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MORE ON THE 2-PRIME IDEALS OF COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with identity. A proper ideal I of R is called 2-prime if for all $a, b \in R$ such that $ab \in I$, then either a^2 or b^2 lies in I. In this paper, we study 2-prime ideals which are generalization of prime ideals. Our study provides an analogous to the prime avoidance theorem and some applications of this theorem. Also, it is shown that if R is a PID, then the families of primary ideals and 2-prime ideals of R are identical. Moreover, a number of examples concerning 2-prime ideals are given. Finally, rings in which every 2-prime ideal is a prime ideal are investigated.

1. Introduction

We assume throughout this paper that all rings are commutative with identity. Let R be a ring and I be an ideal of R. The set of nilpotent elements of R, the set of zero-divisors of R, the set of minimal prime ideal of I, extension and contraction of I under ring homomorphism are denoted by $Nil(R)$, $Z(R)$, $Min_R(I)$, I^e and J^c, respectively. By a proper ideal I of R we mean an ideal with $I \neq R$. For any undefined notation or terminology in commutative ring theory, we refer the reader to [8].

Prime ideals play a central role in commutative ring theory and so this notion has been generalized and studied in several directions. The importance of some of these generalizations is same as the prime ideals, say primary ideals. In a sense they determine how far an ideal is from being prime. For instance, in 1978, Hedstrom and Houston [6] defined the strongly prime ideal, that is a proper ideal P of R such that for $a, b \in K$ with $ab \in P$, either $a \in P$ or $b \in P$ where K is the quotient field of R. In 2003, Anderson and Smith [1] introduced the notion of a weakly prime ideal, i.e., a proper ideal P of R with the property that for $a, b \in R$, $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. So a prime ideal is weakly prime. In 2005, Bhatwadekar and Sharma [4] introduced the notion of almost prime ideal which is also a generalization of prime ideal. A proper ideal I of an integral domain R is said to be almost prime if for $a, b \in R$

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with $ab \in I \setminus I^2$, then either $a \in I$ or $b \in I$, and it is clear that every weakly prime ideal is an almost prime ideal. The notion of 2-absorbing ideals were introduced and investigated in 2007 by Badawi [2]. A nonzero proper ideal I of R is called a 2-absorbing ideal if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. This paper is in this theme and it is devoted to study a generalization of prime ideals so called 2-prime ideals.

A proper ideal I of R is said to be 2-prime if whenever $a, b \in R$ and $ab \in I$, then either a^2 or b^2 lies in I. The concept of 2-prime ideals was first introduced and studied by Beddani and Messirdi in [3] and they uses it to present certain characterization of valuation rings. Clearly, every prime ideal is a 2-prime ideal. However, the converse is not true. For example, $9\mathbb{Z}$ is a 2-prime ideal of R, but it is not prime. For nontrivial 2-prime ideals see Example 2.7.

In Section 2, we classify 2-prime ideals of a PID and we show that the families of primary ideals and 2-prime ideals in a PID are identical (Theorem 2.3). In Proposition 2.4, we give some basic properties of 2-prime ideals. It is easily proved that if J and K are 2-prime ideals of R, then $J \cap K$ and JK need not be a 2-prime ideal of R. By using the technique of efficient covering of ideals, In Theorem 2.9, we prove the 2-prime avoidance theorem for ideals. In Section 3, we investigate all rings in which every 2-prime ideal is prime, i.e., 2-P rings. In Theorem 3.4 we show that if (R, M) is a quasi-local ring, then R is a 2-P ring if and only if $IM = P$, for every minimal prime ideal P over an arbitrary 2-prime ideal I. Finally, it is proved that if (R, M) is a quasi-local ring and $I \in 2\text{-}Min_R(P^2)$, for every P-2-prime ideal I, then R is a 2-P ring if and only if $I = P$, for every ideal $I \in 2\text{-}Min_R(P^2)$ such that $I \subseteq P$ (Corollary 3.8).

2. The 2-prime avoidance theorem

In this section, we study some basic properties of 2-prime ideals and we prove the 2-prime avoidance theorem.

Definition 2.1. Let R be a ring. A proper ideal I of R is called 2-prime if for all $a, b \in R$ such that $ab \in I$, then either a^2 or b^2 lies in I.

