ON OPERATORS T COMMUTING WITH CTC WHERE C IS A CONJUGATION

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ABSTRACT. In this paper, we study the properties of $T$ satisfying $[CTC, T] = 0$ for some conjugation $C$ where $[R, S] := RS - SR$. In particular, we show that if $T$ is normal, then $[CTC, C] = 0$. Moreover, the class of operators $T$ satisfy $[CTC, T] = 0$ is norm closed. Finally, we prove that if $T$ is complex symmetric, then $T$ is binormal if and only if $[C|T|C, |T|] = 0$.

1. Introduction

Let $\mathcal{H}$ be a separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$ and binormal if $T^*T$ and $TT^*$ commute where $T^*$ is the adjoint of $T$.

A conjugation on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ which satisfies $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$. Given $T \in \mathcal{L}(\mathcal{H})$ and a conjugation $C$ on $\mathcal{H}$, let $\mathcal{C}_C(T) := \{ S \in \mathcal{L}(\mathcal{H}) \mid [CTC, S] = 0 \}$ where $[R, S] := RS - SR$.

In this paper, we study the case when $T \in \mathcal{C}_C(T)$, i.e., $[CTC, T] = 0$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be complex symmetric and skew complex symmetric if there exists a conjugation $C$ such that $CTC = T^*$ and $CTC = -T^*$, respectively. In this case, we say that $T$ is (skew) complex symmetric with a conjugation $C$. It is clear that if $T \in \mathcal{C}_C(T)$ is complex symmetric (or skew complex symmetric) with a conjugation $C$, then $T$ is normal. Throughout the paper, we denote the spectrum and the approximate point spectrum of $T \in \mathcal{L}(\mathcal{H})$ by $\sigma(T)$ and $\sigma_a(T)$, respectively. For a set $F \subset \mathbb{C}$, let $F^* = \{ z : z \in F \}$.

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The following examples show that $\mathcal{C}_C(T)$ need not contain complex symmetric operators.

**Example 1.1.** Let $\mathcal{H} = \ell^2$, let $\{e_n\}$ be an orthonormal basis of $\mathcal{H}$ and let $C: \mathcal{H} \to \mathcal{H}$ be the conjugation given by $C(\sum_{n=0}^{\infty} x_n e_n) = \sum_{n=0}^{\infty} \overline{x_n} e_n$ where $\{x_n\}$ is a sequence in $\mathbb{C}$ with $\sum_{n=0}^{\infty} |x_n|^2 < \infty$ and $Ce_n = e_n$ for all $n$. If $W \in \mathcal{L}(\mathcal{H})$ is the weighted shift given by $We_n = \alpha_n e_{n+1}$ for all $n \geq 1$, then it is easy to compute $WCWC e_n = CW CW e_n$ for all $n$. Hence $W \in \mathcal{C}_C(W)$. In particular, if $\alpha_n = 1$ for all $n$, then $W = S$ is the unilateral shift and so $S \in \mathcal{C}_C(S)$. However, $S$ is not complex symmetric.

**Example 1.2.** Let $C$ and $J$ be conjugations on $\mathcal{H}$. Assume that $T = (\begin{smallmatrix} 0 & C \\ J & 0 \end{smallmatrix})$ and $J = (\begin{smallmatrix} 0 & J \\ J & 0 \end{smallmatrix})$ on $\mathcal{H} \oplus \mathcal{H}$. Then $JTJT = TJJT = (\begin{smallmatrix} 1 & J \\ J & 0 \end{smallmatrix})$. Hence $T \in \mathcal{C}_J(T)$ is normal.

**Example 1.3.** Let $\mathcal{H} = \mathbb{C}^n$ and $C(z_1, z_2, z_3, \ldots, z_n) = (\overline{z_1}, \ldots, \overline{z_n})$. If

$$T = \begin{pmatrix} 0 & \lambda_1 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_2 & 0 & \ldots \\ \vdots & \vdots & \ddots & \ddots & \ldots \\ \vdots & \vdots & & 0 & \ldots \\ 0 & 0 & \ldots & \ldots & 0 \end{pmatrix} \quad \text{and} \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

for all nonzero $\lambda_i \in \mathbb{C}$, then $0 = (CTC)T e_1 \neq T(CTC) e_1 = \lambda_1 \cdot \overline{\lambda_{n-1}} \cdot e_1$. Hence $T \notin \mathcal{C}_C(T)$. But, it is clear that $T$ is binormal.

