ON THE FINITENESS OF REAL STRUCTURES OF PROJECTIVE MANIFOLDS

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ABSTRACT. Recently, Lesieutre constructed a 6-dimensional projective variety X over any field of characteristic zero whose automorphism group $\mathrm{Aut}(X)$ is discrete but not finitely generated. As an application, he also showed that X is an example of a projective variety with infinitely many non-isomorphic real structures. On the other hand, there are also several finiteness results of real structures of projective varieties. The aim of this short paper is to give a sufficient condition for the finiteness of real structures on a projective manifold in terms of the structure of the automorphism group. To be more precise, in this paper we show that, when X is a projective manifold of any dimension ≥ 2, if $\mathrm{Aut}(X)$ does not contain a subgroup isomorphic to the non-abelian free group $\mathbb{Z} * \mathbb{Z}$, then there are only finitely many real structures on X, up to $\mathbb{R}$-isomorphisms.

1. Introduction and main results

Our concern of this paper is the finiteness or infiniteness of real structures on a projective variety defined over the complex number field $\mathbb{C}$. The finiteness of real structures on a projective variety is known for several classes of compact smooth manifolds. For example, minimal rational surfaces, minimal algebraic surfaces of non-negative Kodaira dimension, and del Pezzo surfaces are known to have a finite number of the isomorphism classes of real structures (see [8], [4], and [11] for more details). In addition, the finiteness of real structures of projective spaces, abelian varieties, and varieties of general type has been known in the literature (see, e.g., [3], [11], and [12]).

On the other hand, recently in [9] Lesieutre constructed a 6-dimensional projective variety X over any field of characteristic zero whose automorphism group $\mathrm{Aut}(X)$ is discrete but not finitely generated. Further, he showed that X is an example of a projective variety with infinitely many non-isomorphic real structures, and answered a long standing open problem. After that, Dinh and Oguiso showed in [5] that there is a smooth complex projective variety of any

dimension ≥ 2 defined over the complex number field $\mathbb{C}$ whose automorphism group is discrete and not finitely generated. As an application, they also showed that their variety admits infinitely many real structures which are mutually non-isomorphic over the real number field. Moreover, Oguiso showed in [10] that there is a smooth projective surface, birational to some K3 surface, such that the automorphism group is discrete and not finitely generated, over any algebraically closed field of odd characteristic except precisely an algebraic closure of the prime field.

In order to state our main results precisely, we next set up necessary notation and definitions. A *real structure* σ on a projective variety X is an anti-holomorphic involution on X. Two real structures σ and σ' on X are *equivalent* if there exists a $\mathbb{C}$-automorphism φ of X such that $\sigma' \circ \varphi = \varphi \circ \sigma$. Thus, such an automorphism φ of X is equivariant with respect to those two real structures σ and σ', and it corresponds to an $\mathbb{R}$-isomorphism between two $\mathbb{R}$-varieties $X_0 = X/\langle\sigma\rangle$ and $X_0' = X/\langle\sigma'\rangle$. Conversely, any $\mathbb{R}$-isomorphism between two $\mathbb{R}$-varieties X_0 and X_0' corresponds to a $\mathbb{C}$-automorphism between their complexifications that is equivariant with respect to the natural real structures. As a consequence, the set of the equivalence classes of real structures on a projective variety X corresponds bijectively to that of the isomorphism classes of real structures on X. It has been shown in [3, 2.6] that the set of all equivalence classes of real structures on X is given by $H^1(\langle\sigma\rangle, \mathrm{Aut}(X))$. Here $\langle\sigma\rangle$ acts on $\mathrm{Aut}(X)$ by conjugation in such a way that

$$\langle\sigma\rangle \times \mathrm{Aut}(X) \to \mathrm{Aut}(X), \quad (\sigma, \varphi) \mapsto \sigma \circ \varphi \circ \sigma^{-1}.$$

It is obvious that if $\mathrm{Aut}(X)$ is finite, then the number of real structures on X should be finite. In particular, varieties of general type have a finite automorphism group, so the finiteness of real structures on such varieties follows immediately.

For a subgroup G of $GL(m, \mathbb{C})$ with $m \geq 1$, by the well-known Tits alternative theorem in [13] either G contains a subgroup isomorphic to $\mathbb{Z} * \mathbb{Z}$ or G contains a solvable subgroup of finite index. Analogously, a theorem of Tits type for any subgroup G of $\mathrm{Aut}(X)$ of a compact Kähler manifold X has been shown in [15]. See also [6] for a previous result in case of any abelian subgroup G of $\mathrm{Aut}(X)$ such that each non-trivial element of G is of positive entropy.

With these said, our main result is:

Theorem 1.1. *Let X be a projective manifold of dimension $n \geq 2$. If $\mathrm{Aut}(X)$ does not contain a subgroup isomorphic to $\mathbb{Z} * \mathbb{Z}$, then there are only finitely many real structures on X, up to $\mathbb{R}$-isomorphisms.*

Remark 1.2. By the theorem of Tits type for any compact Kähler manifolds in [15] mentioned above, the existence of such an automorphism group $\mathrm{Aut}(X)$ as in Theorem 1.1 implies that there is a subgroup G of $\mathrm{Aut}(X)$ of finite index such that the group G^* given by the image of the representation of G to

$GL(\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{C})$ is solvable and Zariski-connected. Refer to [15] or Theorem 3.1 for more details.