Recall that a proper ideal I of R is called a semiprimary if whenever $a, b \in R$ and $ab \in I$, we have $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Clearly, every 2-prime ideal is a semiprimary ideal of R. It is worth mentioning that if I is a semiprimary ideal of R, then I need not be a 2-prime ideal of R (see the following example).

Example 2.2. Suppose that $R = K[x, y, z]$ is the ring of polynomials over K in indeterminates x, y, z, where K is a field. It follows from [8, Exercice 4.28] that $\sqrt{(x^3, xy, y^3)}$ is a prime ideal of R, and so (x^3, xy, y^3) is a semiprimary ideal of R. But (x^3, xy, y^3) is not 2-prime since $xy \in (x^3, xy, y^3)$ and $x^2 \notin (x^3, xy, y^3)$ and $y^2 \notin (x^3, xy, y^3)$.

In the following theorem, one may see that in a PID every semiprimary ideal of R is 2-prime.

Theorem 2.3. *Let R be a principal ideal domain and I be an ideal of R. Then the following statements are equivalent.*

(1) *I is a semiprimary ideal of R.*

(2) *I is a 2-prime ideal of R.*

(3) *I is a primary ideal of R.*

Proof. (1) $\Leftrightarrow$ (3) Let I be a semiprime ideal of R. Since R is a PID and every prime ideal is maximal, it is known that every semiprimary ideal of R is primary.

(2) $\Leftrightarrow$ (3) We show that I is a 2-prime ideal of R if and only if either $I = (p^n)$, for some positive integer n and an irreducible element p of R or $p = 0$, and so the result follows from [8, Example 4.10]. Suppose that I is a non-zero 2-prime ideal of R. Since R is a principal ideal domain, there exists $r \in R$ such that $I = (r)$. If r is irreducible, then $n = 1$ and we are done. Suppose that r is not an irreducible element. Since R is a unique factorization, r can be expressed in the form

$$r = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m},$$

where $m, n_1, n_2, \ldots, n_m$ are positive integers and p_i's are irreducible elements of R such that p_i and p_j are not associates, if $i \neq j$. Let $a = p_1^{n_1}$ and $b = p_2^{n_2} \cdots p_m^{n_m}$. Then $ab \in I$. Since I is 2-prime, either $a^2 \in I$ or $b^2 \in I$.

If $a^2 = p_1^{2n_1} \in I = (r)$, then there exists $x \in R$ such that

$$a^2 = p_1^{2n_1} = xr = xp_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}, \text{ and so } p_1^{n_1} = xp_2^{n_2} \cdots p_m^{n_m}$$

which implies that $p_j | p_1$ for some $2 \leq j \leq m$. Since p_1 is an irreducible element of R, we conclude that p_1 and p_j are associates, a contradiction.

If $b^2 = (p_2^{n_2} \cdots p_m^{n_m})^2 \in I = (r)$, then there exists $s \in R$ such that

$$(p_2^{n_2} \cdots p_m^{n_m})^2 = sr = sp_1^{n_1} p_2^{n_2} \cdots p_m^{n_m},$$

which implies that $p_1 | (p_2^{n_2} \cdots p_m^{n_m})^2$. Since R is a principal ideal domain, $p_1 | p_j$, for some $2 \leq j \leq m$, a contradiction.

Conversely, suppose that $I = (p^n)$, for some irreducible element $p \in R$ and a positive integer n. Assume $a, b \in R$ and $ab \in I$. Then $a = cp^k$ and $b = vp^i$ for some $c, v \in R$ such that $i + k \geq n$. Assume that $2k < n$ and $2i < n$. Then $2k + 2i < 2n$, a contradiction since $i + k \geq n$. Thus $a^2 \in I$ or $b^2 \in I$, and so I is 2-prime. $\square$

In the following proposition, we present some basic properties of 2-prime ideals.

Proposition 2.4. *Suppose that I is an ideal of R. Then the following statements hold*:

(1) *If I is a 2-prime ideal of R, then $P := \sqrt{I}$ is a prime ideal of R, and we say that I is P-2-prime. Furthermore, P is the smallest prime ideal of R*

which contains I, in that every prime ideal of R which contains I must also contain P.