From Example 1.3, we observe that there exists $T$ such that $T \notin \mathcal{C}_C(T)$, in general.

The aim of this paper is to study some properties of an operator which satisfies $T \in \mathcal{C}_C(T)$ where $C$ is a conjugation on $\mathcal{H}$. In particular, we prove that if $T$ is normal, then $[CTC, C] = 0$. Moreover, the class of operators $T$ satisfy $[CTC, T] = 0$ is norm closed. Finally, we show that if $T$ is complex symmetric, then $T$ is binormal if and only if $[C][T][C][T] = 0$.

### 2. Operators satisfying $T \in \mathcal{C}_C(T)$

In this section, we study several properties about operators which satisfy $T \in \mathcal{C}_C(T)$ where $C$ is a conjugation on $\mathcal{H}$. Remark from [7] that if $T \in \mathcal{L}(\mathcal{H})$ is a complex symmetric operator with a conjugation $C$, then both $\text{Re} \, T$ and $\text{Im} \, T$ are complex symmetric operators with same conjugation $C$. In the following lemma, we consider the previous statement for operators which satisfy $T \in \mathcal{C}_C(T)$.

**Lemma 2.1.** Let $T \in \mathcal{C}_C(T)$. Suppose that $R = \frac{T+C:\mathcal{C}}{2}$ and $S = \frac{T-C:\mathcal{C}}{2i}$. Then $R$ and $S$ belong to $\mathcal{C}_C(T)$ such that $T = R + iS$ and $[R, S] = 0$, $[R, C] = 0$, and $[S, C] = 0$ hold.
Proof. Suppose that $T \in \mathfrak{C}_C(T)$ for a conjugation $C$. Since $R = \frac{T + CT C}{2}$ and $S = \frac{T - CT C}{2i}$, we can easily see that $T = R + iS$ and $RS = SR, CRC = R$ and $CSC = S$ hold. \qed

**Theorem 2.2.** If $T \in \mathcal{L}(\mathcal{H})$ is a normal operator, then $T, T^*, \Re T$, and $\Im T$ are in $\mathfrak{C}_C(T)$ for some conjugation $C$.

Proof. Assume that $T$ is normal. Then $T$ can be written in the form $U[|T|]$, where $U$ may be taken to be unitary such that $U$ and $|T|$ commute with each other by [6, Theorem 7, page 67]. Since $U$ is a unitary operator, by Godić and Lucenko [10], there exist conjugations $C$ and $J$ such that $U = CJ$ and $(CJ)^* = JC$. On the other hand, since $T$ is normal, it follows from [7] that $T$ is complex symmetric. Thus $C[T] = |T|C$ and $J[T] = |T|J$ (see [8, Lemma 1 and Example 2] for more details). Therefore, it is easy to see $CTC\; T = T\; TCT$ by this conjugation $C$. Thus $T \in \mathfrak{C}_C(T)$.

Put $\Re T := \frac{T + CT C}{2}$ and $\Im T := \frac{T - CT C}{2i}$. Since $T$ is normal and $[CTC, T] = 0$, it follows from the Fuglede-Putnam Theorem that $T^* (CTC) = (CTC)T^*$, i.e., $[CTC, T^*] = 0$. Thus $T^* \in \mathfrak{C}_C(T)$. Also, we get that

$$(\Re T)CTC = \frac{1}{2}(TCTC + T^*CTC) = \frac{1}{2}(CTCT + CTCT^*) = CTC(\Re T)$$

and

$$(\Im T)CTC = \frac{1}{2i}(TCTC - T^*CTC) = \frac{1}{2i}(CTCT - CTCT^*) = CTC(\Im T).$$

Hence $T, T^*, \Re T$, and $\Im T$ are in $\mathfrak{C}_C(T)$ for the conjugation $C$. \qed

**Remark 2.3.** The converse of Theorem 2.2 does not hold.

**Example 2.4.** Let $\mathcal{H} = \mathbb{C}^2$ and let $C$ be a conjugation on $\mathcal{H}$ given by $C(x, y) = (\bar{y}, \bar{x})$. Assume that $R = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ on $\mathcal{H}$. Then $CRC = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) = R$. Hence $R \in \mathfrak{C}_C(R)$. However, $R$ is not normal. We also note that $\Re T \notin \mathfrak{C}_C(T)$ and $\Im T \notin \mathfrak{C}_C(T)$.

