Bull. Korean Math. Soc. **57** (2020), No. 1, pp. 81–93
https://doi.org/10.4134/BKMS.b190060
pISSN: 1015-8634 / eISSN: 2234-3016

BACKWARD EXTENSIONS OF BERGMAN-TYPE WEIGHTED SHIFT

Chunji Li, Wentao Qi, and Haiwen Wang

Abstract. Let $m \in \mathbb{N}_0$, $p > 1$ and

$$\alpha^{[m,p]}(x) : \sqrt{x}, \left\{ \sqrt{\frac{(m+n-1)\,p-(m+n-2)}{(m+n)\,p-(m+n-1)}} \right\}_{n=1}^{\infty}.$$

In this paper, we consider the backward extensions of Bergman-type weighted shift $W_{\alpha^{[m,p]}(x)}$. We consider its subnormality, k-hyponormality and positive quadratic hyponormality. Our results include all the results on Bergman weighted shift $W_{\alpha(x)}$ with $m \in \mathbb{N}$ and

$$\alpha(x) : \sqrt{x}, \sqrt{\frac{m}{m+1}}, \sqrt{\frac{m+1}{m+2}}, \sqrt{\frac{m+2}{m+3}}, \ldots.$$

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator T in $\mathcal{L}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if $T^*T \geq TT^*$, and *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $K \supseteq \mathcal{H}$. For $A, B \in \mathcal{L}(\mathcal{H})$, let $[A, B] := AB - BA$. We say that an n-tuple $T = (T_1, \ldots, T_n)$ of operators in $\mathcal{L}(\mathcal{H})$ is *hyponormal* if the operator matrix $([T_j^*, T_i])_{i,j=1}^n$ is positive on the direct sum of n copies of $\mathcal{H}$. For arbitrary positive integer k, an operator $T \in \mathcal{L}(\mathcal{H})$ is (strongly) *k-hyponormal* if $(I, T, \ldots, T^k)$ is hyponormal. It is well-known that T is subnormal if and only if T is ∞-hyponormal. An operator T in $\mathcal{L}(\mathcal{H})$ is said to be *weakly n-hyponormal* if $p(T)$ is hyponormal for any polynomial p with degree less than or equal to n. And an operator T is *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial p. In particular, the weak 2-hyponormality (or weak 3-hyponormality) is referred to as quadratic hyponormality (or cubic hyponormality, resp.), and has been considered in detail in [5], [6], [10], [12], [13], [15], and [16], etc.

Received January 17, 2019; Accepted June 26, 2019.
2010 *Mathematics Subject Classification.* 47B37, 47B20.
Key words and phrases. Bergman-type, subnormal, k-hyponormal, positively quadratically hyponormal, weighted shift.

Let $\{e_n\}_{n=0}^{\infty}$ be the canonical orthonormal basis for Hilbert space $l^2(\mathbb{N}_0)$ $(\mathbb{N}_0 := \mathbb{N} \cup \{0\})$ and let $\alpha := \{\alpha_n\}_{n=0}^{\infty}$ be a bounded sequence of positive numbers. Let W_α be a unilateral weighted shift defined by $W_\alpha e_n := \alpha_n e_{n+1}$ $(n \in \mathbb{N}_0)$. It is well known that W_α is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ $(n \in \mathbb{N}_0)$. The *moments* of W_α are usually defined by $\gamma_0 := 1, \gamma_i := \alpha_0^2 \cdots \alpha_{i-1}^2$ $(i \in \mathbb{N})$.

Berger's Theorem ([4])**.** *W_α is subnormal if and only if there exists a Borel probability measure μ supported in $\left[0, \|W_\alpha\|^2\right]$, with $\|W_\alpha\|^2 \in$ supp μ, such that $\gamma_n = \int t^n d\mu(t)$ $(\forall n \in \mathbb{N}_0)$.*

Let $\alpha(x) : x, \alpha_0, \alpha_1, \alpha_2, \ldots$ $(x > 0)$ be an augmented weight sequence for the given $\alpha = \{\alpha_n\}_{n=0}^{\infty}$. Then we have the following result.

Theorem 1.1 ([3, Prop. 8])**.** *Let W_α be subnormal with associated measure μ. Then $W_{\alpha(x)}$ is subnormal if and only if*
(i) $\frac{1}{t} \in L^1(\mu)$ *and*
(ii) $x^2 \leq \left(\left\|\frac{1}{t}\right\|_{L^1(\mu)}\right)^{-1}$.
In particular, $W_{\alpha(x)}$ is never subnormal when $\mu(\{0\}) > 0$.

Next problem was introduced by Curto and Fialkow ([4], [5]).

The backward extension problem. Let $\alpha(x) : x, \alpha_0, \alpha_1, \ldots$ $(x > 0)$ be an augmented weight sequence for the given $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ and let a weighted shift $W_{\alpha(x)}$ be a backward extension of W_α. Assume that W_α is k-hyponormal for $k \in \mathbb{N} \cup \{\infty\}$. Describe the sets

$$\mathbf{HE}(\alpha; n) = \{x \in \mathbb{R}_+ : W_{\alpha(x)} \text{ is } n\text{-hyponormal}\} \quad (1 \leq n \leq k).$$