(2) *Let P be a prime ideal of R. Then P^2 is a 2-prime ideal of R.*

(3) *Let $f : K \to R$ be a homomorphism of rings. If f is epimorphism and J is a 2-prime ideal of K containing $ker(f)$, then $f(J)$ is a 2-prime ideal of R.*

(4) *Let S be a multiplicatively closed subset of R and $f : R \to S^{-1}R$ denote the natural ring homomorphism. Then the following statements hod*:

(i) *If I is a P-2-prime ideal of R such that $I \cap S = \emptyset$, f is an epimorphism and I containing $ker(f)$, then $I^e := f(I)^{-1}R$ is a P^e-2-prime ideal of $S^{-1}R$. Furthermore, if $S^{-1}I$ is 2-prime and $S \cap Z_R(R/I) = \emptyset$, then I is 2-prime.*

(ii) *If J is a P-2-prime ideal of $S^{-1}R$, then $J^c = f^{-1}(J)$ is a P^c-2-prime ideal of R such that $J^c \cap S = \emptyset$.*

(5) *Let R_1 and R_2 be rings, and let R be the direct product ring $R = R_1 \times R_2$. Then I_1 (resp. I_2) is a 2-prime ideal of R_1 (resp. R_2) if and only if $I_1 \times R_2$ (resp. $R_1 \times I_2$) is a 2-prime ideal of R.*

(6) *If I is P-2-prime, and $a \in R \setminus P$, then $(I : a^2)$ is P-2-prime. In particular, $P = \sqrt{(I : a^2)}$.*

(7) *If I is irreducible and $(I : x) = (I : x^2)$ for every $x \in R \setminus I$, then I is 2-prime.*

Proof. (1) It is clear since every 2-prime ideal is a semipimary ideal of R.

(2) It is clear, as $P^2 \subseteq P$.

(3) Let $x, y \in R$ and $xy \in f(J)$. Since f is surjective, there are $a, b \in K$ such that $x = f(a)$ and $y = f(b)$. Thus $xy = f(ab) \in f(J)$. This means that there is $q \in J$ such that $f(ab) = f(q)$. In other words $ab - q \in kerf$. Since $kerf \subseteq J$, we see that both of $ab - q$ and q are contained in J, and therefore $ab = (ab - q) + q \in J$. But J is a 2-prime ideal, so either $a^2 \in I$ or $b^2 \in I$ and consequently either $f(a^2) = x^2 \in f(I)$ or $f(b^2) = y^2 \in f(J)$. The proof is complete.

(4) (i) Since $I \cap S = \emptyset$, we conclude that $I^e \neq S^{-1}R$. Thus I^e is a proper ideal of $S^{-1}R$. By [8, Lemma 5.24] and [8, Lemma 5.31], $I^e \trianglelefteq S^{-1}R$ and $\sqrt{I^e}$ $= (\sqrt{I})^e = P^e$. Let $x, y \in S^{-1}R$ and $xy \in I^e$. Then $x = \frac{a}{s_1}$ and $y = \frac{b}{s_2}$, for some $a, b \in R$, $s_1, s_2 \in S$, and $\frac{ab}{s_1 s_2} \in I^e$. Hence there exist $c \in f(I)$ and $r \in S$ such that $\frac{ab}{s_1 s_2} = \frac{c}{r}$. Thus $t(abr - cs_1 s_2) = 0$, for some $t \in S$. We observe that $(ta)(rb) \in f(I)$. Since I is a 2-prime ideal of R, it is clear by part (3), $f(I)$ is also a 2-prime ideal of $S^{-1}R$, and we obtain $(ta)^2 \in f(I)$ or $(rb)^2 \in f(I)$. If $(ta)^2 \in f(I)$, then $x^2 = (\frac{a}{s_1})^2 = \frac{t^2 a^2}{(ts_1)^2} \in I^e$, and if $(rb)^2 \in f(I)$, then $y^2 = (\frac{b}{s_2})^2 = \frac{r^2 b^2}{(rs_1)^2} \in I^e$. Thus I^e is a P^e-2-prime ideal of $S^{-1}R$. Now, assume that $a, b \in R$ and $ab \in I$. Then $ab/1 \in S^{-1}I$. Since $S^{-1}I$ is 2-prime, either $a^2/1 \in S^{-1}I$ or $b^2/1 \in S^{-1}I$. If $a^2/1 \in S^{-1}I$, then there exists $s \in S$ such that $sa^2 \in I$. Since $S \cap Z_R(R/I) = \emptyset$, we conclude that $a^2 \in I$. The case $b^2/1 \in S^{-1}I$ is similar. Thus $S^{-1}I$ is 2-prime.

