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DISTRIBUTIONAL CHAOS AND DISTRIBUTIONAL CHAOS IN A SEQUENCE OCCURRING ON A SUBSET OF THE ONE-SIDED SYMBOLIC SYSTEM

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Abstract. The aim of this paper is to show that for the one-sided symbolic system, there exist an uncountable distributively chaotic set contained in the set of irregularly recurrent points and an uncountable distributively chaotic set in a sequence contained in the set of proper positive upper Banach density recurrent points.

1. Introduction

The mathematical term chaos was first introduced by Li and Yorke [6] in 1975, where the authors gave a simple criterion for interval maps to be chaotic, that is, "period three implies chaos". Since then, a large number of scholars have studied the chaotic phenomena involved in different fields. From different points of view of understanding of chaotic phenomena, several chaotic conceptions were introduced by distinct authors. For example, Li in [5] introduced the concept of ω-chaos; Xiong in [17] introduced a new kind of sensitivity called n-sensitivity and for the transitive systems, he proved the existence of n-sensitivity for each positive integer n; Devaney [2] connected the transitivity with the denseness of periodic points of a dynamical system to raise a chaos called Devaney's chaos by other researchers and it was proved by Huang and Ye [4] that Devaney's chaos is stronger than the chaos in the sense of Li-Yorke.

The concept of distributional chaos was introduced by Schweizer and Smital [10] in 1994 for the study of interval dynamic systems. In 2005, the distributional chaos was divided by Balibrea and Smital [1] into three distinct levels, namely DC1, DC2 and DC3. After then, many results on distributional chaos emerged. For example, Dvorakova [3] proved that if f is a DC3 continuous map of a compact metric space, then also f^N is DC3 for every $N > 0$; Oprocha and Wu [9] proved that the average shadowing property trivializes in the case

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of mean equi-continuous systems and that it implies distributional chaos when measure center is non-degenerate.

The concept of distributional chaos in a sequence was introduced in 1999 by Wang [11], which is weaker than distributional chaos. Immediately afterwards, many works on distributional chaos in a sequence were published. For instance, Wang and Yang [15] proved that topologically weakly mixing implies distributional chaos in a sequence; Wang and Peng [14] proved that if f is transitive and not minimal, then there is a factor map which is distributively chaotic in a sequence.

The core issue of dynamical systems is the asymptotic property of orbits. And it is well known that all the important dynamic behaviors of a dynamical system are mainly concentrated on the set of non-wandering points. So the set of wandering points can be regarded as a kind of interference of a system. Whether the set of wandering points is all the interference of a system? The answer is negative since in order to describe the smallest subsystem that maintains all the important dynamic behaviors of the original system, Zhou [18] introduced two new recurrent levels between recurrent points and almost periodic points, which are called weakly almost periodic point and quasi-weakly almost periodic point, respectively. And he pointed out that, in some certain, to study the important dynamic behaviors of a dynamic system, it suffices to study the properties of all weakly almost periodic points of a system.

The symbolic dynamical system (symbolic system for simplicity) is a special dynamical system with extensive usefulness especially in constructing counterexamples. Some symbolic systems have complexly dynamic behaviors. For example, Liao [7] in 1998 constructed in symbolic systems a class of minimal sets displaying distributional chaos. In 2007, Wang [12] proved that there exists an uncountable distributively chaotic set in a sequence in symbolic systems and every point in the chaotic set is weakly almost periodic but not almost periodic. In 2015, Wang [13] strengthened the conclusion of [12] and obtained that the one-sided shift has an uncountable distributively chaotic set contained in the set of all weakly almost periodic points but every point in the chaotic set is not almost periodic.

In this paper, we mainly prove that for the one-sided symbolic system (Σ_N, σ), there exist an uncountable distributively chaotic set contained in the set of irregularly recurrent points and an uncountable distributively chaotic set in a sequence contained in the set of proper positive upper Banach density recurrent points of σ.

2. Preliminaries

We round out the introduction with some notations and known conclusions that will be used in the proofs of main results of this paper.

We say that (X, f) is a dynamical system if X is a compact metric space with a metric d and $f : X \to X$ is a continuous map. Use $\mathbb{N}$ and $\mathbb{N}_0$ to stand for the sets of positive integers and non-negative integers, respectively. Denote by $V(x,\epsilon)$ the open ball centered at $x \in X$ and a radius $\epsilon > 0$, $\overline{V}(x,\epsilon)$ stays for the closure of $V(x,\epsilon)$ in X. The orbit of $x \in X$ under f is denoted by $\mathrm{Orb}(x,f)$ and in this paper, $A - B := \{x : x \in A, x \notin B\}$ denotes the difference set of $A, B \subseteq X$.

Suppose U, V are nonempty open subsets of X and $x \in X$. Write

$$N(U,V) = \left\{n \in \mathbb{N}_0 : U \cap f^{-n}(V) \neq \emptyset\right\}$$

and

$$N(x,U) = \{n \in \mathbb{N}_0 : f^n(x) \in U\}.$$

The upper density of a set $S \subset \mathbb{N}$ is defined as

$$\overline{d}(S) = \limsup_{n\to\infty} \frac{|S \cap \{1,2,\ldots,n\}|}{n}$$

and its lower density is defined as

$$\underline{d}(S) = \liminf_{n\to\infty} \frac{|S \cap \{1,2,\ldots,n\}|}{n},$$

where $|A|$ denotes the cardinality of the set A.