If a weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ is given by $\alpha_n = \sqrt{\frac{n+2}{n+3}}$ $(n \in \mathbb{N}_0)$, then the corresponding weighted shift is called the *Bergman shift* ([2]). In [3], the author showed that if $W_{\alpha(x)}$ is an one-step backward extension of the Bergman shift W_α, then there exists a sequence $\{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{R}_+$ with $\lim_{k\to\infty} \lambda_k = \sqrt{\frac{1}{2}}$ such that $\lambda_k > \lambda_{k+1}$ $(k \geq 1)$ and $\mathbf{HE}(\alpha; k) = (0, \lambda_k]$, where $\lambda_1 = \sqrt{\frac{2}{3}}, \lambda_2 = \frac{3}{4}, \lambda_3 = \sqrt{\frac{8}{15}}, \lambda_4 = \sqrt{\frac{25}{48}}, \ldots$, and $\mathbf{HE}(\alpha, \infty) = \left(0, \sqrt{\frac{1}{2}}\right]$, which distinguishes the classes of k-hyponormal operators from one another. In [14], the authors obtained a formula for k-hyponormal of $W_{\alpha(x)}$ which contributed to the improvement of the study of relationships between subnormality and hyponormality. For $\alpha(x) : \sqrt{x}, \sqrt{\frac{n+2}{n+3}}$ $(n \in \mathbb{N})$, we know that if $\frac{2}{3} < x \leq \frac{13259}{18228}$, then $W_{\alpha(x)}$ is not subnormal but completely semi-weakly hyponormal ([18, Th. 2.3]). Moreover, the authors in [9] considered a Bergman-like shift which is a generalization of Bergman shift and they proved that all Bergman-like shifts are subnormal.

The authors in [11] introduced a class of Bergman-type weighted shift operators and considered its k-hyponormalities. For a positive real number $p > 1$, we consider a weight sequence $\alpha^{[p]} := \{\alpha_k^{[p]}\}_{k\geq 0}$ as follows:

$$\alpha^{[p]} : \sqrt{\frac{1}{p}}, \sqrt{\frac{p}{2p-1}}, \sqrt{\frac{2p-1}{3p-2}}, \sqrt{\frac{3p-2}{4p-3}}, \ldots. \tag{1.1}$$

The corresponding weighted shift $W_{\alpha^{[p]}}$ is called a *Bergman-type shift.* In particular, if $p = 2$, then $\alpha^{[2]} = \left\{\sqrt{\frac{k+1}{k+2}}\right\}$ for $k \geq 0$, i.e., the Bergman-type shift $W_{\alpha^{[2]}}$ is just the Bergman shift. So we can see that the Bergman-type shift with weight $\alpha^{[p]}$ as in (1.1) is a generalized form of Bergman shifts.

In this paper, we consider more generalized form as following ($m \in \mathbb{N}_0$, $p > 1$)

$$\alpha^{[m,p]} : \sqrt{\frac{mp-(m-1)}{(m+1)\,p-(m)}}, \sqrt{\frac{(m+1)\,p-(m)}{(m+2)\,p-(m+1)}}, \ldots, \tag{1.2}$$

and the extended weight sequence

$$\alpha^{[m,p]}(x) : \sqrt{x}, \sqrt{\frac{mp-(m-1)}{(m+1)\,p-(m)}}, \sqrt{\frac{(m+1)\,p-(m)}{(m+2)\,p-(m+1)}}, \ldots, \tag{1.3}$$

with $0 < x \leq \frac{mp-(m-1)}{(m+1)p-(m)}$.

This paper consists of five sections. In Section 2, we give some key lemmas and formulas. In Section 3 and Section 4, we discuss the problem of subnormality and the problem of n-hyponormality for $W_{\alpha^{[m,p]}(x)}$, which improves the results in [7] and [14]. In Section 5, we discuss the problem of positive quadratic hyponormality of $W_{\alpha^{[m,p]}(x)}$.

In this paper, as usual, we denote $\mathbb{N}$, $\mathbb{C}$, and $\mathbb{R}_+$, for the set of positive integers, complex numbers, and nonnegative real numbers, respectively.

2. Preliminaries and notations

In this section, we give essential lemmas and formulas to prove our results. First, we recall Cauchy's double alternant ([17, p. 6]) that the determinant of the matrix with (i,j) entry $\frac{1}{X_i+Y_j}$ is

$$\det_{1\leq i,j\leq n}\left(\frac{1}{X_i+Y_j}\right) = \frac{\prod_{1\leq i<j\leq n}(X_i-X_j)(Y_i-Y_j)}{\prod_{1\leq i,j\leq n}(X_i+Y_j)}. \tag{2.1}$$

According to (2.1), we have the following result.

Proposition 2.1 ([7])**.** *For $\omega \geq 0$, the determinant $A_n(\omega)$ of the matrix with (i,j) entry $\frac{1}{\omega+i+j-1}$ $(1 \leq i,j \leq n)$ is* [1]

$$A_n(\omega) = (1!2!\cdots(n-1)!)^2\,\frac{\Gamma(\omega+1)\,\Gamma(\omega+2)\cdots\Gamma(\omega+n)}{\Gamma(n+\omega+1)\,\Gamma(n+\omega+2)\cdots\Gamma(2n+\omega)}.$$

[1] $\Gamma(\cdot)$ is the gamma function.

Recall that a weighted shift W_α is *quadratically hyponormal* if $W_\alpha + sW_\alpha^2$ is hyponormal for any $s \in \mathbb{C}$ ([5]), i.e., $D(s) := [(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2] \geq 0$, for any $s \in \mathbb{C}$. Let $\{e_i\}_{i=0}^\infty$ be an orthonormal basis for $\mathcal{H}$ and let P_n be the orthogonal projection on $\vee_{i=0}^n \{e_i\}$. For $s \in \mathbb{C}$ we let