(ii) It is an immediate consequent from part (3) of [3, Proposition 1.3] and [8, Exercise 2.43] that J^c is P^c-2-prime.

(5) $\Rightarrow$) Suppose that I_1 is a 2-prime ideal of R_1. Let $(a,b)(c,d) \in I_1 \times R_2$ for some $(a,b),(c,d) \in R$. Then $ac \in I_1$. Since I_1 is 2-prime, either $a^2 \in I_1$ or $c^2 \in I_1$. Hence either $(a,b)^2 \in I_1 \times R_2$ or $(c,d)^2 \in I_1 \times R_2$. Thus $I_1 \times R_2$ is a 2-prime ideal of R.

$\Leftarrow$) Let $I_1 \times R_2$ be a 2-prime ideal of R, and let $ab \in I_1$ for some $a, b \in R_1$. Then $(a,1)(b,1) \in I_1 \times R_2$. Hence $(a,1)^2 \in I_1 \times R_2$ or $(b,1)^2 \in I_1 \times R_2$. Therefore $a^2 \in I_1$ or $b^2 \in I_1$. Thus I_1 is a 2-prime ideal of R_1.

(6) Let $b \in (I : a^2)$. Then $ba^2 \in I$ and $a \notin P$. Since I is a P-2-prime ideal, we get $b^2 \in I$ and hence $b \in \sqrt{I} = P$. Thus $I \subseteq (I : a^2) \subseteq P$ and so

$$P = \sqrt{I} \subseteq \sqrt{(I : a^2)} \subseteq \sqrt{P} = P.$$

Hence $\sqrt{(I : a^2)} \subseteq \sqrt{P} = P \subset R$. This also means that $(I : a^2)$ is proper. Now suppose that $c, d \in R$ and $cd \in (I : a^2)$ but $d^2 \notin (I : a^2)$. Then $cda^2 = (ca)(da) \in I$. Since $d^2a^2 \notin I$ and I is P-2-prime, we deduce that $(ca)^2 = c^2a^2 \in I$. Therefore, $(I : a^2)$ is P-2-prime.

(7) Let I be irreducible and let $xy \in I$ be such that $x^2 \notin I$, for some $x, y \in R$. If $x \in I$ or $y \in I$, then there is nothing to prove. Assume that $x \notin I$ and $y \notin I$. We show that $y^2 \in I$. Suppose to the contrary, $y^2 \notin I$. Let $a \in (I + x^2) \cap (I + y^2)$. Then there are $c, d \in I$ and $s, t \in R$ such that $a = c + sx^2 = d + ty^2$. Hence $ax = cx + sx^3 = dx + ty^2x \in I$. Thus $sx^3 \in I$, and since $(I : x) = (I : x^2)$, we conclude that $sx^2 \in I$. Therefore, $a = c + sx^2 \in I$. This shows that $(I + x^2) \cap (I + y^2) \subseteq I$, and hence $(I + x^2) \cap (I + y^2) = I$, a contradiction. Thus I is a 2-prime ideal of R. $\square$

Corollary 2.5. (1) *Let $f : K \to R$ be the inclusion homomorphism of rings. If J is a 2-prime ideal of R, then $J \cap K$ is a 2-prime ideal of K.*

(2) *Let $I \subseteq J$ be ideals of R. Then J is a 2-prime ideal of R if and only if J/I is a 2-prime ideal of R/I.*

Corollary 2.6. *Let I be an ideal of ring R and X is an indeterminate. Then the following statements hold.*

(1) *$\langle I, X\rangle$ is a 2-prime ideal of $R[X]$ if and only if I is a 2-prime ideal of R.*

(2) *If $I[X]$ is a 2-prime ideal of $R[X]$, then I is a 2-prime ideal of R.*

Proof. (1) By part (2) of Corollary 2.5 and taking the isomorphism $\langle I, X\rangle/\langle X\rangle \cong I$ in $\langle R, X\rangle/\langle X\rangle \cong R$, we conclude that $\langle I, X\rangle$ is a 2-prime ideal of $R[X]$ if and only if I is a 2-prime ideal of R.