This theorem generalizes a result [2, Theorem 1] of Benzerga which has been proved only for complex rational surfaces.

Corollary 1.3. *Let X be a smooth complex rational surface, and let $\mathrm{Aut}^*(X)$ be the image of the natural morphism from $\mathrm{Aut}(X)$ to $O(\mathrm{Pic}(X)) \subset GL(\rho(X), \mathbb{Z})$, where $\rho(X)$ denotes the rank of the Picard group $\mathrm{Pic}(X)$. If $\mathrm{Aut}^*(X)$ does not contain a subgroup isomorphic to $\mathbb{Z} * \mathbb{Z}$, then X has finitely many real structures, up to $\mathbb{R}$-isomorphisms.*

By the results of Gromov [7] and Yomdin [14], for any automorphism $\varphi \in \mathrm{Aut}(X)$ one may define the *topological entropy* $h(\varphi)$ by the logarithm of the spectral radius of the automorphism φ^* of the Néron-Severi group $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{C}$. Clearly the topological entropy $h(\varphi)$ can be zero (or null) or positive. By [1, Theorem 1], if a rational surface X has an infinite number of non-equivalent real structures, then X is a blown-up of the projective space $\mathbb{P}^2$ at r points with $r \geq 10$ and has at least one automorphism of positive entropy. Theorem 1.1 shows that the converse of [1, Theorem 1] does not hold, in general. That is, the existence of an automorphism of positive entropy on a projective manifold of any dimension ≥ 2 does not necessarily imply the finiteness of real structures, up to $\mathbb{R}$-isomorphisms.

We organize this paper, as follows. In Section 2, we set up some basic notation and collect a few important facts necessary for the proof of Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.1.

2. Group cohomology of finite groups

The aim of this section is to briefly review some fundamental definitions and facts necessary for the proof of Theorem 1.1, for the sake of reader's convenience.

To begin with, we recall some definitions and results about non-abelian group cohomology of finite groups (see [1, Section 2] for more details).

Let K be a finite group. A *K-group* A is a group on which K acts by automorphisms in such a way that for any $\sigma \in K$

$$\sigma(ab) = \sigma(a)\sigma(b), \quad a, b \in A.$$

In particular, if A is abelian, then it is called a *K-module*. The 0-th cohomology group $H^0(K, A)$ of K with coefficients in A is defined by the set A^K of the fixed points of this action of K.

For a K-group A and each $a \in A$, there is a map, denoted by the same letter a, given by

$$K \to A, \quad \sigma \mapsto a_\sigma := \sigma(a).$$

The map $a : K \to A$ is called a *1-cocycle* if for any $\sigma, \tau \in K$ we have

$$a_{\sigma\tau} = a_\sigma (a_\tau)_\sigma = a_\sigma \sigma(a_\tau).$$

If $K = \langle \sigma \rangle \cong \mathbb{Z}/2$, then the cocycle condition is equivalent to that of an element $a_\sigma \in A$ such that $a_\sigma \sigma(a_\sigma) = e$, where e denotes the identity element of A. The set of all 1-cocycles is denoted by $Z^1(K, A)$. Two cocycles a and b in $Z^1(K, A)$ are *equivalent*, denoted by $a \sim b$, if there is an element $c \in A$ such that for any $\sigma \in K$

$$b_\sigma = c^{-1} a_\sigma c_\sigma.$$

The first cohomology group $H^1(K, A)$ of K with coefficients in A is defined by $H^1(K, A) = Z^1(K, A)/\sim$.

If

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is a short exact sequence of K-groups, then we have the following long exact sequence of pointed sets

$$0 \to A^K \to B^K \to C^K \to H^1(K, A) \xrightarrow{f^*} H^1(K, B) \xrightarrow{g^*} H^1(K, C) \to \cdots .$$

Thus, if $H^1(K, C)$ and $H^1(K, A)$ are both finite, then $H^1(K, B)$ should be also finite.

For the proof of Theorem 1.1, we also need the following fact (see [1, Theorem 2.4]).

Lemma 2.1. *Let K be a finite group, and let A be a K-group. Then the following statements hold*:

(1) *If A is a complex linear algebraic group with a real structure σ, then $H^1(\langle \sigma \rangle, A)$ is finite for the natural action of $\langle \sigma \rangle$ on A.*

(2) *If A contains a subgroup of finite index isomorphic to $\mathbb{Z}^k$ for some non-negative $k \in \mathbb{Z}$, then $H^1(K, A)$ is finite, independent of the action of K on A. In particular, if A is finite, i.e., $k = 0$, then $H^1(K, A)$ is always finite.*

3. Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1, and continue to use the notation in Section 2.

For the proof of Theorem 1.1, we first recall the theorem of Tits type for compact Kähler manifolds as in [15, Theorem 1.1].