$$\begin{aligned} D_n(s) &= P_n[(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2]P_n \\ &= \begin{bmatrix} q_0 & r_0 & 0 & \cdots & 0 & 0 \\ \overline{r_0} & q_1 & r_1 & \cdots & 0 & 0 \\ 0 & \overline{r_1} & q_2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & q_{n-1} & r_{n-1} \\ 0 & 0 & 0 & \cdots & \overline{r_{n-1}} & q_n \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} &q_k := u_k + |s|^2 v_k, \quad r_k := w_k \bar{s}, \\ &u_k := \alpha_k^2 - \alpha_{k-1}^2, \quad v_k := \alpha_k^2 \alpha_{k+1}^2 - \alpha_{k-2}^2 \alpha_{k-1}^2, \\ &w_k := \alpha_k^2 (\alpha_{k+1}^2 - \alpha_{k-1}^2)^2 \ \text{ for } k \geq 0, \end{aligned}$$

and $\alpha_{-1} = \alpha_{-2} := 0$. Hence, W_α is quadratically hyponormal if and only if $D_n(s) \geq 0$ for every $s \in \mathbb{C}$ and every $n \in \mathbb{N}_0$. We consider $d_n(\cdot) := \det D_n(\cdot)$ which is a polynomial in $t := |s|^2$ of degree $n+1$, with Maclaurin expansion $d_n(t) := \sum_{i=0}^{n+1} c(n,i)\, t^i$. It is easy to find the following recursive relations ([5])

$$\begin{cases} d_0(t) = q_0, \\ d_1(t) = q_0 q_1 - |r_0|^2, \\ d_{n+2}(t) = q_{n+2} d_{n+1}(t) - |r_{n+1}|^2 d_n(t) \quad (n \geq 0). \end{cases}$$

Also, we can obtain the followings

$$\begin{aligned} &c(0,0) = u_0, \quad c(0,1) = v_0, \\ &c(1,0) = u_1 u_0, \quad c(1,1) = u_1 v_0 + u_0 v_1 - w_0, \quad c(1,2) = v_1 v_0, \end{aligned}$$

and

$$\begin{aligned} c(n+2,i) = u_{n+2} c(n+1,i) + v_{n+2} c(n+1,i-1) - w_{n+1} c(n,i-1) \\ (n \geq 0, \text{ and } 0 \leq i \leq n+1). \end{aligned}$$

In particular, for any $n \in \mathbb{N}_0$, we have

$$c(n,0) = u_0 u_1 \cdots u_n, \quad c(n,n+1) = v_0 v_1 \cdots v_n.$$

Furthermore, we can obtain the results as following.

Lemma 2.2. *Let $\rho := v_2(u_0 v_1 - w_0) + v_0(u_1 v_2 - w_1)$. Then for any $n \geq 4$, we have*

$$c(n,n) = u_n c(n-1,n) + (u_{n-1} v_n - w_{n-1})\, c(n-2,n-1)$$

$$+\sum_{i=1}^{n-3} v_n v_{n-1} \cdots v_{i+3} \left(u_{i+1} v_{i+2} - w_{i+1}\right) c\left(i, i+1\right) + v_n v_{n-1} \cdots v_3 \rho.$$

Lemma 2.3. *Let* $\tau := u_0 \left(u_1 v_2 - w_1\right)$. *Then for any* $n \geq 4$, *we have*

$$\begin{aligned} c\left(n, n-1\right) = {} & u_n c\left(n-1, n-1\right) + \left(u_{n-1} v_n - w_{n-1}\right) c\left(n-2, n-2\right) \\ & + \sum_{i=1}^{n-3} v_n v_{n-1} \cdots v_{i+3} \left(u_{i+1} v_{i+2} - w_{i+1}\right) c\left(i, i\right) + v_n v_{n-1} \cdots v_3 \tau. \end{aligned}$$

Lemma 2.4. *For any* $n \geq 5$ *and* $0 \leq i \leq n-2$, *we have*

$$\begin{aligned} c\left(n, i\right) = {} & u_n c\left(n-1, i\right) + \left(u_{n-1} v_n - w_{n-1}\right) c\left(n-2, i-1\right) \\ & + \sum_{j=1}^{n-3} v_n v_{n-1} \cdots v_{j+3} \left(u_{j+1} v_{j+2} - w_{j+1}\right) c\left(j, j+i-n+1\right) \\ & + v_n v_{n-1} \cdots v_5 c\left(i-n+5, 0\right) \left(u_{i-n+6} v_{i-n+7} - w_{i-n+6}\right). \end{aligned}$$

To detect the positivity of $d_n\left(t\right)$, we need the following concepts.

Definition 1 ([5])**.** Let $\alpha : \alpha_0, \alpha_1, \ldots$ be a positive weight sequence. We say that W_α is *positively quadratically hyponormal* if $c\left(n, i\right) \geq 0$ for all $n, i \in \mathbb{N}_0$, with $0 \leq i \leq n+1$, and $c\left(n, n+1\right) > 0$ for all $n \in \mathbb{N}_0$.

Definition 2 ([1, Def. 3.1])**.** Let $\alpha : \alpha_0, \alpha_1, \ldots$ be a positive weight sequence.
(1) A weighted shift W_α has *property* $B\left(k\right)$ if $u_{n+1} v_n \geq w_n$ $\left(n \geq k\right)$.
(2) A weighted shift W_α has *property* $C\left(k\right)$ if $v_{n+1} u_n \geq w_n$ $\left(n \geq k\right)$.

We give the result as following.

Proposition 2.5 ([3, Coro. 5])**.** *Let* W_α *be any unilateral weighted shift. Then* W_α *is* 2*-hyponormal if and only if* $\theta_k := u_k v_{k+1} - w_k \geq 0$, $\forall k \in \mathbb{N}$.

It is well-known that if W_α is 2-hyponormal or positively quadratically hyponormal, then W_α is quadratically hyponormal. By Proposition 2.5, and Lemma 2.2, Lemma 2.3 and Lemma 2.4, we have the following result.