(2) It is clear by part (1) of Corollary 2.5. $\square$

It is worth mentioning that if J and K are 2-prime ideals of R, then $J \cap K$ and JK need not be 2-prime ideals of R (See the following example).

Example 2.7. In the ring of integers $\mathbb{Z}$, $2\mathbb{Z}$ and $3\mathbb{Z}$ are 2-prime ideals but $2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$ is not 2-prime.

Suppose that $R = \mathbb{Z}[y] + 3x\mathbb{Z}[x, y]$. Then $J = yR$ and $K = 3x\mathbb{Z}[x, y]$ are 2-prime ideals of R. Let $I = JK$. Then $(3x)y \in I$. Clearly $9x^2 \notin I$ and $y^2 \notin I$. Hence I is not a 2-prime ideal of R.

Next, we state the 2-prime avoidance theorem for 2-prime ideals of R. First we need the following lemma.

Let $I, I_1, I_2, \dots, I_n$ be ideals of R. A covering $I \subseteq I_1 \cup I_2 \cup \dots \cup I_n$ is said to be efficient precisely when I is not contained in the union of any $n-1$ of the ideals $I_1, I_2, \dots, I_n$. Analogously we shall say that $I = I_1 \cup I_2 \cup \dots \cup I_n$ is an efficient union if none of the I_k may be excluded (See [5, 7]).

Lemma 2.8. *Let $I \subseteq I_1 \cup I_2 \cup \dots \cup I_n$ $(n \geq 2)$ be an efficient covering. If $I \cap \sqrt{I_i} \nsubseteq I \cap \sqrt{I_j}$ for every $i \neq j$, then no I_j is 2-prime, for every $j \in \{1, \dots, n\}$.*

Proof. Suppose to the contrary, I_j is a 2-prime ideal of R, for some $j \in \{1, \dots, n\}$. It is easy to see that $I = (I \cap \sqrt{I_1}) \cup (I \cap \sqrt{I_2}) \cup \dots \cup (I \cap \sqrt{I_n})$ is an efficient covering. Thus there exists an element $x_j \in I \setminus \sqrt{I_j}$, for every $j \in \{1, \dots, n\}$. Since $I = (I \cap I_1) \cup (I \cap I_2) \cup \dots \cup (I \cap I_n)$ is an efficient union, we conclude that $(\bigcap_{i \neq j} I_i) \cap I \subseteq I_j \cap I$, by [5, Lemma 1]. By hypothesis, $\sqrt{I_i} \nsubseteq \sqrt{I_j}$, for every $i \neq j$. Hence there exists $y_i \in \sqrt{I_i} \setminus \sqrt{I_j}$ for every $i \neq j$. Let $y = \prod_{i \neq j} y_i$. Then $y = \prod_{i \neq j} y_i \in \sqrt{I_i}$ but $y = \prod_{i \neq j} y_i \notin \sqrt{I_j}$. Therefore, there exist positive integers $a_1, a_2, \dots, a_n$, where $y_1^{a_1} \in I_1$, $y_2^{a_2} \in I_2, \dots, y_n^{a_n} \in I_n$. Suppose that $l = \max\{a_1, a_2, \dots, a_n\}$. Then $y^l \in I_i$ for every $i \neq j$ but $y^l \notin I_j$. Hence, $y^l x_j \in I \cap I_i$ for every $i \neq j$, but $y^l x_j \notin I \cap I_j$, otherwise, assume that $y^l x_j \in I_j \cap I$. Since I_j is 2-prime, we have either $y^{2l} \in I_j$ or $x_j^2 \in I_j$ which is impossible as neither $y \in \sqrt{I_j}$ nor $x_j \in \sqrt{I_j}$ (as by part 1 of Proposition 2.4, $\sqrt{I_j}$ is a prime ideal of R). Therefore, $y^l x_j \notin I \cap I_j$ and this contradicts the fact that $(\bigcap_{i \neq j} I_i) \cap I \subseteq I_j \cap I$. The proof is complete. □

Now, we present the 2-prime avoidance theorem.