The upper Banach density of S is defined as

$$BD^+(S) = \limsup_{|I|\to\infty} \frac{|S \cap I|}{|I|},$$

where I ranges over interval segments over $\mathbb{N}$. The lower Banach density of S can be similarly defined.

A point $x \in X$ is called a recurrent point of f if for any $\epsilon > 0$ there is a positive integer n such that

$$f^n(x) \in V(x,\epsilon).$$

Denote by $R(f)$ the set of all recurrent points of f.

Definition 2.1 ([18])**.** A point $x \in X$ is called a weakly almost periodic point of f if for any $\epsilon > 0$ there is an integer $N_\epsilon > 0$ such that

$$|\{r : f^r(x) \in V(x,\epsilon),\ 0 \le r < nN_\epsilon\}| \ge n, \quad \forall n > 0.$$

Definition 2.2 ([18])**.** A point $x \in X$ is called a quasi-weakly almost periodic point of f if for any $\epsilon > 0$ there are an integer $N_\epsilon > 0$ and a subsequence $\{n_i\}$ of positive integers such that

$$|\,\{r : f^r(x) \in V(x,\epsilon),\ 0 \le r < n_i N_\epsilon\}\,| \ge n_i, \quad \forall i > 0.$$

Definition 2.3 ([16])**.** A point $x \in X$ is called a positive upper Banach density recurrent point of f if for any $\epsilon > 0$, $N(x, V(x,\epsilon))$ has positive upper Banach density.

Denote by $W(f)$, $QW(f)$ and $BD^+(f)$ the sets of all weakly almost periodic points, all quasi-weakly almost periodic points and all positive upper Banach density recurrent points of f, respectively. Clearly, $W(f) \subseteq QW(f) \subseteq BD^+(f) \subseteq R(f)$, and there are examples in [18] and [16] showing that all the inclusion relations above are proper. And in [8], the author called each point in $QW(f) - W(f)$ an irregularly recurrent point of f. In the paper, we also use this term and call each point in $BD^+(f) - QW(f)$ a proper positive upper Banach density recurrent point of f. It is well known that for the one-sided symbolic system (Σ_N, σ) (see the following introduction), the sets of irregularly recurrent points and proper positive upper Banach density recurrent points of σ are non-empty.

In order to decide whether a point is weakly almost periodic or quasi-weakly almost periodic, Zhou [18] posed the following useful lemmas.

Lemma 2.4 ([18])**.** *Let $x \in R(f)$. Then $x \in W(f)$ if and only if for any $\epsilon > 0$, $N(x, V(x, \epsilon))$ has positive lower density.*

Lemma 2.5 ([18])**.** *Let $x \in R(f)$. Then $x \in QW(f)$ if and only if for any $\epsilon > 0$, $N(x, V(x, \epsilon))$ has positive upper density.*

Next, we recall the conception of distributional chaos.

Let $x, y \in X$. For any real $t > 0$, let

$$F^*_{xy}(t) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{[0,t)}(d(f^i(x), f^i(y))) \text{ and}$$
$$F_{xy}(t) = \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{[0,t)}(d(f^i(x), f^i(y))),$$

where χ_A is the characteristic function of the set A. Both $F^*_{xy}(t)$ and $F_{xy}(t)$ are non-decreasing functions and may be viewed as cumulative probability distribution functions satisfying $F^*_{xy}(t) = F_{xy}(t) = 0$ for all $t < 0$ (see [10] for details).

Definition 2.6 ([10])**.** A pair of points $(x, y) \in X \times X$ is said to be distributively chaotic if $F^*_{xy}(t) = 1$ for all $t > 0$ and $F_{xy}(\epsilon) = 0$ for some $\epsilon > 0$. f is said to distributively chaotic if there exists an uncountable set $D \subset X$ such that any two different points in D are distributively chaotic.

Let $N \geq 2$, $S = \{0, 1, \ldots, N-1\}$ and $\Sigma_N = \{(x_1 x_2 \cdots) : x_i \in S, i = 1, 2, \ldots\}$. Define $\rho : \Sigma_N \times \Sigma_N \to \mathbb{R}$ as follows: for any $x = (x_1 x_2 \cdots)$, $y = (y_1 y_2 \cdots) \in \Sigma_N$,

$$\rho(x, y) = \begin{cases} 0, & \text{if } x = y, \\ \frac{1}{N^k}, & \text{if } x \neq y, \text{ where } k = \min\{n \geq 1 : x_n \neq y_n\} - 1. \end{cases}$$

It is easy to verify that ρ is a metric on Σ_N. The space (Σ_N, ρ) is compact and is called the one-sided symbolic space.

Define $\sigma : \Sigma_N \to \Sigma_N$ by $\sigma(x_1x_2\cdots) = (x_2x_3\cdots)$ for each $(x_1x_2\cdots) \in \Sigma_N$.