Proposition 2.6 ([1, Coro. 3.5])**.** *Let* W_α *be any unilateral weighted shift. If* W_α *is* 2*-hyponormal, then* W_α *is positively quadratically hyponormal.*

3. Subnormality of $W_{\alpha^{[m,p]}(x)}$

For weighted shifts in (1.2), we have

$$\gamma_0 = 1, \ \gamma_n = \frac{mp - \left(m-1\right)}{\left(m+n\right) p - \left(m+n-1\right)}, \left(n \geq 1\right) \tag{3.1}$$

and

$$\int_0^1 t^n \mathrm{d}\left(t^{\frac{m(p-1)+1}{p-1}}\right) = \frac{m\left(p-1\right)+1}{p-1} \int_0^1 t^{n+\frac{m(p-1)+1}{p-1}-1} \mathrm{d}t$$

$$= \frac{mp-(m-1)}{(m+n)\,p-(m+n-1)} = \gamma_n,$$

i.e., $\mathrm{d}\mu = \mathrm{d}\Big(t^{\frac{m(p-1)+1}{p-1}}\Big) = t^{\frac{m(p-1)+1}{p-1}-1}\mathrm{d}t$ is a *representing measure* for $\gamma := \{\gamma_n\}_{n=0}^{\infty}$ as in (3.1). Hence, we can obtain the result as following.

Theorem 3.1. *Let* $m \geq 1$ *and* $\alpha^{[m,p]}(x)$ *be as in* (1.3). *Then* $W_{\alpha^{[m,p]}(x)}$ *is subnormal if and only if* $0 < x \leq \frac{(m-1)p-(m-2)}{mp-(m-1)}$.

Proof. Indeed,

$$\int_0^1 \frac{1}{t}\mathrm{d}\mu = \int_0^1 \frac{1}{t}\mathrm{d}\left(t^{\frac{m(p-1)+1}{p-1}}\right) = \frac{mp-(m-1)}{(m-1)p-(m-2)}.$$

Thus by Theorem 1.1, we have our conclusion. □

By Theorem 3.1, we have the results as following.

Corollary 3.2. *Let*

$$\alpha^{[1,p]}(x) : \sqrt{x}, \sqrt{\frac{p}{2p-1}}, \sqrt{\frac{2p-1}{3p-2}}, \sqrt{\frac{3p-2}{4p-3}}, \ldots.$$

Then $W_{\alpha^{[1,p]}(x)}$ *is subnormal if and only if* $0 < x \leq \frac{1}{p}$.

Corollary 3.3 ([14, Exa. 3.2])**.** *Let* $m \geq 1$ *and*

$$\alpha^{[m,2]}(x) : \sqrt{x}, \sqrt{\frac{m+1}{m+2}}, \sqrt{\frac{m+2}{m+3}}, \sqrt{\frac{m+3}{m+4}}, \ldots.$$

Then $W_{\alpha^{[m,2]}(x)}$ *is subnormal if and only if* $0 < x \leq \frac{m}{m+1}$.

Corollary 3.4. *Let* $m \geq 1$ *and*

$$\alpha^{[m,3]}(x) : \sqrt{x}, \sqrt{\frac{2m+1}{2m+3}}, \sqrt{\frac{2m+3}{2m+5}}, \sqrt{\frac{2m+5}{2m+7}}, \ldots.$$

Then $W_{\alpha^{[m,3]}(x)}$ *is subnormal if and only if* $0 < x \leq \frac{2m-1}{2m+1}$.

4. k-hyponormality of $W_{\alpha^{[m,p]}(x)}$

First, we give the following result.

Proposition 4.1. (1) *Let* $\alpha^{[0,2]}(x) : \sqrt{x}, \sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots.$ *Then* $W_{\alpha^{[0,2]}(x)}$ *is* n*-hyponormal if and only if* $0 < x \leq \frac{1}{2\left(\sum_{i=1}^{n}\frac{1}{i}\right)}$. *In particular,* $W_{\alpha^{[0,2]}(x)}$ *is not subnormal for any* $x > 0$.

(2) *Let* $1 < p \neq 2$, *and* $\alpha^{[0,p]}(x) : \sqrt{x}, \sqrt{\frac{1}{p}}, \sqrt{\frac{p}{2p-1}}, \sqrt{\frac{2p-1}{3p-2}}, \ldots.$ *Then* $W_{\alpha^{[0,p]}(x)}$ *is* n*-hyponormal if and only if*

$$0 < x \leq \frac{(2-p)\prod_{l=0}^{n-1}\left[lp-(l-1)\right]^2}{\prod_{l=0}^{n-1}\left[lp-(l-1)\right]^2-(n!)^2\,(p-1)^{2n}}.$$

Proof. See [11, Th. 3.2, Th. 4.1]. □

For $m \geq 1$, we have:

Theorem 4.2. *Let $\alpha^{[m,p]}(x)$ be as in (1.3). Then $W_{\alpha^{[m,p]}(x)}$ $(m \geq 1)$ is n-hyponormal if and only if*

$$0 < x \leq \frac{(m-1)(p-1)+1}{m(p-1)+1} \frac{\prod_{l=0}^{n-1}\left[(m+l)p-(m+l-1)\right]^2}{\prod_{l=0}^{n-1}\left[(m+l)p-(m+l-1)\right]^2-(n!)^2(p-1)^{2n}}.$$

