G) \geq c$, which exists by Corollary 3.2 of [2], and we also pick any integer $l \geq 2$. We construct the graph H on $G \times \{1, \dots, l\}$ by declaring two pairs (u, i) and (v, j) adjacent if and only if either u, v are adjacent in G or when $u = v$ and i, j are consecutive.

Let us note that, for any pair of non-adjacent vertices u, v of G, there is a vertex w of G that is adjacent to both u and v. Therefore, if a pair of vertices (u, i) and (v, j) of H are not adjacent to each other, then they are both adjacent to $(w, 1)$; this proves that the diameter of H is at most two. In order to prove that $c(H) \geq c(G)$, we need to provide a strategy that allows the robber to win against $c(G) - 1$ cops when played on H (which is easy – the robber can just forget about the second index and play his winning strategy as on G; see [17] for more advanced results on *Cops and Robbers* played on graph products).

Finally, let us see that $tc(G) = 1$, that is, the robber can be caught by a single teleporting cop on H. In order to do this, the cop determines his initial position arbitrarily, and when the robber appears at some (u, i), the cop takes an arbitrary empty vertex of the form (u, j) and then goes only to vertices of the path $P_u = \{(u, 1), \dots, (u, l)\}$. From now on, the robber cannot leave P_u and gets captured without forcing the cop to use his teleporting device. $\square$

In fact, our counterexample is a bit stronger than requested in Conjecture 2.1 – namely, our cop can win even if he is allowed to teleport to a non-adjacent vertex exactly once. It would be interesting to see how much power the cops lose when restricted to at most one use of the teleporting device each.

More formally, we consider the game defined by the same rules as *Cops and Robbers* with the exception that each cop is allowed to move not only to any vertex adjacent to his current position, but also to *teleport* to any other vertex not occupied by the robber unless he has already used this additional option earlier in the game. We define the *restricted teleporting cop number* $rtc(G)$ as the smallest number of cops needed to eventually catch the robber under the rules described in this paragraph.

Problem 2.3. Can we have $rtc(G) \neq tc(G)$? If yes, can $rtc(G)$ be arbitrarily large for graphs G satisfying $tc(G) = 1$?

Acknowledgements. The author is grateful to the anonymous reviewer for helpful comments and to the editors for rapid processing of the paper. The author is financially supported by the Russian Science Foundation (project No. 19-71-00063).

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