Obviously, σ is continuous and so (Σ_N, σ) is a dynamical system which is called the one-sided symbolic system. If Y is a non-empty closed invariant subset of Σ_N, then (Y, σ) is called a sub-shift of (Σ_N, σ).

Every $A \in \bigcup_{n=1}^{\infty} S^n$ is called a tuple of S, where

$$S^n = \{(x_1x_2\cdots x_n) : x_i \in S, 1 \le i \le n\}$$

for each $n \ge 1$. We say that a tuple $A = (a_1a_2\cdots a_n)$ of S occurs in the tuple $B = (b_1b_2\cdots b_m)$ of S, denoted by $A \prec B$, if there exists $0 \le i < m - n$ such that $a_j = b_{i+j}$ for each $j = 1, 2, \ldots, n$, where $n < m$ and $n, m \in \mathbb{N}$. Meanwhile, n is called the length of the tuple $A = (a_1a_2\cdots a_n)$, denoted by $|A|$. If $B = (b_1b_2\cdots b_m)$ and $C = (c_1c_2\cdots c_l)$ are two tuples of S, then $BC = (b_1b_2\cdots b_mc_1c_2\cdots c_l)$. What's more, if $A_1, A_2, \ldots$ are infinitely many tuples of S, then $(A_1A_2\cdots)$ is an element of Σ_N.

Let $A = (a_1a_2\cdots a_n)$ be a tuple of $\{0,1\}$. Denote $\bar{A} = (\bar{a}_1\bar{a}_2\cdots\bar{a}_n)$ and call it the inverse of A, where

$$\bar{a}_i = \begin{cases} 0, & \text{if } a_i = 1, \\ 1, & \text{if } a_i = 0 \end{cases}$$

for $i = 1, 2, \ldots, n$. Clearly, $|\bar{A}| = |A|$ and $\bar{\bar{A}} = A$. See [19] for more details of symbolic systems.

3. Distributional chaos occurring on the set of irregularly recurrent points of the one-sided symbolic system

In order to prove the main result of this section, a lemma is firstly given.

Lemma 3.1 ([7]). *For each $N \ge 2$, there exists an uncountable subset E of Σ_N such that for any two different points $x = (x_1x_2\cdots)$, $y = (y_1y_2\cdots)$ of E, $x_n = y_n$ for infinitely many n and $x_m \ne y_m$ for infinitely many m.*

The following is the main result of this section.

Theorem 3.2. *Let $N \ge 2$ and (Σ_N, σ) be the one-sided symbolic system. Then there exists an uncountable distributively chaotic set of σ contained in the set of irregularly recurrent points of σ.*

Proof. For simplicity, we only prove the result for the case of $N = 2$ since the proofs of other cases are completely similar.

Let $A_1 = 01$ and

$$A_2 = D_1 \overbrace{0\cdots0}^{|D_1|^2} \overbrace{1\cdots1}^{|D_1|^3} \overbrace{D_1\cdots D_1}^{m_{11}},$$

here $D_1 = A_1\bar{A}_1$ (or $\bar{A}_1A_1$) and $m_{11} = |D_1 \overbrace{0\cdots0}^{|D_1|^2} \overbrace{1\cdots1}^{|D_1|^3}|^2$.

Let

$$A_3 = D_2 \overbrace{0\cdots0}^{|D_2|^2}\overbrace{1\cdots1}^{|D_2|^3}\overbrace{D_1\cdots D_1}^{m_{21}}\overbrace{D_2\cdots D_2}^{m_{22}},$$

where D_2 is exactly a finite arrangement of all tuples of

$$\mathcal{B}_2 = \left\{B_1B_2 : B_i \in \left\{A_i, \bar{A}_i\right\}, i = 1, 2\right\}$$

and $m_{21} \geq |D_2 \overbrace{0\cdots0}^{|D_2|^2}\overbrace{1\cdots1}^{|D_2|^3}|^2$, $m_{22} = |D_2 \overbrace{0\cdots0}^{|D_2|^2}\overbrace{1\cdots1}^{|D_2|^3}\overbrace{D_1\cdots D_1}^{m_{21}}|^2$.

Assume that D_i and A_i are defined successfully for all $1 \leq i \leq k$, $k \in \mathbb{N}$, put

$$A_{k+1} = D_k \overbrace{0\cdots0}^{|D_k|^2}\overbrace{1\cdots1}^{|D_k|^3}\overbrace{D_1\cdots D_1}^{m_{k1}}\overbrace{D_2\cdots D_2}^{m_{k2}}\cdots\overbrace{D_k\cdots D_k}^{m_{kk}},$$

where D_n is exactly a finite arrangement of all tuples of $\mathcal{B}_n$ for all $n \leq k$, here

$$\mathcal{B}_n = \left\{B_1B_2\cdots B_k : B_i \in \left\{A_i, \bar{A}_i\right\}, i = 1, 2, \ldots, n\right\}$$

and for $1 \leq i \leq k$,

$$m_{ki} \geq |D_k \overbrace{0\cdots0}^{|D_k|^2}\overbrace{1\cdots1}^{|D_k|^3}\overbrace{D_1\cdots D_1}^{m_{k1}}\overbrace{D_2\cdots D_2}^{m_{k2}}\cdots\overbrace{D_{i-1}\cdots D_{i-1}}^{m_{k,i-1}}|^2,$$

$$m_{kk} = |D_k \overbrace{0\cdots0}^{|D_k|^2}\overbrace{1\cdots1}^{|D_k|^3}\overbrace{D_1\cdots D_1}^{m_{k1}}\overbrace{D_2\cdots D_2}^{m_{k2}}\cdots\overbrace{D_{k-1}\cdots D_{k-1}}^{m_{k,k-1}}|^2.$$

So by induction, B_k is defined well for all $k \in \mathbb{N}$.