Proof. We know that $W_{\alpha^{[m,p]}(x)}$ is n-hyponormal if and only if the following Hankel matrix

$$M_{n+1}^{[m,p]}(x) := \begin{bmatrix} \frac{1}{x} & 1 & \gamma_1 & \cdots & \gamma_{n-1} \\ 1 & \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{n-1} & \gamma_n & \gamma_{n+1} & \cdots & \gamma_{2n-1} \end{bmatrix} \geq 0,$$

where $\gamma_i = \frac{mp-(m-1)}{(m+i)p-(m+i-1)}$ $(1 \leq i \leq 2n-1)$. Since

$$\begin{aligned} D_{n+1}^{[m,p]}(x) :=& \det M_{n+1}^{[m,p]}(x) \\ =& (mp-(m-1))^{n+1} \\ & \times \left[\left(\frac{1}{(mp-(m-1))\,x} - \frac{1}{(m-1)p-(m-2)}\right) D_n^{[1]}(p) + D_{n+1}^{[2]}(p)\right], \end{aligned}$$

where (by Proposition 2.1)

$$\begin{aligned} D_n^{[1]}(p) &= \begin{vmatrix} \frac{1}{(m+1)p-m} & \frac{1}{(m+2)p-(m+1)} & \cdots & \frac{1}{(m+n)p-(m+n-1)} \\ \frac{1}{(m+2)p-(m+1)} & \frac{1}{(m+3)p-(m+2)} & \cdots & \frac{1}{(m+n+1)p-(m+n)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(m+n)p-(m+n-1)} & \frac{1}{(m+n+1)p-(m+n)} & \cdots & \frac{1}{(m+2n-1)p-(m+2n-2)} \end{vmatrix}_n \\ &= \frac{1}{(p-1)^n} \frac{(1!2!\cdots(n-1)!)^2}{\left(\prod_{l=1}^{n}\prod_{s=1}^{n}\left(\frac{1}{p-1}+m+l+s-1\right)\right)}, \end{aligned}$$

and

$$\begin{aligned} D_{n+1}^{[2]}(p) &= \begin{vmatrix} \frac{1}{(m-1)p-(m-2)} & \frac{1}{mp-(m-1)} & \cdots & \frac{1}{(m+n-1)p-(m+n-2)} \\ \frac{1}{mp-(m-1)} & \frac{1}{(m+1)p-m} & \cdots & \frac{1}{(m+n)p-(m+n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(m+n-1)p-(m+n-2)} & \frac{1}{(m+n)p-(m+n-1)} & \cdots & \frac{1}{(m+2n-1)p-(m+2n-2)} \end{vmatrix}_{n+1} \\ &= \frac{1}{(p-1)^{n+1}} \frac{(1!2!\cdots n!)^2}{\left(\prod_{l=0}^{n}\left(\frac{1}{p-1}+m+l-1\right)^{l+1}\right)\left(\prod_{l=n+1}^{2n}\left(\frac{1}{p-1}+m+l-1\right)^{2n-l+1}\right)}, \end{aligned}$$

we know that $D_{n+1}^{[m,p]}(x) \geq 0$ if and only if

$$\begin{aligned} 0 < x &\leq \frac{1}{mp-(m-1)} \frac{[(m-1)p-(m-2)]D_n^{[1]}(p)}{D_n^{[1]}(p)-[(m-1)p-(m-2)]D_{n+1}^{[2]}(p)} \\ &= \frac{(m-1)(p-1)+1}{m(p-1)+1} \frac{\prod_{l=0}^{n-1}[(m+l)p-(m+l-1)]^2}{\prod_{l=0}^{n-1}[(m+l)p-(m+l-1)]^2-(n!)^2(p-1)^{2n}}. \end{aligned}$$

Thus we have our conclusion. □

By Theorem 4.2, we have the following results.

Corollary 4.3 ([11, Th. 4.1])**.** *Let* $\alpha^{[1,p]}(x) : \sqrt{x}, \sqrt{\frac{p}{2p-1}}, \sqrt{\frac{2p-1}{3p-2}}, \sqrt{\frac{3p-2}{4p-3}}, \ldots.$ *Then* $W_{\alpha^{[1,p]}(x)}$ *is n-hyponormal if and only if*

$$0 < x \leq \frac{1}{p} \frac{\prod_{l=1}^{n}[lp-(l-1)]^2}{\prod_{l=1}^{n}[lp-(l-1)]^2-(n!)^2(p-1)^{2n}}.$$

Corollary 4.4 ([8, Exa. 8])**.** *Let* $\alpha^{[1,2]}(x) : \sqrt{x}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \ldots.$ *Then* $W_{\alpha^{[1,2]}(x)}$ *is n-hyponormal if and only if* $0 < x \leq \frac{1}{2}\frac{(n+1)^2}{n(n+2)}$.

5. Positive quadratic hyponormality of $W_{\alpha^{[m,p]}(x)}$

Let $\alpha^{[m,p]}(x)$ $(m \in \mathbb{N}_0, p > 1)$ be as in (1.3). Then we have the following result.

Lemma 5.1. $u_{n+1}v_n = w_n\,(\forall n \geq 3)$ *and* $u_n v_{n+1} > w_n\,(\forall n \geq 2)$.

Proof. Since $\alpha_n^2 = \frac{(m+n-1)p-(m+n-2)}{(m+n)p-(m+n-1)}\,(n \geq 1)$, we have

$$\begin{aligned} u_{n+1} &= \frac{(p-1)^2}{((m+n+1)p-(m+n))((m+n)p-(m+n-1))}, \\ v_n &= \frac{4(p-1)^2}{((m+n+1)p-(m+n))((m+n-1)p-(m+n-2))}, \\ w_n &= \frac{4(p-1)^4}{\begin{pmatrix} ((m+n-1)p-(m+n-2))((m+n)p-(m+n-1)) \\ \times((m+n+1)p-(m+n))^2 \end{pmatrix}}, \end{aligned}$$

thus $u_{n+1}v_n = w_n\,(\forall n \geq 3)$. And since

$$\begin{aligned} u_n &= \frac{(p-1)^2}{((m+n)p-(m+n-1))((m+n-1)p-(m+n-2))}, \\ v_{n+1} &= \frac{4(p-1)^2}{((m+n+2)p-(m+n+1))((m+n)p-(m+n-1))}, \end{aligned}$$

for $n \geq 2$, we have

$$u_n v_{n+1} - w_n$$

$$= \frac{4(p-1)^6}{\left(\begin{array}{c}((m+n-1)p-(m+n-2))((m+n+2)p-(m+n+1)) \\ \times((m+n)p-(m+n-1))^2((m+n+1)p-(m+n))^2\end{array}\right)} > 0.$$

Thus we have our conclusion. □

By Lemma 5.1, we know that $W_{\alpha^{[m,p]}(x)}$ has the properties $B(3)$ and $C(2)$. Hence, by [1, Th. 3.9], we obtain the result as following.