Next, we state some basic properties of an operator $T \in \mathfrak{C}_C(T)$.

**Theorem 2.5.** Let $C$ be a conjugation on $\mathcal{H}$. Then the following statements hold.

(i) If $T \in \mathfrak{C}_C(T)$, then $f(T) \in \mathfrak{C}_C(T)$ for every function $f$ analytic on $\sigma(T)$.

(ii) If $T \in \mathfrak{C}_C(T)$ is invertible, then $T^{-1} \in \mathfrak{C}_C(T)$.

(iii) If $T_1, T_2 \in \mathfrak{C}_C(T)$, then $T_1 + T_2, \alpha T_1, T_1T_2$, and $T_2T_1$ are in $\mathfrak{C}_C(T)$ for any $\alpha \in \mathbb{C}$.

(iv) The class $\mathfrak{C}_C(T)$ is closed in norm.

Proof. (i) If $T \in \mathfrak{C}_C(T)$, then $p(T) \in \mathfrak{C}_C(T)$ for every polynomial $p$. If $T$ is a function analytic on $\sigma(T)$, then there exists $\{p_n\}$, sequence of polynomials, such that $\{p_n\}$ converges uniformly to $f$ on $\sigma(T)$. Since $p_n(T) \in \mathfrak{C}_C(T)$, it follows that $f(T) \in \mathfrak{C}_C(T)$. 
Thus we obtain
\[ TCT^{-1} = T^{-1}(TCT)T^{-1} = T^{-1}(CTCT)T^{-1} = T^{-1}CTC. \]

Hence \( T^{-1} \in \mathcal{C}_C(T) \).

(iii) Since \( T_1, T_2 \in \mathcal{C}_C(T) \), we have \((T_1 + T_2)CTC = CTC(T_1 + T_2)\) and \(T_1T_2(CTC) = T_1(CTC)T_2 = (CTC)T_1T_2\). Therefore \( T_1 + T_2 \) and \( T_1T_2 \) are in \( \mathcal{C}_C(T) \). Similarly, \( T_2T_1 \) is in \( \mathcal{C}_C(T) \).

(iv) If \( \{S_n\} \) is a sequence of operators such that
\[ S_n \in \mathcal{C}_C(T) \quad \text{and} \quad \lim_{n \to \infty} \|S_n - T\| = 0, \]
then we obtain
\[
\|TCTC - CTCT\| \leq \|TCTC - S_nCTC\| + \|CTCS_n - CTCT\|
\leq \|T - S_n\||CTC\| + \|CTC\||S_n - T\| \to 0 \quad \text{as} \ n \to \infty.
\]
Hence \( T \in \mathcal{C}_C(T) \) and so the class \( \mathcal{C}_C(T) \) is closed in norm.

From Theorem 2.5, we observe that \( \mathcal{C}_C(T) \) is a Banach space.

**Corollary 2.6.** If \( N \in \mathcal{L}(\mathcal{H}) \) is normal, then \( f(N) \in \mathcal{C}_C(N) \) for every function \( f \) analytic on \( \sigma(N) \). In particular, if \( N \) is invertible, then \( N^{-1} \in \mathcal{C}_C(N) \).

**Proof.** The proof follows from Theorems 2.2 and 2.5.

**Proposition 2.7.** Let \( T \in \mathcal{C}_C(T) \) for some conjugation \( C \). Then the following statements hold.

(i) \( T^* \in \mathcal{C}_C(T^*) \) and \( T^{-1} \in \mathcal{C}_C(T^{-1}) \) if \( T^{-1} \) exists.

(ii) If \( X \in \mathcal{L}(\mathcal{H}) \) is invertible with \([X, C] = 0\), then \( X^{-1}TX \in \mathcal{C}_C(X^{-1}TX) \).