By Lemma 3.1, take an uncountable subset E of Σ_2 satisfying that for any two different points $x = (x_1x_2\cdots)$ and $y = (y_1y_2\cdots)$ in E, $x_n = y_n$ for infinitely many n and $x_m \neq y_m$ for infinitely many m.

Let

$$\mathcal{B} = \left\{(B_1B_2\cdots) \in \Sigma_2 : B_i \in \left\{A_i, \bar{A}_i\right\}, i \geq 1\right\} \subseteq \Sigma_2$$

and define $\phi : E \to \mathcal{B}$ by $\phi(x) = (B_1B_2\cdots)$ for all $x = (x_1x_2\cdots) \in E$, where

$$B_i = \begin{cases} A_i, & \text{if } x_i = 1, \\ \bar{A}_i, & \text{if } x_i = 0 \end{cases}$$

for each $i \in \mathbb{N}$.

Write $\tilde{S} = \phi(E)$. Obviously, $\tilde{S} \subseteq \mathcal{B}$. Since E is uncountable and ϕ is injective, $\tilde{S}$ is uncountable.

Next it suffices to prove the following claims.

(1) $\tilde{S} \subseteq QW(\sigma)$;

(2) $\tilde{S} \cap W(\sigma) = \emptyset$;

(3) $\tilde{S}$ is an uncountable distributively chaotic set of (Σ_2, σ).

Now we prove claim (1): Clearly $\tilde{S} \subseteq R(\sigma)$. Let $y = (B_1B_2\cdots)$ be a point in $\tilde{S}$. Put $a_n = |B_1B_2\cdots B_n|$ for every $n \in \mathbb{N}$. Then for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $\frac{1}{2^{a_k}} < \epsilon$. Set for all $i \geq 1$,

$$n_i = |B_1B_2\cdots B_{k+i}D_{k+i}\overbrace{0\cdots 0}^{|D_{k+i}|^2}\overbrace{1\cdots 1}^{|D_{k+i}|^3}\overbrace{D_1\cdots D_1}^{m_{k+i,1}}\overbrace{D_2\cdots D_2}^{m_{k+i,2}}\cdots\overbrace{D_k\cdots D_k}^{m_{k+i,k}}|.$$

Then we have

$$\begin{aligned} n_i &= |B_1\cdots B_{k+i}| + |D_{k+i}\overbrace{0\cdots 0}^{|D_{k+i}|^2}\overbrace{1\cdots 1}^{|D_{k+i}|^3}\overbrace{D_1\cdots D_1}^{m_{k+i,1}}\cdots\overbrace{D_{k-1}\cdots D_{k-1}}^{m_{k+i,k-1}}| \\ &\quad + |D_k|m_{k+i,k} \\ &\leq |B_1B_2\cdots B_{k+i}| + m_{k+i,k} + |D_k|\times m_{k+i,k} \\ &\leq (2+|D_k|)\times m_{k+i,k}. \end{aligned}$$

Since for any fixed $i \in \mathbb{N}$, $B_1B_2\cdots B_i \prec D_i$, we obtain that

$$|N(y,V(y,\epsilon))\cap\{1,2,\ldots,n_i\}| \geq m_{k+i,k}.$$

So

$$\begin{aligned} \overline{d}(N(y,V(y,\epsilon))) &= \limsup_{n\to\infty}\frac{|N(y,V(y,\epsilon))\cap\{1,2,\ldots,n\}|}{n} \\ &\geq \limsup_{i\to\infty}\frac{|N(y,V(y,\epsilon))\cap\{1,2,\ldots,n_i\}|}{n_i} \\ &\geq \frac{m_{k+i,k}}{(2+|D_k|)\times m_{k+i,k}} \\ &= \frac{1}{2+|D_k|} > 0, \end{aligned}$$

which implies that $\tilde{S} \subseteq QW(\sigma)$.

Next we prove Claim (2). For any $y = (B_1B_2\cdots) \in \tilde{S}$ there exists $x = (x_1x_2\cdots) \in E$ such that $\phi(x) = y$. For convenience, write $y = (y_1y_2\cdots y_n\cdots)$.

If $y_1 = 1$, then we consider the following two cases.