Theorem 2.9 (2-prime avoidance theorem)**.** *Let $I_1, \dots, I_n$ be ideals of R and at most two of $I_1, \dots, I_n$ are not 2-prime. Suppose that I is an ideal of R such that $I \subseteq I_1 \cup I_2 \cup \ldots \cup I_n$ and $I \cap \sqrt{I_i} \nsubseteq I \cap \sqrt{I_j}$, for every $i \neq j$. Then $I \subseteq I_j$, for some $j \in \{1, \dots, n\}$.*

Proof. Let $I \subseteq I_1 \cup I_2 \cup \dots \cup I_n$ be a covering such that at least $n-2$ of the ideals $I_1, I_2, \dots, I_n$ are 2-prime. Without loss of generality, one may reduce the covering to an efficient covering. If $n = 2$, then it is obvious. Suppose that $n > 2$. Since the covering is efficient and $I \cap \sqrt{I_i} \nsubseteq I \cap \sqrt{I_j}$ for every $i \neq j$, by Lemma 2.8, $n < 2$. Hence $n = 1$ and so $I \subseteq I_j$, for some $j \in \{1, \dots, n\}$. □

In the light of Theorem 2.9, we state the following corollaries.

Corollary 2.10. *Let $I = \langle r_1, r_2, \dots, r_s \rangle$ be a finitely generated ideal of R, for some $r_1, r_2, \dots, r_s \in R$. Let $I_1, \dots, I_n$ be 2-prime ideals of R, $I \nsubseteq \sqrt{I_i}$, for* *every $i \in \{1, \cdots, n\}$ and $I \cap \sqrt{I_i} \nsubseteq I \cap \sqrt{I_j}$, for every $i \neq j$. Then there exist $b_2, \ldots, b_s \in R$ such that $\alpha = r_1 + b_2 r_2 + \cdots + b_s r_s \notin \bigcup_{i=1}^{n} I_i$.*

Proof. We prove the corollary by induction on n. If $n = 1$, then the result is clear. So suppose that $n > 1$ and the result has been proved for smaller values than n. Then there exist $a_2, \ldots, a_s \in R$ such that $x = r_1 + a_2 r_2 + \cdots + a_s r_s \notin \bigcup_{i=1}^{n-1} I_i$. If $x \notin I_n$, then $x \notin \bigcup_{i=1}^{n} I_i$ and so there is nothing to prove. Hence suppose that $x \in I_n$. If $r_2, \ldots, r_s \in \sqrt{I_n}$, then $r_1 \in \sqrt{I_n}$, a contradiction, as $I \nsubseteq \sqrt{I_n}$. Thus we assume $r_i \notin \sqrt{I_n}$, for some i. Without loss of generality, suppose that $r_2 \notin \sqrt{I_n}$. By the hypothesis, $\sqrt{I_i} \nsubseteq \sqrt{I_n}$, for every $i \neq n$. Hence, there exists $y_i \in \sqrt{I_i} \setminus \sqrt{I_n}$, for every $i \neq n$. Therefore, there exist positive integers $k_1, k_2, \ldots, k_{n-1}$, where $y_1^{k_1} \in I_1$, $y_2^{k_2} \in I_2, \ldots, y_{n-1}^{k_{n-1}} \in I_{n-1}$. Let $l = \max\{k_1, k_2, \ldots, k_{n-1}\}$ and $y = \prod_{i=1}^{n-1} y_i$. Then $y^l \in I_i$, for every $i \neq n$ but $y^l \notin I_n$. Therefore, $y \in \sqrt{I_i} \setminus \sqrt{I_n}$, for every $i \neq n$. Let $\alpha = r_1 + (a_2 + y^l) r_2 + \cdots + a_s r_s$. We consider two cases. **Case one**: Suppose that $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$. By the 2-prime avoidance theorem (Theorem 2.9), $I \subseteq I_j$, for some $j \in \{1, \ldots, n\}$, which is a contradiction. **Case two**: Suppose that $I \nsubseteq I_1 \cup I_2 \cup \cdots \cup I_n$. Then by a similar argument as above, we assume $r_2 \notin \sqrt{I_n}$. Hence $\alpha = x + y^l r_2 \notin \bigcup_{i=1}^{n} I_i$ and so the proof is complete. $\square$

Corollary 2.11. *Let $I_1, I_2, \ldots, I_n$ be 2-prime ideals of R, I be an ideal of R and $I \cap \sqrt{I_i} \nsubseteq I \cap \sqrt{I_j}$, for every $i \neq j$. If $r \in R$ and $Rr + I \nsubseteq \bigcup_{i=1}^{n} I_i$, then there exists $x \in I$ such that $r + x \notin \bigcup_{i=1}^{n} I_i$.*