(iii) If \( R \in \mathcal{L}(\mathcal{H}) \) is unitarily equivalent to \( T \), i.e., \( R = UTU^* \) where \( U \) is unitary, then \( R \in \mathcal{C}_D(R) \) for a conjugation \( D = UCU^* \).

(iv) \([CT^nC, T^m] = 0\) for all \( n, m \in \mathbb{N} \).

**Proof.** (i) If \( T \in \mathcal{C}_C(T) \), then it is clear that \( T^* \in \mathcal{C}_C(T^*) \). If \( T \in \mathcal{C}_C(T) \) is invertible, then \( T(CTC) = (CTC)T \) implies
\[ CT^{-1}CT^{-1} = [T(CTC)]^{-1} = [(CTC)]^{-1}T^{-1}CT^{-1}C. \]

(ii) If \( X \) is an invertible with \( X = CXC \), then we obtain
\[
C(X^{-1}TX)C(X^{-1}TX) = CX^{-1}TXX^{-1}CTX
= CX^{-1}TCTX = X^{-1}CTCTX
= X^{-1}TCTCX = X^{-1}TXX^{-1}CTCX
= X^{-1}TXCX^{-1}TCX = (X^{-1}TX)C(X^{-1}TX)C.
\]
Hence \( X^{-1}TX \in \mathcal{C}_C(X^{-1}TX) \).

(iii) Since \([CTC, T] = 0\), \( R = UTU^* \), and \( D = UCU^* \), it follows that \([DRD, R] = U[CTC, T]U^* = 0\). Hence \( R \in \mathcal{C}_D(R) \) for the conjugation \( D \).
(iv) It is clear that $CTC = T^2C$ and $CT^2T = TCT^2C$. Assume that $CT^kCT^j = T^jCT^kC$ for all $k \leq n$ and $j \leq m$. Then we have

\begin{align*}
(1) \quad C & T^{n+1}CT^m = CTCCT^mCT^m = CTCCT^mCT^mC = T^{n+1}CCT^mC \\
(2) \quad C & T^nCT^{m+1} = CTCCT^mT = T^mCT^nCT = T^{m+1}CT^mC.
\end{align*}

Since (1) and (2) hold for $n+1$ and $m+1$, it holds $CT^nCT^m = T^mCT^nC$ for every $n, m \in \mathbb{N}$.

Let us recall that $\mathcal{H}_1 \otimes \mathcal{H}_2$ denotes the completion (endowed with a sensible uniform cross-norm) of the algebraic tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ where $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable complex Hilbert spaces. For operators $T \in \mathcal{L}(\mathcal{H}_1)$ and $S \in \mathcal{L}(\mathcal{H}_2)$, we define the tensor product operator $T \otimes S$ on $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by

$$(T \otimes S)(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j) = \sum_{j=1}^{n} \alpha_j Tx_j \otimes Sy_j.$$ 

Then it is well known that $T \otimes S \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. The definition of $T \otimes S$ is extended from these finite linear combinations of simple tensors to the whole space. It was known from [3] that if $C_1$ and $C_2$ are conjugations on $\mathcal{H}$, we define $C_1 \otimes C_2$ on $\mathcal{H} \otimes \mathcal{H}$ by

$$(C_1 \otimes C_2)(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j) = \sum_{j=1}^{n} \alpha_j C_1 x_j \otimes C_2 y_j.$$ 

Then $C_1 \otimes C_2$ is a conjugation on $\mathcal{H} \otimes \mathcal{H}$.

We also observe the following easy consequences. So we omit its proof.

**Proposition 2.8.** Let $C, C_1, C_2$ be conjugations on $\mathcal{H}$. Then the following statements hold.

(i) If $T_i \in C_{\mathcal{C}}(T, \mathcal{H})$ for conjugations $C_i$ with $i = 1, 2$, respectively, then $T_1 \oplus T_2 \in C_{\mathcal{C}}(T_1 \oplus T_2)$ for a conjugation $C_1 \oplus C_2$.

(ii) Let $T \in C_{\mathcal{C}}(T)$ and $S \in C_{\mathcal{C}}(S)$. If $[T, S] = 0$ and $[CTC, S] = 0$, then $T + S \in C_{\mathcal{C}}(T + S)$ and $TS \in C_{\mathcal{C}}(TS)$ for a conjugation $C$.