Case 1: if '1' appears infinite many times in x, that is, there exists a sequence $\{n_i\}$ of positive integers such that $x_{n_i} = 1$ for each $i \in \mathbb{N}$. Then $B_{n_i} = A_{n_i}$ for all $i \in \mathbb{N}$. Note that when

$$j \in \left[a_{n_i-1} + |D_{n_i-1}|,\ a_{n_i-1} + |D_{n_i-1}| + |D_{n_i-1}|^2 - 1\right],$$

the first term of $\sigma^j(y)$ is '0', i.e., $(\sigma^j(y))_1 = 0$, hence $\rho(y,(\sigma^j(y))) = 1$.

Put $m_i = |B_1\cdots B_{n_i-1}| + |D_{n_i-1}| + |D_{n_i-1}|^2$. We have

$$\begin{aligned} \underline{d}(N(y,V(y,\frac{1}{3}))) &= \liminf_{n\to\infty}\frac{|N(y,V(y,\frac{1}{3}))\cap\{1,2,\ldots,n\}|}{n} \\ &\leq \liminf_{i\to\infty}\frac{|N(y,V(y,\frac{1}{3}))\cap\{1,2,\ldots,m_i\}|}{m_i} \\ &\leq \liminf_{i\to\infty}\frac{|B_1\cdots B_{n_i-1}| + |D_{n_i-1}|}{|B_1\cdots B_{n_i-1}| + |D_{n_i-1}| + |D_{n_i-1}|^2} \end{aligned}$$

$$\leq \lim_{i\to\infty} \frac{2|D_{n_i-1}|}{|D_{n_i-1}| + |D_{n_i-1}|^2} = 0,$$

which draws that $y \notin W(\sigma)$.

Case 2: if there are only a finite number of '1' appearing in x, i.e., there are infinitely many '0' appearing in x, then there exists a sequence $\{n_s\}$ of positive integers such that $x_{n_s} = 0$ for every $s \in \mathbb{N}$ and $B_{n_s} = \bar{A}_{n_s}$ for all $s \in \mathbb{N}$. So when

$$j \in \left[a_{n_s-1} + |D_{n_s-1}| + |D_{n_s-1}|^2,\ a_{n_s-1} + |D_{n_s-1}| + |D_{n_s-1}|^2 + |D_{n_s-1}|^3 - 1\right],$$

the first term of $\sigma^j(y)$ is '0', i.e., $(\sigma^j(y))_1 = 0$, hence $\rho(y, (\sigma^j(y))) = 1$.

Put $m_s = |B_1 \cdots B_{n_s-1}| + |D_{n_s-1}| + |D_{n_s-1}|^2 + |D_{n_s-1}|^3$, then

$$\begin{aligned}
\underline{d}(N(y, V(y, \frac{1}{3}))) &= \liminf_{n\to\infty} \frac{|(N(y, V(y, \frac{1}{3})) \cap \{1, 2, \ldots, n\}\,|}{n} \\
&\leq \liminf_{s\to\infty} \frac{|N(y, V(y, \frac{1}{3})) \cap \{1, 2, \ldots, m_s\}\,|}{m_s} \\
&\leq \liminf_{s\to\infty} \frac{|B_1 \cdots B_{n_s-1}| + |D_{n_s-1}| + |D_{n_s-1}|^2}{|B_1 \cdots B_{n_s-1}| + |D_{n_s-1}| + |D_{n_s-1}|^2 + |D_{n_s-1}|^3} \\
&\leq \lim_{s\to\infty} \frac{3|D_{n_s-1}|^2}{|D_{n_s-1}|^3} = 0.
\end{aligned}$$

Therefore, $y \notin W(\sigma)$.

If the first term of y is '0', i.e., $y_1 = 0$, we can similarly prove Claim (2), here we omit it.

Proof of Claim (3):

Let $x = (B_1 B_2 \cdots)$ and $y = (C_1 C_2 \cdots)$ be two different points in $\tilde{S}$, where $B_i, C_i \in \left\{A_i, \bar{A}_i\right\}$, $i \in \mathbb{N}$. By the definition of $\tilde{S}$, there are two sequences $\{p_i\}$ and $\{q_i\}$ of positive integers with $p_i \to \infty$ and $q_i \to \infty$ such that $B_{p_i} = C_{p_i}$ and $B_{q_i} = \bar{C}_{q_i}$ for all $i \in \mathbb{N}$. For simplicity, put for all $j \in \mathbb{N}$,

$$\delta_{xy}(j) = \rho(\sigma^j(x), \sigma^j(y)).$$

Firstly, it is easy to see that given $p_i > 1$, the first a_{p_i-1} terms of $\sigma^j(x)$ and $\sigma^j(y)$ are same for all $a_{p_i-1} \leq j \leq a_{p_i} - a_{p_i-1}$, so $\delta_{xy}(j) \leq \frac{1}{2^{a_{p_i-1}}}$. Thus for given $t > 0$, $\sigma_{xy}(j) \leq \frac{1}{2^{a_{p_i-1}}} < t$ provided i is large enough. That is $\chi_{[0,t)}(\delta_{xy}(j)) = 1$ when $j \in [a_{p_i-1}, a_{p_i} - a_{p_i-1}]$. Furthermore,