Proof. Suppose that $r \in \bigcap_{i=1}^{k} I_i$ but $r \notin \bigcup_{i=k+1}^{n} I_i$. If $k = 0$, then $r = r + 0 \notin \bigcup_{i=1}^{n} I_i$ and so we are done. Thus assume that $1 \leq k$. By the hypothesis, $I \cap \sqrt{I_i} \nsubseteq I \cap \sqrt{I_j}$ for every $i \neq j$, and so prime avoidance theorem implies that $I \nsubseteq \bigcup_{i=1}^{k} \sqrt{I_i}$. Hence, there exists $a \in I \setminus \bigcup_{i=1}^{k} \sqrt{I_i}$. We show that $\bigcap_{i=k+1}^{n} I_i \nsubseteq \bigcup_{i=1}^{k} \sqrt{I_i}$. Suppose that $\bigcap_{i=k+1}^{n} I_i \subseteq \bigcup_{i=1}^{k} \sqrt{I_i}$. By the 2-prime avoidance theorem we get $\bigcap_{i=k+1}^{n} I_i \subseteq \sqrt{I_j}$, for some $j \in \{1, \ldots, k\}$. This implies that $\sqrt{\bigcap_{i=k+1}^{n} I_i} = \bigcap_{i=k+1}^{n} \sqrt{I_i} \subseteq \sqrt{I_j}$ for some $j \in \{1, \ldots, k\}$. Since $\sqrt{I_i}$'s are prime, we conclude that $\sqrt{I_i} \subseteq \sqrt{I_j}$ where $i \in \{k+1, \ldots, n\}$, $j \in \{1, \ldots, k\}$. Thus, $I \cap \sqrt{I_i} \subseteq I \cap \sqrt{I_j}$ with $i \neq j$, which contradicts the hypothesis. Thus there exists $b \in \bigcap_{i=k+1}^{n} I_i \setminus \bigcup_{i=1}^{k} \sqrt{I_i}$. If for every $\alpha \in \bigcap_{i=k+1}^{n} I_i \setminus \bigcup_{i=1}^{k} I_i$ we have $\alpha \in \bigcup_{i=k+1}^{n} \sqrt{I_i} \setminus \bigcup_{i=1}^{k} I_i$, then we get $(\bigcap_{i=k+1}^{n} I_i \setminus \bigcup_{i=1}^{k} I_i) \subseteq (\bigcup_{i=k+1}^{n} \sqrt{I_i} \setminus \bigcup_{i=1}^{k} I_i)$, and so $\bigcap_{i=k+1}^{n} I_i \subseteq \bigcup_{i=k+1}^{n} \sqrt{I_i}$ which is also a contradiction. Thus, we can assume that $b \in \bigcap_{i=k+1}^{n} I_i \setminus \bigcup_{i=1}^{k} \sqrt{I_i}$. Let $x = ab$. Then $x \in I$. We also have $x \in \bigcap_{i=k+1}^{n} I_i$, but $x \notin \bigcup_{i=1}^{k} I_i$, because otherwise $x = ab \in I_i$ for some $i \in \{1, \ldots, k\}$. Since I_i is 2-prime, either $a^2 \in I_i$ or $b^2 \in I_i$ for some $i \in \{1, \ldots, k\}$, a contradiction. Thus $x \in \bigcap_{i=k+1}^{n} I_i \setminus \bigcup_{i=1}^{k} I_i$. Now $r \in \bigcap_{i=1}^{k} I_i \setminus \bigcup_{i=k+1}^{n} I_i$ shows that $r + x \notin \bigcup_{i=1}^{n} I_i$. $\square$

3. 2-P rings

In this section we study rings in which every 2-prime ideal is prime.

Definition 3.1. Let R be a ring. We say that R is a 2-P ring if every 2-prime ideal of R is prime.

The prove Theorems 3.4 and 3.7, the following lemma is needed.

Lemma 3.2. *Let (R, M) be a quasi-local ring and P be a prime ideal of R. Then PM is a 2-prime ideal of R. Furthermore, PM is prime if and only if $PM = P$.*

Proof. Let $a, b \in R$ and $ab \in PM \subseteq P$. Clearly, $a \in P$ or $b \in P$. We assume that $a \in P$. Since a is not a unit, we conclude that $a \in M$. Hence, $a^2 \in PM$, as desired.

