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ALMOST SURE AND COMPLETE CONSISTENCY OF THE ESTIMATOR IN NONPARAMETRIC REGRESSION MODEL FOR NEGATIVELY ORTHANT DEPENDENT RANDOM VARIABLES

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Abstract. In this paper, the author considers the nonparametric regression model with negatively orthant dependent random variables. The wavelet procedures are developed to estimate the regression function. For the wavelet estimator of unknown function $g(\cdot)$, the almost sure consistency is derived and the complete consistency is established under the mild conditions. Our results generalize and improve some known ones for independent random variables and dependent random variables.

1. Introduction

Consider the estimation of a standard nonparametric regression model involving a regression function $g(\cdot)$ which is defined on $[0, 1]$.

$$Y_i = g(x_i) + \vartheta_i, \quad (1 \le i \le n), \tag{1.1}$$

where x_i are nonrandom design points, x_i's are denoted x_i and taken to be ordered $0 \le x_1 \le \cdots \le x_n \le 1$, ϑ_i are random errors.

It is widely recognized that as an important method in data analysis, regression function estimation is widely applied in filtering and prediction in communications and control systems, classification and pattern recognition, and econometrics. So the model (1.1) has got widely studies. The model (1.1) has been used popularly in solving practical problems and variety of estimation methods have been used to get access to estimators of $g(\cdot)$.

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For example, Georgiev [5] proposed the weighted estimator of nonparametric regression functions $\widehat{g}(x) = \sum_{i=1}^{n} W_i(x)Y_i$, where the weight function $W_i(x),\ i = 1, \ldots, n$, depends on the fixed design points $x_1, \ldots, x_n$ and on the number of observations. In the independent case, the weighted estimator has been considered by many researchers, such as Georgiev and Greblicki [7], Müller [14], Georgiev [6] and the references therein. In various dependence cases, the weighted estimator has also been investigated extensively, such as, Liang and Jing [13] for asymptotic properties with NA errors, Yang [28] for asymptotic normality with α-mixing errors, Li et al. [10] for Berry-Esseen bounds with linear process generated by φ-mixing errors, Shen et al. [16] for consistency with NSD errors, Wang and Si [23] for consistency with NOD errors, Bao et al. [2] for consistency with END errors, Zhang et al. [29] for consistency with WOD errors, and so on.

Since the 1990s, wavelet techniques attracted wide attention, many scholars have tried to use wavelet methods in statistics. And the great adaptability to the degree of smoothness of the underlying unknown curve is the major advantage of the wavelet method. It is widely known that the hypotheses of degrees of smoothness of the underlying function in the wavelet approach are less restrictive. Hall and Patil [8] have clearly shown the wavelet estimators has extraordinary local adaptability in handling discontinuities, and demonstrated explicitly that discontinuities of the unknown curve with a negligible effect on performance of wavelet curve estimator. Besides, compared to kernel estimate, which tends to emphasize less on some fine local details of the curve like change point, wavelet estimate can catch the local details and the global characteristics of the curve from data. Owing to the existing efficient algorithms for wavelet methods, the computation speed becomes another practical and important advantage of using wavelet methods over the kernel methods. Antoniadis et al. [1] proposed a wavelet estimator of $g(\cdot)$ which has been studied extensively, and subsequently many authors adopted wavelet methods to estimate various models. One may refer to Xue and Liu [27] for wavelet estimator in semiparametric model with i.i.d. errors, Liang [12] for wavelet estimator in heteroscedastic model with α-mixing errors, Zhou et al. [30] for wavelet estimator in varying coefficient model with α-mixing errors, Ding et al. [3] for wavelet estimator in nonparametric model with END errors, Ding et al. [4] for wavelet estimator in heteroscedastic semiparametric model with φ-mixing errors, and so on.

Recall a nonparametric wavelet estimator of $g(\cdot)$ proposed by Antoniadis et al. [1]

$$\widehat{g}(x) = \sum_{i=1}^{n} Y_i \int_{\Gamma_i} E_m(x,s)ds, \tag{1.2}$$

where $\Gamma_i = [s_{i-1}, s_i),\ s_0 = 0,\ s_n = 1,\ s_i = (x_i + x_{i+1})/2,\ i = 1, \ldots, n$. Hence $x_i \in \Gamma_i$ for $1 \leq i \leq n$.