$$\begin{aligned}
\frac{1}{a_{p_i} - a_{p_i-1}} \sum_{j=1}^{a_{p_i} - a_{p_i-1}} \chi_{[0,t)}(\delta_{xy}(j)) &\geq \frac{1}{a_{p_i} - a_{p_i-1}} \sum_{j=a_{p_i-1}+1}^{a_{p_i} - a_{p_i-1}} \chi_{[0,t)}(\delta_{xy}(j)) \\
&= \frac{a_{p_i} - a_{p_i-1} - a_{p_i-1}}{a_{p_i} - a_{p_i-1}} \\
&= 1 - \frac{a_{p_i-1}}{a_{p_i} - a_{p_i-1}}
\end{aligned}$$

$$
\begin{aligned}
&= 1 - \frac{|B_1 \cdots B_{p_i-1}|}{|B_{p_i}|} \\
&\geq 1 - \frac{|D_{p_i-1}|}{m_{p_i-1,p_i-1} \times |D_{p_i-1}|} \to 1
\end{aligned}
$$

as $i \to \infty$. This proves $F^*_{xy}(t) = 1$.

Secondly, it is easy to see that for a given $q_i > 1$, the first a_{q_i-1} terms of $\sigma^j(x)$ and $\sigma^j(y)$ are distinct correspondingly for all $a_{q_i-1} \leq j \leq a_{q_i} - a_{q_i-1}$, so $\delta_{xy}(j) = 1$. Therefore for any $\epsilon \in (0,1]$, $\chi_{[0,\epsilon)}(\delta_{xy}(j)) = 0$ for all j with $a_{q_i-1} \leq j \leq a_{q_i} - a_{q_i-1}$. Furthermore,

$$
\begin{aligned}
\frac{1}{a_{q_i} - a_{q_i-1}} \sum_{j=1}^{a_{q_i}-a_{q_i-1}} \chi_{[0,\varepsilon)}(\delta_{xy}(j)) &\leq \frac{1}{a_{q_i} - a_{q_i-1}} \sum_{j=1}^{a_{q_i-1}} \chi_{[0,\varepsilon)}(\delta_{xy}(j)) \\
&\leq \frac{a_{q_i-1}}{a_{q_i} - a_{q_i-1}} \\
&= \frac{|B_1 \cdots B_{q_i-1}|}{|B_{q_i}|} \\
&\leq \frac{|D_{q_i-1}|}{m_{q_i-1,q_i-1}|D_{q_i-1}|} \\
&= \frac{1}{m_{q_i-1,q_i-1}} \to 0
\end{aligned}
$$

when $i \to \infty$. This shows $F_{xy}(\epsilon) = 0$ for all $\epsilon \in (0,1]$.

So $(x,y) \in \tilde{S} \times \tilde{S}$ is a distributively chaotic pair of σ. The arbitrariness of x and y implies that $\tilde{S}$ is an uncountable distributively chaotic set of (Σ_2, σ). $\square$

4. Distributional chaos in a sequence occurring on the set of proper positive upper Banach density recurrent points of the one-sided symbolic system

In this section, we consider the distributional chaos occurring on the set of proper positive upper Banach density recurrent points of the one-sided symbolic system. We prove that there exists an uncountable distributively chaotic set in a sequence of σ contained in the set of proper positive upper Banach density recurrent points. But we don't know whether there also exists distributively chaotic phenomenon of σ occurring on such a set.

At first, we review the notion of distributional chaos in a sequence introduced by Wang [11].

Let (X,f) be a dynamical system, $\{p_i\}$ be a strictly increasing sequence of positive integers, $x, y \in X$ and $t > 0$. Write

$$
F^*_{xy}(t, \{p_i\}) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{[0,t)}(d(f^{p_i}(x), f^{p_i}(y)))
$$

and

$$F_{xy}(t, \{p_i\}) = \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{[0,t)}(d(f^{p_i}(x), f^{p_i}(y))).$$

Definition 4.1 ([11]). If $D \subseteq X$ for any $x, y \in D$ with $x \neq y$, $F^*_{xy}(t, \{p_i\}) = 1$ for all $t > 0$ and $F_{xy}(\epsilon, \{p_i\}) = 0$ for some $\epsilon > 0$, then D is said to be a distributively chaotic set with respect to $\{p_i\}$ for f, and (x, y) is said to be a distributively chaotic point pair with respect to $\{p_i\}$. Denote by DCR$(f, \{p_i\})$ the set of all distributively chaotic point pairs with respect to $\{p_i\}$ for f.

f is said to be distributively chaotic in a sequence if f has an uncountable distributively chaotic set with respect to some sequence of positive integers.

Let $\{p_i\}$ be an increasing sequence of positive integers. Set

$$PR(f, \{p_i\}) = \{(x, y) \in X \times X : \forall\epsilon > 0,\ \exists i \in \mathbb{N} \text{ s.t. } d(f^{p_i}(x), f^{p_i}(y)) < \epsilon\}$$

and call it the proximal relation of f with respect to $\{p_i\}$. The asymptotic relation and distal relation of f with respect to $\{p_i\}$ are defined, respectively, as

$$AR(f, \{p_i\}) = \left\{(x, y) \in X \times X : \lim_{i\to\infty} d(f^{p_i}(x), f^{p_i}(y)) = 0\right\}$$

and

$$DR(f, \{p_i\}) = X \times X - PR(f, \{p_i\}).$$

Lemma 4.2 ([2]). *If both $\{p_i\}$ and $\{q_i\}$ are infinitely increasing subsequences of $\{m_i\}$, a sequence of positive integers, then there exists an infinitely increasing subsequence $\{t_i\}$ of $\{m_i\}$ such that*

$$AR(f, \{p_i\}) \cap DR(f, \{q_i\}) \subseteq DCR(f, \{t_i\}).$$

Next we present the main result of this section as follows.