Now let PM be a prime ideal of R and $x \in P$. Clearly, $PM \subseteq P \subseteq M$. Hence, $x^2 \in PM$. Since PM is prime, $x \in PM$, i.e., $PM = P$. □

Corollary 3.3. *Let (R, M) be a local 2-P ring. Then R is a field.*

Proof. By Lemma 3.2, M^2 is 2-prime. Since R is a 2-P ring, we deduce that $M^2 = M$. Now, the result follows from Nakayama's lemma (see [8, Theorem 8.24]). □

Theorem 3.4. *Let (R, M) be a quasi-local ring. Then R is a 2-P ring if and only if $IM = P$, for every minimal prime ideal P over an arbitrary 2-prime ideal I. In particular, M is an idempotent ideal if R is a 2-P ring.*

Proof. $\Rightarrow$) Let R be a 2-P ring and P be a minimal prime over a 2-prime ideal I. Then I is prime and $P = I$. By Lemma 3.2, IM is 2-prime and hence IM is prime. Again by Lemma 3.2, $IM = I$, the proof is complete.

$\Leftarrow$) Let I be a 2-prime ideal of R. By Proposition 2.4(1), $I \subseteq \sqrt{I} = P$. Since $IM = P$, we deduce that $P = IM \subseteq I \cap M = I$ and so $I = P$ is prime, as desired. □

Proposition 3.5. *Suppose that R is a 2-P ring. Then $P^2 = P$, for every prime ideal P of R.*

Proof. Suppose that R is a 2-P ring and P is a prime ideal of R. By part (2) of Proposition 2.4, P^2 is a 2-prime ideal of R. Since R is 2-P ring, P^2 is a prime ideal of R. It is easily seen that $P^2 = P$. □

Definition 3.6. Let I be an ideal and P be a 2-prime ideal of a ring R. We say that P is a minimal 2-prime ideal over I if there is no a 2-prime ideal Q of R such that $I \subseteq Q \subset P$. We denote the set of minimal 2-prime ideals over I by 2-$Min_R(I)$.

Theorem 3.7. *Let (R, M) be a quasi-local ring, P a prime ideal of R and $(\sqrt{I})^2 \subseteq I$, for every 2-prime ideal I of R. Then the following statements are equivalent:*

(1) *For every ideal* $I \in 2\text{-}Min_R(P^2)$*, if* $P \in Min_R(I)$*, then* $IM = P$;
(2) *For every ideal* $I \in 2\text{-}Min_R(P^2)$ *such that* $I \subseteq P$*, we have* $I = P$.

Proof. $\Rightarrow$) Let $I \in 2\text{-}Min_R(P^2)$ and $I \subseteq P$. We claim that $P \in Min_R(I)$. Suppose there exists a prime ideal Q such that $I \subseteq Q \subseteq P$. Clearly,

$$P^2 \subseteq I \subseteq Q \subseteq P.$$

Let $x \in P$. Then $x^2 \in P^2$ and hence, $x^2 \in Q$. Since Q is prime, we get $x \in Q$. Therefore, $P = Q$ and so the claim is proved. Clearly, $IM \subseteq I \subseteq P$. Now by Part (1) $IM = P$ and so $I = P$.

$\Leftarrow$) Suppose that $I \in 2\text{-}Min_R(P^2)$ such that P is a minimal prime ideal over I. Since $\sqrt{I}$ is a prime ideal of R and since $P \in Min_R(I)$, we conclude that $\sqrt{I} = P$. Hence by hypothesis, $(\sqrt{I})^2 = P^2 \subseteq I \subseteq P$ and so (by (2)) $I = P$. Since $P^2 \subseteq PM \subseteq I = P$ and PM is 2-prime (by Lemma 3.2), we conclude that $PM = IM = P$. □

We close this paper with the following corollary.

Corollary 3.8. *Let* (R, M) *be a quasi-local ring and* $I \in 2\text{-}Min_R(P^2)$*, for every* P*-2-prime ideal* I*. Then* R *is a* 2-P *ring if and only if* $I = P$*, for every ideal* $I \in 2\text{-}Min_R(P^2)$ *such that* $I \subseteq P$.

Proof. The proof follows from Theorems 3.4 and 3.7. □

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