Let $\phi(\cdot)$ be a given scaling function in the Schwarz space with order l. A multiresolution analysis of $L^2(\mathbb{R})$ consists of an increasing sequence of the closed subspace $\{W_m, m \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the integer set and $L^2(\mathbb{R})$ is a set of square integral functions over the real line. Since $\{\phi(x-k), k \in \mathbb{Z}\}$ is an orthogonal family of $L^2(\mathbb{R})$ and W_0 is the subspace spanned, if we define

$$\phi_{mk}(x) = 2^{m/2}\phi(2^m x - k),\ k \in \mathbb{Z},$$

then $\{\phi_{0k}, k \in \mathbb{Z}\}$ is an orthogonal basis of W_0, and $\{\phi_{mk}, k \in \mathbb{Z}\}$ is an orthogonal basis of W_m. The associated integral kernel of W_m is given by

$$E_m(x,s) = 2^m E_0(2^m x, 2^m s) = 2^m \sum_{k\in\mathbb{Z}} \phi(2^m x - k)\phi(2^m s - k).$$

Let us introduce the concept of negatively orthant dependence as follows.

Definition 1.1. A finite collection of random variables $X_1, X_2, \ldots, X_n$ is said to be negatively orthant dependent (NOD, for short) if

$$\mathrm{P}(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leq \prod_{i=1}^{n} \mathrm{P}(X_i > x_i)$$

and

$$\mathrm{P}(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) \leq \prod_{i=1}^{n} \mathrm{P}(X_i \leq x_i)$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$. An infinite sequence $\{X_n, n \geq 1\}$ is said to be NOD if every finite subcollection is NOD.

An array of random variables $\{X_{ni}, i \geq 1, n \geq 1\}$ is called rowwise NOD if for every $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is NOD.

The notion of NOD sequences was first introduced by Joag-Dev and Proschan [9]. Indeed, independent random sequences are NOD. Joag-Dev and Proschan [9] showed that NA random sequences are NOD. They also proposed an example in which $Z = (Z_1, Z_2, Z_3, Z_4)$ possesses NOD, but does not possess NA. Thus we know that NOD is weaker than NA. So studying the basic properties, limiting behavior and some applications of NOD sequences are of great interest. A number of results for NOD sequences have been investigated by many authors. For example, Wang et al. [22] studied some exponential inequalities and asymptotic approximation of inverse moment, Wang et al. [19] studied the convergence rate for the strong law of large numbers, Wang et al. [20] studied some properties and some strong limit results of the weighted sums, Li et al. [11] studied the Bahadur representation for sample quantile, Wang et al. [21] studied complete convergence for arrays of rowwise NOD, Shen [15] studied strong convergence rate for weighted sums of arrays of rowwise NOD, Wang and Si [23] studied the complete consistency of estimator of nonparametric regression model, Wang and Hu [18] studied asymptotic properties of least square estimators in the simple linear errors-in-variables regression model, Wang et al. [24] studied complete moment convergences, and so forth. However, few literature investigates consistency results for the wavelet estimator of the nonparametric regression model with NOD random variables. The main purpose of the paper is to investigate the consistency results for the estimator of the nonparametric regression model with NOD random variables.

The rest of the paper is organized as follows. In the next section we introduce assumptions and main results. Section 3 gives some preliminary lemmas, which are used in the proofs of the main results. The proofs of the main results are collected in Section 4.

Throughout the paper for any function g, we use $c(g)$ to denote all continuity points of the regression function g on $[0,1]$. Let C and M denote positive constants which may be different in various places. $a_n = O(b_n)$ or $\ll$ and $a_n = o(b_n)$ stand for $a_n \leq Cb_n$ and $\lim_{n\to\infty} a_n/b_n = 0$, respectively. Let $I(\cdot)$ be the indicator function, $\log x = \log \max(x, e)$. $\doteq$ means that each term of the left equation is respectively denoted as the right signs. All limits are taken as the sample size n tends to ∞, unless specified otherwise.