(iii) If $T \in C_{\mathcal{C}}(T)$ and $S \in C_{\mathcal{C}}(S)$ for conjugations $C_1$ and $C_2$, respectively, then $T \otimes S \in C_{\mathcal{C}}(T \otimes S)$ for a conjugation $C_1 \otimes C_2$.

For the next result, we need the following lemma.

**Lemma 2.9** ([11, Lemma 3.21]). Let $T \in \mathcal{L}(\mathcal{H})$ and let $C$ be a conjugation on $\mathcal{H}$. Then $\sigma(CTC) = \sigma(T)^*$ and $\alpha(TCT) = \alpha(T)^*$.

If $T$ satisfies $CTC = T$, then $\sigma(T) = \sigma(T)^*$ from Lemma 2.9, that is, $\sigma(T)$ is a symmetric set with the real line. For a commuting pair $T = (T_1, T_2) \in \mathcal{L}(\mathcal{H})^2$, $\sigma_T(T_1, T_2)$ (or $\sigma_T(T)$) and $\sigma_ja(T_1, T_2)$ (or $\sigma_ja(T)$) denote the Taylor spectrum and the joint approximate point spectrum of $(T_1, T_2)$, respectively. We explain
the Taylor spectrum for a commuting 2-tuple \( T = (T_1, T_2) \) case. Consider the following chain complex \( E(T) \) as follows;

\[
E(T) : 0 \to H \xrightarrow{\delta_1^T} H \oplus H \xrightarrow{\delta_2^T} H \to 0,
\]

where \( \delta_1^T(x) := (-T_2x) \oplus (T_1x) \) and \( \delta_2^T(x_1 \oplus x_2) := T_1x_1 + T_2x_2 \). Then it is easy to see that \( \delta_2^T \circ \delta_1^T = 0 \). The commuting 2-tuple \( T = (T_1, T_2) \) is said to be non-singular if the chain complex \( E(T) \) is exact, i.e., \( \ker \delta_2^T = \{0\} \), image \( \delta_1^T = \ker \delta_2^T \) and image \( \delta_2^T = H \). It is well known that \( T \) is non-singular if and only if

\[
\alpha(T) = \left( \begin{array}{cc} T_1 & T_2 \\ -T_2 & T_1 \end{array} \right)
\]

is invertible on \( H \oplus H \) (see [14]). For \( z = (z_1, z_2) \in \mathbb{C}^2 \), let \( T - z = (T_1 - z_1, T_2 - z_2) \). Then we define the Taylor spectrum \( \sigma_T(T) \) of \( T = (T_1, T_2) \) as

\[
\sigma_T(T) = \{ \lambda \in \mathbb{C}^2 : \|x_n\| = 1 \text{ and } (T - \lambda_1)x_n \to 0 \text{ and } (S - \lambda_2)x_n \to 0 \text{ as } n \to \infty \}.
\]

It is well known \( \sigma_{ja}(T, S) \subset \sigma_T(T, S) \) (see [1] and [13]).

**Proposition 2.10.** Let \( T \in \mathcal{C}_C(T) \). Then there exist commuting operators \( R \) and \( S \) such that the following statements hold:

(i) \( T = R + iS \) and \( (T, R, S) \) is a commuting 3-tuple.

(ii) \( \sigma(R) \) and \( \sigma(S) \) are symmetric sets with the real line.

(iii) If \( \lambda \in \sigma(T) \), then there exist \( \alpha \in \sigma(R) \) and \( \beta \in \sigma(S) \) such that \( \lambda = \alpha + i\beta \).

(iv) If \( \alpha \in \sigma(R) \), then there exist \( \lambda \in \sigma(T) \) and \( \beta \in \sigma(S) \) such that \( \lambda = \alpha + i\beta \).

(v) If \( \beta \in \sigma(S) \), then there exist \( \lambda \in \sigma(T) \) and \( \alpha \in \sigma(R) \) such that \( \lambda = \alpha + i\beta \).

**Proof.** The proofs of (i) and (ii) follow from Theorem 2.2 and Lemma 2.9.

(iii) Since \( (R, S) \) is a commuting pair and \( T = R + iS \), the proof follows from the spectral mapping theorem for \( f(a, b) = a + ib \) of the Taylor spectrum.