Proposition 5.2. *Let $\alpha^{[m,p]}(x)$ be as in (1.3). $W_{\alpha^{[m,p]}(x)}$ is positively quadratically hyponormal if and only if*

$$c(1,1), c(2,1), c(2,2), c(3,2), c(3,3), c(4,3)$$

are all nonnegative.

By Proposition 5.2, we obtain the following result.

Theorem 5.3. *Let $p \geq 2$ and $\alpha^{[0,p]}(x) : \sqrt{x}, \sqrt{\frac{1}{p}}, \sqrt{\frac{p}{2p-1}}, \sqrt{\frac{2p-1}{3p-2}}, \sqrt{\frac{3p-2}{4p-3}}, \ldots.$ Then $W_{\alpha^{[0,p]}(x)}$ is positively quadratically hyponormal if and only if*

$$0 < x \leq \frac{16p^3-25p^2+6p+4}{32p^4-96p^3+120p^2-75p+20}.$$

Proof. In fact,

$$\begin{aligned}
c(1,1) &= x\frac{p-(2p-1)x}{p(2p-1)} > 0, \\
c(2,1) &= \frac{2}{p}x(p-1)^2\frac{1-px}{(2p-1)(3p-2)} \geq 0, \\
c(2,2) &= x(p-1)^2\frac{p+2(1-px)}{p^2(2p-1)(3p-2)} > 0, \\
c(3,2) &= x(p-1)^4\frac{(4p+3)-2\left(4p^2-4p+3\right)x}{p^2(3p-2)(4p-3)(2p-1)^2}, \\
c(3,3) &= x(p-1)^2\frac{\left(4p^3-4p^2-p+2\right)-(2p-1)\left(4p^2-6p+3\right)x}{p^2(3p-2)(4p-3)(2p-1)^2}, \\
c(4,3) &= x(p-1)^4\frac{\left(16p^3-25p^2+6p+4\right)-\left(32p^4-96p^3+120p^2-75p+20\right)x}{p^2(4p-3)(5p-4)(2p-1)^2(3p-2)^2}.
\end{aligned}$$

And

$$\begin{aligned}
c(3,2) \geq 0 &\Longleftrightarrow 0 < x \leq c_{32} := \frac{4p+3}{8p^2-8p+6}, \\
c(3,3) \geq 0 &\Longleftrightarrow 0 < x \leq c_{33} := \frac{4p^3-4p^2-p+2}{(2p-1)(4p^2-6p+3)}, \\
c(4,3) \geq 0 &\Longleftrightarrow 0 < x \leq c_{43} := \frac{16p^3-25p^2+6p+4}{32p^4-96p^3+120p^2-75p+20}.
\end{aligned}$$

It is easy to see that $\min\left\{\frac{1}{p}, c_{32}, c_{33}, c_{43}\right\} = c_{43}$. Thus, by Proposition 5.2, we obtain the result. $\square$

Corollary 5.4 ([1, Exa. 4.3])**.** *Let* $\alpha^{[0,2]}(x) : \sqrt{x}, \sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots$. *Then* $W_{\alpha^{[0,2]}(x)}$ *is positively quadratically hyponormal if and only if* $0 < x \leq \frac{22}{47}$.

For a weighted shift with first two equal weights, we have the result as following.

Proposition 5.5 ([1, Coro. 3.10])**.** *Let* W_α *be a weighted shift with* $\alpha_0 = \alpha_1$. *If* W_α *has property* $B(3)$, *then* W_α *is positively quadratically hyponormal if and only if* $c(3,2) \geq 0$ *and* $c(4,3) \geq 0$.

By Lemma 5.1, since $W_{\alpha^{[m,p]}(x)}$ has the properties $B(3)$, we let

$$\alpha_0 = \alpha_1 = \sqrt{\frac{mp-(m-1)}{(m+1)p-m}}, \quad \alpha_k = \sqrt{\frac{(m+k-1)p-(m+k-2)}{(m+k)p-(m+k-1)}}\,(k \geq 2),$$

then we obtain

$$c(3,2) = \frac{((m-1)p-(m-2))(p-1)^4(m(p-1)+1)^2}{(m(p-1)+p)^4(m(p-1)+2p-1)^2(m(p-1)+3p-2)} \geq 0,$$

and

$$c(4,3) = \frac{(p-1)^6(m(p-1)+1)^2 f(p)}{\left(\begin{array}{c}(m(p-1)+p)^4(m(p-1)+2p-1)^2(m(p-1)+3p-2)^2\\ \times(m(p-1)+4p-3)(m(p-1)+5p-4)\end{array}\right)},$$

where

$$f(p) = \left(3m^2+7m-16\right)p^2 + \left(39-8m-6m^2\right)p + \left(3m^2+m-20\right).$$

(i) If $m = 1$, then $f(p) = -6p^2+25p-16$. Since $f(p) \geq 0 \Longleftrightarrow \frac{25}{12} - \frac{1}{12}\sqrt{241}\,(\approx 0.789\,65) < p \leq \frac{1}{12}\sqrt{241} + \frac{25}{12}\,(\approx 3.377)$, we know that $c(4,3) \geq 0 \Longleftrightarrow 1 < p \leq \frac{1}{12}\sqrt{241} + \frac{25}{12}$.