2. Assumptions and main results

In order to list some restrictions for ϕ and g, we give two definitions here.

Definition 2.1. A father wavelet ϕ is said to be β-regular ($S_\beta, \beta \in \mathbb{N}$) if for any $\jmath \leq \beta$ and any integer $\jmath$, then $\left|\frac{d^\jmath \varphi}{dx^\jmath}\right| \leq C_k(1+|x|)^{-k}$, where C_k is a generic constant depending only on k.

Definition 2.2. A function space $H^\nu (\nu > 1/2)$ is said to be Sobolev space with order ν, i.e., if $y \in H^\nu$, then $\int (1+w^2)^\nu |\widehat{y}(w)|^2 dw < \infty$, where $\widehat{y}$ is the Fourier transform of y.

For convenience, the assumptions used in this paper are listed below.

(A1) $g(\cdot) \in H^\nu$, and $g(\cdot)$ satisfies the Lipschitz condition of order $\gamma > 0$;

(A2) Scaling function $\phi(\cdot)$ is β-regular with $\beta \geq \nu$, has a compact support and satisfies the Lipschitz condition with order 1 and $|\widehat{\phi}(\varepsilon) - 1| = O(\varepsilon)$ as $\varepsilon \to 0$, where $\widehat{\phi}$ is the Fourier transform of ϕ;

(A3) $\max_{1\leq i\leq n} |s_i - s_{i-1}| = O(n^{-1})$, $m \to \infty$ and $2^{-m}n \to \infty$ as $n \to \infty$;

(A4) $2^m = O(n^{\frac{1}{3}})$.

Based on the assumptions above, we can get the following results.

Theorem 2.1. *Let $\{\vartheta_i, 1 \leq i \leq n\}$ be a sequence of NOD random variables with* $\mathrm{E}\vartheta_i = 0$ *and* $\sup_{1\leq i\leq n} \mathrm{E}|\vartheta_i|^t < \infty$ *for some* $t > \frac{3}{2}$. *Assume further that assumptions* (A1)-(A4) *hold true, then* $\forall\ x \in c(g)$,

$$g_n(x) \to g(x) \quad a.s., \ \text{as } n \to \infty.$$

Theorem 2.2. *Let $\{\vartheta_i, 1 \leq i \leq n\}$ be a mean zero NOD sequence, which is stochastically dominated by a random variable X with* $\mathrm{E}|X|^3 < \infty$. *Assume further that assumptions* (A1)-(A4) *hold true, then* $\forall\ x \in c(g)$,

$$g_n(x) \to g(x) \quad \text{completely}, \ \text{as } n \to \infty.$$

Remark 2.1. Assumptions (A1)-(A3) are general basic assumptions of wavelet estimation, which have been used by many authors, one can refer to Antoniadis et al. [1], Liang [12], Li et al. [10], and so on. (A4) is assumed in Xue and Liu [27]. So we can see that the assumptions in this paper are suitable and reasonable.

3. Some lemmas

In this section, we will present some important lemmas which will be used to prove the above main results.

Lemma 3.1 (Taylor et al. [17]). *Let random variables $X_1, X_2, \ldots, X_n$ be NOD. If $f_1, f_2, \ldots, f_n$ are all nondecreasing (or all nonincreasing) functions, then random variables $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ are also NOD.*

Lemma 3.2 (Li et al. [10]). *Suppose that assumptions* (A1)-(A3) *hold. Then*

(i) $\left|\int_{\Gamma_i} E_m(x,s)ds\right| = O\left(\frac{2^m}{n}\right),\ i = 1, 2, \ldots,$

(ii) $\sup\limits_x \int_0^1 |E_m(x,s)|ds \le C,$

(iii) $\sum\limits_{i=1}^n \left|\int_{\Gamma_i} E_m(x,s)ds\right| \le C,$

(iv) $\sum\limits_{i=1}^n \left(\int_{\Gamma_i} E_m(x,s)ds\right)^2 = O\left(\frac{2^m}{n}\right).$