(iv) Since \( (T, S) \) is a commuting pair and \( R = -T + iS \), the proof follows from the spectral mapping theorem for \( g(a, b) = -a + ib \) of the Taylor spectrum.

(v) The proof follows from a similar method of (iv).

**Remark 2.11.** The statements (iii), (iv) and (v) hold for the approximate point spectra \( \sigma_a(T), \sigma_a(R) \) and \( \sigma_a(S) \). Please see [1] for the spectral mapping theorem for the joint approximate point spectrum.

\[\square\]
For an operator $T \in \mathcal{L}(\mathcal{H})$ and a conjugation $C$, we define the operator $\alpha_m(T; C)$ by

$$\alpha_m(T; C) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} C^{m-j} C \cdot T^j.$$ 

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an $[m, C]$-symmetric operator if $\alpha_m(T; C) = 0$. See [4] for properties of $[m, C]$-symmetric operators.

**Proposition 2.12.** If $T \in C_C(T)$ is an $[m, C]$-symmetric operator, then the following statements hold.

(i) $CTC - T$ is $m$-nilpotent, i.e., $(CTC - T)^m = 0$.

(ii) $\sigma_T(CTC, T) = \{(\lambda, \lambda) : \lambda \in \sigma(T)\}$. In this case, it holds $\sigma(CTC) = \sigma(T) = \sigma(T)^\ast$. Moreover, it holds $\sigma_{ja}(CTC, T) = \{(\lambda, \lambda) : \lambda \in \sigma_a(T)\}$.

**Proof.** (i) Since $T$ commutes with $CTC$, the proof follows that $0 = \alpha_m(T; C)$.

(ii) Since $(CTC, T)$ is a commuting pair, by the spectral mapping theorem of the Taylor spectrum, it holds $f(\sigma_T(CTC, T)) = \sigma(CTC - T)$, where $f(\mu, \lambda) = \mu - \lambda$. By Proposition 2.12, we have $\sigma(CTC - T) = \{0\}$ and hence $\mu = \lambda$. Thus $\sigma_T(CTC, T) = \{(\lambda, \lambda) : \lambda \in \sigma(T)\}$ and we have $\sigma(CTC) = \sigma(T) = \sigma(T)^\ast$ from [11]. Since $\sigma_{ja}(CTC, T) = \{(\lambda, \lambda) : \lambda \in \sigma_a(T)\}$, the proof follows from the spectral mapping theorem of the joint approximate point spectrum.

For an operator $T \in \mathcal{L}(\mathcal{H})$, $T$ is said to be normaloid if $r(T) = \|T\|$, where $r(T)$ is the spectral radius of $T$. Then we have the following corollary.

**Corollary 2.13.** Let $T \in C_C(T)$ be an $[m, C]$-symmetric operator. If $CTC - T$ is normaloid, then $CTC - T = 0$.

**Proof.** By Proposition 2.12, we have $\sigma(CTC - T) = \{0\}$. Since $CTC - T$ is normaloid, it holds $CTC - T = 0$.

For an operator $T \in \mathcal{L}(\mathcal{H})$ and a conjugation $C$, we define the operator $\lambda_m(T; C)$ by

$$\lambda_m(T; C) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} C^{m-j} C \cdot T^{m-j}.$$ 

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an $[m, C]$-isometric operator if $\lambda_m(T; C) = 0$. See [3] for properties of $[m, C]$-isometric operators.

**Proposition 2.14.** If $T \in C_C(T)$ is an $[m, C]$-isometric operator, then the following statements hold.

(i) $CTCT - I$ is $m$-nilpotent, i.e., $(CTCT - I)^m = 0$. 

\( (ii) \quad \sigma_T(CTC, T) = \{(1/\lambda, \lambda) : \lambda \in \sigma(T)\}. \) In this case, it holds \( \sigma(CTC) = \{(1/\lambda, \lambda) : \lambda \in \sigma_a(T)\}. \) Moreover, it holds \( \sigma_{ja}(CTC, T) = \{(1/\lambda, \lambda) : \lambda \in \sigma_a(T)\}. \)

**Proof.** (i) Since \( T \) commutes with \( CTC \), we have \( \lambda_m(T; C) = (CTC T - I)^m \). Hence, we have \( (CTC T - I)^m = 0 \).