Theorem 4.3. *Let $N \geq 2$ and (Σ_N, σ) be the one-sided symbolic system. Then there exists an uncountable distributively chaotic set in a sequence of σ contained in the set of proper positive upper Banach density recurrent points of σ.*

Proof. We only prove the result for the case of $N = 2$ because the proofs of other cases can be proved similarly.

Fix arbitrarily $a = (a_1a_2\cdots) \in \Sigma_2$, denote $[a]_n = a_1a_2\cdots a_n$. Let

$$B_a = \{([a]_1[b]_1[a]_2[b]_2\cdots[a]_n[b]_n\cdots) : b = (b_1b_2\cdots) \in \Sigma_2\},$$

then B_a is an uncountable subset of Σ_2.

(1) For any $e = (e_1e_2\cdots) \in B_a$, put $Q_1 = 1e_1$ and $Q_2 = Q_1\overbrace{0\cdots0}^{|Q_1|^2}Q_1e_2$ and

$$Q_3 = Q_2\overbrace{0\cdots0}^{|Q_2|^2}Q_1Q_1\overbrace{0\cdots0}^{|Q_2|^2}Q_2Q_2e_3.$$

By induction, for each $n \in \mathbb{N}$ with $n \geq 3$, set

$$Q_n = Q_{n-1} \overbrace{0 \cdots 0}^{|Q_{n-1}|^2} \overbrace{Q_1 \cdots Q_1}^{n-1} \overbrace{0 \cdots 0}^{|Q_{n-1}|^2} \overbrace{Q_2 \cdots Q_2}^{n-1} \cdots \overbrace{0 \cdots 0}^{|Q_{n-1}|^2} \overbrace{Q_{n-1} \cdots Q_{n-1}}^{n-1} e_n.$$

Then Q_n is defined well for each $n \in \mathbb{N}$. Set $x(e) = \lim_{n\to\infty}(Q_n 000\cdots)$. Let $J = \{x(e) : e \in B_a\}$. Obviously, J is an uncountable set.

(2) Next we prove that $J \subseteq BD^+(\sigma)$.

Clearly $J \subseteq R(\sigma)$. Take $x \in J$ and let V be a neighborhood of x, then there exists $k \in \mathbb{N}$ such that for each $y \in \Sigma_2$, if y begins with Q_k, then $y \in V$. Notice that

$$x = (Q_1 \cdots \overbrace{Q_k \cdots Q_k}^{k} \cdots \overbrace{Q_k \cdots Q_k}^{k+1} \cdots \overbrace{Q_k \cdots Q_k}^{k+2} \cdots),$$

take $n_i = |Q_1 \cdots \overbrace{Q_k \cdots Q_k}^{k+i}|$ and $I_i = [n_i - (k+i)|Q_k|,\ n_i]$. Then $|N(x,V) \cap I_i| \geq k+i$ and

$$\begin{aligned}
\limsup_{|I|\to\infty} \frac{|N(x,V)\cap I|}{|I|} &\geq \limsup_{i\to\infty} \frac{|N(x,V)\cap I_i|}{|I_i|} \\
&\geq \lim_{i\to\infty} \frac{k+i}{|Q_k| \times (k+i)} \\
&\geq \frac{1}{|Q_k|} > 0.
\end{aligned}$$

So $x \in BD^+(\sigma)$.

(3) We will prove that every $x \in J$ is not quasi-weakly almost periodic.

Take $x \in J$ and $V_0 = V(x, \frac{1}{3})$. At first, we claim that for each $n \in \mathbb{N}$, there is $k \in \mathbb{N}$ such that $|Q_k| \leq n < |Q_{k+1}|$ and

$$\frac{|N(x,V_0) \cap \{1,2,\ldots,n\}\,|}{n} \leq \frac{|N(x,V_0) \cap \{1,2,\ldots,|Q_k|\}\,|}{|Q_k|}.$$

In fact, write $N_k = |N(x,V_0) \cap \{1,2,\ldots,|Q_k|\}\,|$. When $0 < i \leq k$ and

$$|Q_k| + i|Q_k|^2 + k(|Q_1| + \cdots + |Q_{i-1}|) < n \leq |Q_k| + i|Q_k|^2 + k(|Q_1| + \cdots + |Q_i|),$$

then we have

$$\frac{|N(x,V_0) \cap \{1,2,\ldots,n\}\,|}{n} \leq \frac{N_k + k(|Q_1| + \cdots + |Q_i|)}{|Q_k| + i|Q_k|^2 + k(|Q_1| + \cdots + |Q_{i-1}|)}.$$