Lemma 3.3 (Antoniadis et al. [1]). *Suppose that assumptions* (A1)-(A3) *hold. Then*

$$\mathrm{E}\widehat{g}(x) - g(x) = O(n^{-\gamma}) + O(\eta_m),$$

where

$$\eta_m = \begin{cases} (1/2^m)^{\nu-1/2} & 1/2 < \nu < 3/2, \\ \sqrt{m}/2^m & \nu = 3/2, \\ 1/2^m & \nu > 3/2. \end{cases}$$

Lemma 3.4 (Wu [26]). *Let $p \ge 2$ and $\{X_n, n \ge 1\}$ be a sequence of NOD random variables with $\mathrm{E}X_n = 0$ and $\mathrm{E}|X_n|^p < \infty$ for every $n \ge 1$. Then there exists a positive constant C_p depending only on p such that for every $n \ge 1$,*

$$\mathrm{E}\left(\max_{1\le j\le n}\left|\sum_{i=1}^j X_i\right|^p\right) \le C\log^p n\left\{\sum_{i=1}^n \mathrm{E}|X_i|^p + \left(\sum_{i=1}^n \mathrm{E}X_i^2\right)^{p/2}\right\}.$$

Lemma 3.5 (Shen [15]). *Let $\{X_n, n \ge 1\}$ be a sequence of NOD random variables with zero means and finite second moments. Denote $S_n = \sum_{i=1}^n X_i$ and $B_n^2 = \sum_{i=1}^n \mathrm{E}X_i^2$ for each $n \ge 1$. Then for all $x > 0$ and $y > 0$,*

$$\begin{aligned} &\mathrm{P}(|S_n| \ge x) \\ &\le 2\mathrm{P}\left(\max_{1\le i\le n}|X_i| \ge y\right) + 2\exp\left\{-\frac{x^2}{2(xy+B_n^2)}\left(1 + \frac{2}{3}\log\left(1 + \frac{xy}{B_n^2}\right)\right)\right\}. \end{aligned}$$

For convenience, we give the definition of stochastic domination here.

Definition 3.1. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$\mathrm{P}(|X_n| > x) \leq C\mathrm{P}(|X| > x)$$

for all $x \geq 0$ and $n \geq 1$.

By the definition of stochastic domination and integration by parts, we can get the following property for stochastic domination. For the details of the proof, one can refer to Wu [25].

Lemma 3.6 (Wu [25]). *Let $\{X_n, n \geq 1\}$ be an array of random variables which is stochastically dominated by a random variable X. Then, for any $\alpha > 0$ and $b > 0$, the following two statements hold*:

$$\begin{aligned} &\mathrm{E}|X_n|^\alpha I(|X_n| \leq b) \leq C_1 \left(\mathrm{E}|X|^\alpha I(|X| \leq b) + b^\alpha \mathrm{P}(|X| > b)\right), \\ &\mathrm{E}|X_n|^\alpha I(|X_n| > b) \leq C_2 \mathrm{E}|X|^\alpha I(|X| > b), \end{aligned}$$

where C_1 and C_2 are positive constants. Consequently, $\mathrm{E}|X_n|^\alpha \leq C\mathrm{E}|X|^\alpha$, *where C is a positive constant.*

4. Proofs of the main results

Proof of Theorem 2.1. In view of (1.1) and (1.2) we have

$$\begin{aligned} |\widehat{g}(x) - g(x)| &\leq |\widehat{g}(x) - \mathrm{E}\widehat{g}(x)| + |\mathrm{E}\widehat{g}(x) - g(x)| \\ &= \left|\sum_{i=1}^n \int_{\Gamma_i} E_m(x,s) ds \vartheta_i\right| + |\mathrm{E}\widehat{g}(x) - g(x)|. \end{aligned} \tag{4.1}$$