(ii) Since \( (CTC, T) \) is a commuting pair, by the spectral mapping theorem of the Taylor spectrum, it holds

\[ f(\sigma(CTC, T)) = \sigma(CTC T - I), \]

where \( f(\mu, \lambda) = \mu \cdot \lambda - 1 \). By Proposition 2.14(i), we have \( \sigma(CTC T - I) = \{0\} \) and hence \( \mu \cdot \lambda = 1 \). Therefore, \( \sigma_T(CTC, T) = \{(1/\lambda, \lambda) : \lambda \in \sigma(T)\} \) and we have \( \sigma(CTC) = \{(1/\lambda, \lambda) : \lambda \in \sigma(T)\} \). By the same way, we get

\[ \sigma_{ja}(CTC, T) = \{(1/\lambda, \lambda) : \lambda \in \sigma_a(T)\}. \]

Finally, we focus on the binormality of \( T \) when \( T \in C_C(T) \) for a conjugation \( C \) on \( H \).

**Lemma 2.15.** Let \( T \in \mathcal{L}(H) \) and let \( C \) be a conjugation on \( H \). If \( (T^*T)C = C(TT^*) \), then \( T \) is binormal if and only if \( |T| \in C_C(|T|) \).

**Proof.** Let \( T \) be binormal. Then \( |T^*||T| = |T||T^*| \). Since \( (T^*T)C = C(TT^*) \), it follows that \( |T^*| = C|T| \). Therefore \( |T|C|T|C = C|T|C|T| \). Thus \( |T| \in C(|T|) \).

Conversely, if \( |T| \in C(|T|) \), then \( |T|C|T|C = C|T|C|T| \) implies \( |T^*||T| = |T||T^*| \). Thus \( T \) is binormal. \( \square \)

It is well known that normal operators are binormal. The **Duggal transform** \( \tilde{T}^D \) of \( T \) is given by \( \tilde{T}^D := |T|U \) where \( U \) is the appropriate partial isometry satisfying \( \ker(U) = \ker(T) \) and \( \ker(U^*) = \ker(T^*) \) (see [5]).

**Theorem 2.16.** Let \( T \in \mathcal{L}(H) \) be complex symmetric with a conjugation \( C \). Suppose that \( T = U[T] \) is the polar decomposition of \( T \) where \( U = CJ \) and \( J \) is a partial conjugation supported on \( \overline{\text{ran}([T])} \), which commutes with \( T \). Then the following statements are equivalent.

(i) \( T \) is binormal.

(ii) \( |T| \in C_C(|T|) \).

(iii) \( ||\tilde{T}^D||, |T| \] = 0 where \( \tilde{T}^D := |T|U \) is the Duggal transform of \( T \).

**Proof.** (i) \( \Leftrightarrow \) (ii) Let \( T = U[T] \) be the polar decomposition of \( T \). By [8], \( U = CJ \) where \( C \) and \( J \) are conjugations and \( J \) commutes with \( |T| \). Since \( T \) is complex symmetric with the conjugation \( C \), it follows that \( (T^*T)C = |T|^2C = C(TT^*) \). Hence the proof follows from Lemma 2.15.

(i) \( \Leftrightarrow \) (iii) Let \( \tilde{T}^D := |T|U \) be the Duggal transform of \( T \). If \( T \) is binormal, then \( \tilde{T}^D \) is binormal by [12] and so \( ||\tilde{T}^D||, |(\tilde{T}^D)^*| \] = 0. Since \( T \) is complex symmetric with the conjugation \( C \), it follows that \( (T^*T)C = C(TCTC) = \)
Corollary 2.17. By [9, Corollary 3], there exist a conjugation $C$ by $T$ such that $T°C$ is complex symmetric. Hence by [8, Theorem 2], there exist a conjugation $C$ on $\mathcal{H}$ and a partial conjugation $J$ supported on $\text{ran}[T]$ such that $T = CJ[T]$ and $J[T] = [T]J$. On the other hand, since $T^2$ is normal, it follows from the Fuglede-Putnam Theorem that $(T^2)T^* = T^*(T^2)$. Hence

$$[T^*T, TT^*] = T^*TT^* - TT^*T = TTT^*T^* - TTT^*T^* = 0$$

and so $T$ is binormal (also see [2]). Therefore, $|C[T]C, |T]\] = 0$ for this conjugation $C$.