(ii) If $m \geq 2$, then $f(p) \geq 0$. In fact, from $f(p) = 0$, we obtain the two distinct roots

$$\bar{p} := r(m) = \frac{6m^2+8m-39 \pm \sqrt{241}}{6m^2+14m-32}.$$

Since $r'(m) = \frac{1}{\left(m \pm \frac{1}{6}\sqrt{241}+\frac{7}{6}\right)^2} > 0$ and $\bar{p} \to 1$ (as $m \to \infty$), we know that $\bar{p} < 1$. Thus, if $m \geq 2$ and $p > 1$, then $f(p) \geq 0$.

Thus by Proposition 5.5 and [16, Prop. 3.4], we obtain the results as following.

Theorem 5.6. *Let* $m \geq 2$ *and* $\alpha^{[m,p]}(x)$ *be as in* (1.3)*. Then the followings are equivalent.*

(1) $W_{\alpha^{[m,p]}(x)}$ *is positively quadratically hyponormal*;

(2) $W_{\alpha^{[m,p]}(x)}$ *is quadratically hyponormal*;

(3) $0 < x \le \frac{mp-(m-1)}{(m+1)p-(m)}$.

Theorem 5.7. *Let* $\alpha^{[1,p]}(x) : \sqrt{x}, \sqrt{\frac{p}{2p-1}}, \sqrt{\frac{2p-1}{3p-2}}, \sqrt{\frac{3p-2}{4p-3}}, \ldots$. *If* $1 < p \le \frac{25+\sqrt{241}}{12} (\approx 3.377)$, *then the followings are equivalent.*

(1) $W_{\alpha^{[1,p]}(x)}$ *is positively quadratically hyponormal*;
(2) $W_{\alpha^{[1,p]}(x)}$ *is quadratically hyponormal*;
(3) $0 < x \le \frac{p}{2p-1}$.

By Theorem 5.7, we obtain the results as following.

Corollary 5.8 ([3, Prop. 7])**.** *Let* $\alpha^{[1,2]}(x) : \sqrt{x}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \ldots$. *Then the followings are equivalent.*

(1) $W_{\alpha^{[1,2]}(x)}$ *is positively quadratically hyponormal*;
(2) $W_{\alpha^{[1,2]}(x)}$ *is quadratically hyponormal*;
(3) $0 < x \le \frac{2}{3}$.

Corollary 5.9. *Let* $\alpha^{[1,3]}(x) : \sqrt{x}, \sqrt{\frac{3}{5}}, \sqrt{\frac{5}{7}}, \sqrt{\frac{7}{9}}, \ldots$. *Then the followings are equivalent.*

(1) $W_{\alpha^{[1,3]}(x)}$ *is positively quadratically hyponormal*;
(2) $W_{\alpha^{[1,3]}(x)}$ *is quadratically hyponormal*;
(3) $0 < x \le \frac{3}{5}$.

Furthermore, for $m = 1$, and $p > \frac{25+\sqrt{241}}{12}$, by direct computation, we obtain

$$c(1,1) = px\frac{(2p-1)-(3p-2)x}{(2p-1)(3p-2)} > 0,$$
$$c(2,1) = 2\frac{x}{2p-1}(p-1)^2\frac{p-(2p-1)x}{(3p-2)(4p-3)} \ge 0,$$

and

$$c(2,2) = px(p-1)^2\frac{(4p-1)-2(2p-1)x}{(3p-2)(4p-3)(2p-1)^2},$$
$$c(3,2) = x(p-1)^4\frac{p(11p-4)-2\left(11p^2-12p+4\right)x}{(4p-3)(5p-4)(2p-1)^2(3p-2)^2},$$
$$c(3,3) = px(p-1)^2\frac{\left(16p^3-31p^2+20p-4\right)-(3p-2)\left(7p^2-10p+4\right)x}{(4p-3)(5p-4)(2p-1)^2(3p-2)^2},$$
$$c(4,3) = x(p-1)^4\frac{p\left(44p^3-98p^2+71p-16\right)-\left(94p^4-277p^3+312p^2-160p+32\right)x}{(5p-4)(6p-5)(2p-1)^2(3p-2)^2(4p-3)^2}.$$

Thus

$$c(2,2) \ge 0 \Longleftrightarrow 0 < x \le c_{22} := \frac{(4p-1)}{2(2p-1)},$$

$$c\,(3,2) \geq 0 \Longleftrightarrow 0 < x \leq c_{32} := \frac{p\,(11p-4)}{2\,(11p^2-12p+4)},$$
$$c\,(3,3) \geq 0 \Longleftrightarrow 0 < x \leq c_{33} := \frac{\left(16p^3-31p^2+20p-4\right)}{(3p-2)\,(7p^2-10p+4)},$$
$$c\,(4,3) \geq 0 \Longleftrightarrow 0 < x \leq c_{43} := \frac{p\left(44p^3-98p^2+71p-16\right)}{94p^4-277p^3+312p^2-160p+32}.$$

It is easy to see that $\min\{c_{22}, c_{32}, c_{33}, c_{43}\} = c_{43}$. Thus, by Proposition 5.2, we obtain the results as following.