Theorem 3.1. *Let X be a compact Kähler manifold of dimension $n \geq 2$, and let G be a subgroup of $\mathrm{Aut}(X)$. Then only one of the following two statements holds*:

(1) *G contains a subgroup isomorphic to the non-abelian free group $\mathbb{Z} * \mathbb{Z}$.*

(2) *There is a subgroup G_1 of G of finite index such that the group G_1^* given by the image of the representation of G_1 to $GL(H^{1,1}(X))$ is solvable and Zariski-connected. Moreover, the set*

$$N(G_1) = \{g \in G_1 \,|\, h(g) = 0\}$$

consisting of all elements $g \in G_1$ with null entropy is a normal subgroup of G_1 and the quotient group $G_1/N(G_1)$ is a free abelian group $\mathbb{Z}^k$ of rank $k \leq n-1$.

In case of a projective manifold X, we may use $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ instead of $H^{1,1}(X)$ in the above theorem. To be more precise, let G be a group of automorphisms of a projective manifold (or, more generally, variety) X. Then G acts on certain cohomological spaces such as the Néron-Severi group $\mathrm{NS}(X)$ by pulling back divisor classes, and thus we have the following homomorphism

$$p : G \to GL(\mathrm{NS}(X)) = GL(\rho(X), \mathbb{Z}),$$

where $\rho(X)$ denote the rank of $\mathrm{NS}(X)$, called the *Picard number* of X.

From now on, we let G^* (resp. $G^{\#}$) denote the image (resp. kernel) of the homomorphism p from G to $GL(\mathrm{NS}(X))$. In particular, if $G = \mathrm{Aut}(X)$, then we have the following short exact sequence

(3.1) $$0 \to \mathrm{Aut}^{\#}(X) \to \mathrm{Aut}(X) \xrightarrow{p} \mathrm{Aut}^*(X) \to 0,$$

where $\mathrm{Aut}^{\#}(X)$ denotes the kernel of p. The group $\mathrm{Aut}^{\#}(X)$ is a linear algebraic group, while $\mathrm{Aut}^*(X)$ is a discrete subgroup of $GL(\rho(X), \mathbb{Z})$ (see [1, Introduction] for more details).

Now, we are ready to prove Theorem 1.1, as follows.

Proof of Theorem 1.1. As before, let X be a projective manifold of dimension $n \geq 2$. Assume that $\mathrm{Aut}(X)$ does not contain a subgroup isomorphic to the free product $\mathbb{Z} * \mathbb{Z}$. Then it follows from Theorem 3.1 that there is a subgroup G of $\mathrm{Aut}(X)$ of finite index such that $G/N(X)$ is isomorphic to $\mathbb{Z}^k$ for some $k \leq n-1$. Here $N(G)$ is a normal subgroup of $\mathrm{Aut}(X)$ which consists of all elements of G with null entropy.

Let σ be a real structure on X. If there is no such a real structure on X, we are done.

We need the following lemma.

Lemma 3.2. *The first cohomology group $H^1(\langle \sigma \rangle, \mathrm{Aut}^*(X))$ is finite.*

Proof. By assumption and Theorem 3.1(2), $\mathrm{Aut}^*(X)$ contains a subgroup isomorphic to

$$G^* \cong G/N(X) \cong \mathbb{Z}^k$$

of finite index for some integer k with $0 \leq k \leq n-1$. Thus it follows from Lemma 2.1(2) that $H^1(\langle \sigma \rangle, \mathrm{Aut}^*(X))$ should be finite. This completes the proof. $\square$

Next, as in (3.1) we consider the short exact sequence

(3.2) $$0 \to \mathrm{Aut}^{\#}(X) \to \mathrm{Aut}(X) \xrightarrow{p} \mathrm{Aut}^*(X) \to 0.$$

Since X is a projective manifold, it follows from Lemma 2.1(1) (or [1, Theorem 1.2]) that $\mathrm{Aut}^{\#}(X)$ is a complex linear algebraic group and admits a real

structure induced from the real structure σ on X. Thus, by Lemma 2.1(1), $H^1(\langle\sigma\rangle, \mathrm{Aut}^\#(X))$ is finite with respect to the induced action of σ on $\mathrm{Aut}^\#(X)$.

Finally, we consider the long exact sequence

$$\cdots \to H^0(\langle\sigma\rangle, \mathrm{Aut}^*(X))$$

(3.3)
$$\to H^1(\langle\sigma\rangle, \mathrm{Aut}^\#(X)) \to H^1(\langle\sigma\rangle, \mathrm{Aut}(X)) \xrightarrow{p^*} H^1(\langle\sigma\rangle, \mathrm{Aut}^*(X))$$
$$\to \cdots$$

induced from the short exact sequence (3.2). Since $H^1(\langle\sigma\rangle, \mathrm{Aut}^*(X))$ is shown to be finite by Lemma 3.2, it is immediate to see from (3.3) together with the finiteness of $H^1(\langle\sigma\rangle, \mathrm{Aut}^\#(X))$ that

$$H^1(\langle\sigma\rangle, \mathrm{Aut}(X))$$

should be also finite. This implies that the number of real structures of X is finite, up to $\mathbb{R}$-isomorphisms. This completes the proof of Theorem 1.1.   □

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