As some applications of Theorem 2.16, we get the following corollary.

Corollary 2.17. Let $T \in \mathcal{L}(\mathcal{H})$ be such that $T^2$ is normal. Then $|T| \in C_C(|T|)$.

Proof. By [9, Corollary 3], $T$ is complex symmetric. Hence by [8, Theorem 2], there exist a conjugation $C$ on $\mathcal{H}$ and a partial conjugation $J$ supported on $\text{ran}[T]$, such that $T = CJ[T]$ and $J[T] = [T]J$. On the other hand, since $T^2$ is normal, it follows from the Fuglede-Putnam Theorem that $(T^2)T^* = T^*(T^2)$. Hence

$$|T^*T, TT^*| = T^*TT^* - TT^*T = TTT^*T^* - TTT^*T^* = 0$$

and so $T$ is binormal (also see [2]). Therefore, $|C[T]C, |T]\] = 0$ for this conjugation $C$. □

As some applications of Theorem 2.16, we provide examples of complex symmetric operators which are binormal or non-binormal.

Example 2.18. Let $T = (\frac{1}{1})$ on $\mathbb{C}^2$. Then $T$ is complex symmetric with the conjugation $C$ defined by $C(z_1, z_2) = (\bar{z}_2, \bar{z}_1)$ for $z_1, z_2 \in \mathbb{C}$. Since $|T| = \frac{1}{\sqrt{2}} (\frac{1}{1})$, it follows that

$$C|T|C[T] = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \text{ and } |T|C[T]C = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}.$$ 

Hence $T$ is not binormal by Theorem 2.16.

Example 2.19. Let $\mathcal{H} = \ell^2$ and let $C$ be the canonical conjugation given by $C(\sum_{n=0}^{\infty} x_n e_n) = \sum_{n=0}^{\infty} \overline{x_n} e_n$ with $Ce_n = e_n$ for all $n$. Assume that $T = (\frac{2}{0})$ on $\mathcal{H} \oplus \mathcal{H}$, where $S \in \mathcal{L}(\mathcal{H})$ is the unilateral shift. Then $S$ and $S^*$ commute with the conjugation $C$. Denote the conjugation $C$ given by $C = (\begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix})$. Then we obtain that

$$CT^* - TC = \begin{pmatrix} C & CS^* \\ CS & S^*C \end{pmatrix} - \begin{pmatrix} C & S^*C \\ SC & 0 \end{pmatrix} = 0.$$ 

Hence $T$ is a complex symmetric operator (cf. [9]). Moreover, since $T = (\frac{S}{0})$, it follows that

$$T^*T = \begin{pmatrix} SS^* & S \\ S & 2I \end{pmatrix} \text{ and } TT^* = \begin{pmatrix} 2I & S^* \\ S & SS^* \end{pmatrix}.$$ 

Therefore we have

$$TT^*T = \begin{pmatrix} 2SS^* + S^2 & 2S + 2S^* \\ S^2S^* + SS^* & S^2 + 2SS^* \end{pmatrix}.$$
and
\[ T^*TT^* = \begin{pmatrix} S^2 + 2SS^* & SS^* + S^2S^* \\ 2S + 2S^* & S^2 + 2SS^* \end{pmatrix}. \]
Hence \( T \) is not binormal. On the other hand, if \( S \) is the unilateral shift on \( H \), then \( T = S^* \oplus S \) is binormal and complex symmetric. Indeed, in this case, we have \( T^*T = \begin{pmatrix} SS^* & 0 \\ 0 & C \end{pmatrix} \), \( |T|C = \begin{pmatrix} 0 & CSS^* \\ C & 0 \end{pmatrix} \), and \( C|T| = \begin{pmatrix} 0 & SS^* \\ SS^* & 0 \end{pmatrix} \). Hence \( [|T|, C|T|C] = 0 \) and so \( T \in C(|T|) \). Therefore \( T \) is binormal by Theorem 2.16.

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