Since

$$\begin{aligned}
&\frac{N_k + k(|Q_1| + \cdots + |Q_i|)}{|Q_k| + i|Q_k|^2 + k(|Q_1| + \cdots + |Q_{i-1}|)} - \frac{N_k}{|Q_k|} \\
= &\frac{k|Q_k|(|Q_1| + \cdots + |Q_i|) - iN_k|Q_k|^2 - kN_k(|Q_1| + \cdots + |Q_{i-1}|)}{(|Q_k| + i|Q_k|^2 + k(|Q_1| + \cdots + |Q_{i-1}|))|Q_k|} \\
\leq &\frac{|Q_k|(k(|Q_1| + \cdots + |Q_i|) - i|Q_k|) - kN_k(|Q_1| + \cdots + |Q_{i-1}|)}{(|Q_k| + i|Q_k|^2 + k(|Q_1| + \cdots + |Q_{i-1}|))|Q_k|} \\
\leq &\frac{-kN_k(|Q_1| + \cdots + |Q_{i-1}|)}{(|Q_k| + i|Q_k|^2 + k(|Q_1| + \cdots + |Q_{i-1}|))|Q_k|} \leq 0,
\end{aligned}$$

we obtain that

$$\frac{|N(x, V_0) \cap \{1, 2, \ldots, n\}\,|}{n} \leq \frac{N_k}{|Q_k|}.$$

When $0 \leq i \leq k-1$ and

$$|Q_k| + i|Q_k|^2 + k(|Q_1| + \cdots + |Q_i|) < n \leq |Q_k| + (i+1)|Q_k|^2 + k(|Q_1| + \cdots + |Q_i|),$$

we have

$$\frac{|N(x, V_0) \cap \{1, 2, \ldots, n\}\,|}{n} \leq \frac{N_k + k(|Q_1| + \cdots + |Q_i|)}{|Q_k| + i|Q_k|^2 + k(|Q_1| + \cdots + |Q_i|)}.$$

By the similar argument,

$$\frac{|N(x, V_0) \cap \{1, 2, \ldots, n\}\,|}{n} \leq \frac{N_k}{|Q_k|}.$$

Therefore,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{|N(x, V_0) \cap \{1, 2, \ldots, n)\}|}{n} \\
\leq &\limsup_{k\to\infty} \frac{|N(x, V_0) \cap \{1, 2, \ldots, |Q_k|\}|}{|Q_k|} \\
\leq &\limsup_{k\to\infty} \frac{|Q_k| + k(|Q_1| + \cdots + |Q_k|)}{k|Q_k|^2} \\
= &\ 0
\end{aligned}$$

which yields that x is not a quasi-weakly almost periodic point of (Σ_2, σ).

(4) We will prove that J is a distributively chaotic set of (Σ_2, σ) with respect to some sequence of positive integers.

Firstly, we take $n_i = |Q_{i^2}| - i$ for each $i \in \mathbb{N}$, then for any $x(e) \in J$, where $e \in B_a$, we have

$$\sigma^{n_i}(x(e)) = ([a]_i \cdots)$$

and

$$\rho(\sigma^{n_i}(x(e)), a) \leq \frac{1}{2^i}.$$

Hence

$$\lim_{i\to\infty} \rho(\sigma^{n_i}(x(e)), a) = 0.$$

By the arbitrariness of x, we obtain that for any $x, y \in J$,

$$\lim_{i\to\infty} \rho(\sigma^{n_i}(x), \sigma^{n_i}(y)) = 0,$$

i.e., $(x, y) \in AR(\sigma, \{n_i\})$.

Secondly, put $q_i = |Q_{i^2+i}| - i$ for each $i \in \mathbb{N}$, then for any $x(e) \in J$, where $e \in B_a$, we have

$$\sigma^{q_i}(x(e)) = ([b]_i \cdots).$$

For all $x, y \in J$ with $x \neq y$, there exist $\overline{\beta}, \overline{\gamma} \in B_a$ such that $x = x(\overline{\beta})$ and $y = y(\overline{\gamma})$. Without loss of generality, assume that $\overline{\beta} = ([a]_1[\beta]_1[a]_2[\beta]_2 \cdots)$ and $\overline{\gamma} = ([a]_1[\gamma]_1[a]_2[\gamma]_2 \cdots)$. By the constructions of J and B_a, it is not hard to see that

$$(\beta_1\beta_2 \cdots) = \beta \neq \gamma = (\gamma_1\gamma_2 \cdots).$$

Therefore

$$\lim_{i\to\infty} \rho(\sigma^{q_i}(x), \sigma^{q_i}(y)) = \rho(\gamma, \beta) > 0,$$

i.e., $(x, y) \in DR(\sigma, \{q_i\})$. Hence

$$J \times J \subset AR(\sigma, \{n_i\}) \cap DR(\sigma, \{q_i\}).$$

By Lemma 4.2, there exists $\{t_i\} \subset \{n_i\} \cup \{q_i\}$ such that $J \times J \subseteq DCR(\sigma, \{t_i\})$, so J is a distributively chaotic set of (Σ_2, σ) with respect to $\{t_i\}$. □

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