Theorem 5.10. *Let* $\alpha^{[1,p]}(x): \sqrt{x}, \sqrt{\frac{p}{2p-1}}, \sqrt{\frac{2p-1}{3p-2}}, \sqrt{\frac{3p-2}{4p-3}}, \ldots,$ *and* $p > \frac{25+\sqrt{241}}{12}$. *Then* $W_{\alpha^{[1,p]}(x)}$ *is positively quadratically hyponormal if and only if*

$$0 < x \leq \frac{p\left(44p^3-98p^2+71p-16\right)}{94p^4-277p^3+312p^2-160p+32}.$$

Remark. If $p > \frac{25+\sqrt{241}}{12}$, then $\frac{p\left(44p^3-98p^2+71p-16\right)}{94p^4-277p^3+312p^2-160p+32} < \frac{p}{2p-1}$.

Corollary 5.11. *Let* $\alpha^{[1,4]}(x): \sqrt{x}, \sqrt{\frac{4}{7}}, \sqrt{\frac{7}{10}}, \sqrt{\frac{10}{13}}, \ldots.$ *Then* $W_{\alpha^{[1,4]}(x)}$ *is positively quadratically hyponormal if and only if* $0 < x \leq \frac{379}{670}\,(\approx 0.565\,67)$.

References

[1] J. Y. Bae, I. B. Jung, and G. R. Exner, *Criteria for positively quadratically hyponormal weighted shifts*, Proc. Amer. Math. Soc. **130** (2002), no. 11, 3287–3294. https://doi.org/10.1090/S0002-9939-02-06493-6

[2] Y. B. Choi, J. K. Han, and W. Y. Lee, *One-step extension of the Bergman shift*, Proc. Amer. Math. Soc. **128** (2000), no. 12, 3639–3646. https://doi.org/10.1090/S0002-9939-00-05516-7

[3] R. E. Curto, *Quadratically hyponormal weighted shifts*, Integral Equations Operator Theory **13** (1990), no. 1, 49–66. https://doi.org/10.1007/BF01195292

[4] R. E. Curto and L. A. Fialkow, *Recursively generated weighted shifts and the subnormal completion problem*, Integral Equations Operator Theory **17** (1993), no. 2, 202–246. https://doi.org/10.1007/BF01200218

[5] ———, *Recursively generated weighted shifts and the subnormal completion problem. II*, Integral Equations Operator Theory **18** (1994), no. 4, 369–426. https://doi.org/10.1007/BF01200183

[6] R. E. Curto and I. B. Jung, *Quadratically hyponormal weighted shifts with two equal weights*, Integral Equations Operator Theory **37** (2000), no. 2, 208–231. https://doi.org/10.1007/BF01192423

[7] R. E. Curto and S. H. Lee, *Quartically hyponormal weighted shifts need not be 3-hyponormal*, J. Math. Anal. Appl. **314** (2006), no. 2, 455–463. https://doi.org/10.1016/j.jmaa.2005.04.020

[8] R. E. Curto, S. H. Lee, and W. Y. Lee, *A new criterion for k-hyponormality via weak subnormality*, Proc. Amer. Math. Soc. **133** (2005), no. 6, 1805–1816. https://doi.org/10.1090/S0002-9939-04-07727-5

[9] R. E. Curto, Y. T. Poon, and J. Yoon, *Subnormality of Bergman-like weighted shifts*, J. Math. Anal. Appl. **308** (2005), no. 1, 334–342. https://doi.org/10.1016/j.jmaa.2005.01.028

[10] Y. Dong, G. Exner, I. B. Jung, and C. Li, *Quadratically hyponormal recursively generated weighted shifts*, in Recent advances in operator theory and applications, 141–155, Oper. Theory Adv. Appl., 187, Birkhäuser, Basel, 2009. https://doi.org/10.1007/978-3-7643-8893-5_7

[11] Y. Dong, M. R. Lee, and C. Li, *New results on k-hyponormality of backward extensions of subnormal weighted shifts*, J. Appl. Math. Inform. **37** (2019), no. 1-2, 73–83. https://doi.org/10.14317/jami.2019.073

[12] G. Exner, J. Y. Jin, I. B. Jung, and J. E. Lee, *Weak Hamburger-type weighted shifts and their examples*, J. Math. Anal. Appl. **462** (2018), no. 2, 1357–1380. https://doi.org/10.1016/j.jmaa.2018.02.045

[13] G. Exner, I. B. Jung, and S. S. Park, *Weakly n-hyponormal weighted shifts and their examples*, Integral Equations Operator Theory **54** (2006), no. 2, 215–233. https://doi.org/10.1007/s00020-004-1360-2

[14] I. B. Jung and C. Li, *A formula for k-hyponormality of backstep extensions of subnormal weighted shifts*, Proc. Amer. Math. Soc. **129** (2001), no. 8, 2343–2351. https://doi.org/10.1090/S0002-9939-00-05844-5

[15] I. B. Jung and S. S. Park, *Cubically hyponormal weighted shifts and their examples*, J. Math. Anal. Appl. **247** (2000), no. 2, 557–569. https://doi.org/10.1006/jmaa.2000.6879

[16] ———, *Quadratically hyponormal weighted shifts and their examples*, Integral Equations Operator Theory **36** (2000), no. 4, 480–498. https://doi.org/10.1007/BF01232741

[17] C. Krattenthaler, *Advanced determinant calculus*, Sém. Lothar. Combin. **42** (1999), Art. B42q, 67 pp.

[18] C. Li and M. R. Lee, *Existence of non-subnormal completely semi-weakly hyponormal weighted shifts*, Filomat **31** (2017), no. 6, 1627–1638. https://doi.org/10.2298/FIL1706627L

Chunji Li
Department of Mathematics
Northeastern University
Shenyang 110-004, P. R. China
Email address: lichunji@mail.neu.edu.cn

Wentao Qi
Department of Mathematics
Northeastern University
Shenyang 110-004, P. R. China
Email address: 20151490@stu.neu.edu.cn

Haiwen Wang
Department of Mathematics
Northeastern University
Shenyang 110-004, P. R. China
Email address: whw_maths@hotmail.com