We see from Lemma 3.3 that

$$|\mathrm{E}\widehat{g}(x) - g(x)| = O(n^{-\gamma}) + O(\eta_m).$$

Note that since $m \to \infty$, then $\eta_m = \begin{cases} (1/2^m)^{\nu-1/2}, & 1/2 < \nu < 3/2 \\ \sqrt{m}/2^m, & \nu = 3/2 \\ 1/2^m, & \nu > 3/2 \end{cases} \to 0,$
and for $\gamma > 0$, then $n^{-\gamma} \to 0$ as $n \to \infty$, which gives that

$$|\mathrm{E}\widehat{g}(x) - g(x)| \to 0 \quad \text{as} \quad n \to \infty. \tag{4.2}$$

Hence, by (4.1) and (4.2), we can see that in order to prove the Theorem 2.1, it suffices to show that

$$\sum_{i=1}^n \int_{\Gamma_i} E_m(x,s) ds \vartheta_i \to 0 \quad \text{a.s.,} \quad \text{as} \quad n \to \infty. \tag{4.3}$$

In the following, for fixed design point $x \in [0,1]$, without loss of generality, we may assume that the weights $\int_{\Gamma_i} E_m(x,s)ds > 0$. For any $\varepsilon > 0$, we choose small positive constant such that $\kappa < \frac{2}{3}$ and some large $N \geq 1$, one can write

$$\begin{aligned} X_i^{(1)} \doteq & -n^{-\kappa} I\left(\int_{\Gamma_i} E_m(x,s)ds\vartheta_i < -n^{-\kappa}\right) \\ & + \int_{\Gamma_i} E_m(x,s)ds\vartheta_i I\left(\left|\int_{\Gamma_i} E_m(x,s)ds\vartheta_i\right| \leq n^{-\kappa}\right) \\ & + n^{-\kappa} I\left(\int_{\Gamma_i} E_m(x,s)ds\vartheta_i > n^{-\kappa}\right), \end{aligned}$$

$$X_i^{(2)} \doteq \left(\int_{\Gamma_i} E_m(x,s)ds\vartheta_i - n^{-\kappa}\right) I\left(n^{-\kappa} < \int_{\Gamma_i} E_m(x,s)ds\vartheta_i < \frac{\varepsilon}{N}\right),$$

$$X_i^{(3)} \doteq \left(\int_{\Gamma_i} E_m(x,s)ds\vartheta_i + n^{-\kappa}\right) I\left(-n^{-\kappa} > \int_{\Gamma_i} E_m(x,s)ds\vartheta_i > -\frac{\varepsilon}{N}\right),$$

$$\begin{aligned} X_i^{(4)} \doteq & \left(\int_{\Gamma_i} E_m(x,s)ds\vartheta_i - n^{-\kappa}\right) I\left(\int_{\Gamma_i} E_m(x,s)ds\vartheta_i \geq \frac{\varepsilon}{N}\right) \\ & + \left(\int_{\Gamma_i} E_m(x,s)ds\vartheta_i + n^{-\kappa}\right) I\left(\int_{\Gamma_i} E_m(x,s)ds\vartheta_i \leq -\frac{\varepsilon}{N}\right), \end{aligned}$$

$$\sum_{i=1}^n \int_{\Gamma_i} E_m(x,s)ds\vartheta_i = \sum_{i=1}^n X_i^{(1)} + \sum_{i=1}^n X_i^{(2)} + \sum_{i=1}^n X_i^{(3)} + \sum_{i=1}^n X_i^{(4)}. \tag{4.4}$$

Then, in order to prove $\sum_{i=1}^n \int_{\Gamma_i} E_m(x,s)ds\vartheta_i \to 0$ a.s., it suffices to show that $\sum_{i=1}^n X_i^{(j)} \to 0$ a.s., $j = 1,2,3,4$.

Because of $\mathrm{E}\vartheta_i = 0$ and $\mathrm{E}|\vartheta_i|^t < \infty$ for some $t > \frac{3}{2}$, so it follows from Lemma 3.2 and (A4) that

$$\begin{aligned} \left|\mathrm{E}\left(\sum_{i=1}^n X_i^{(1)}\right)\right| \leq & \sum_{i=1}^n \left\{ n^{-\kappa} \mathrm{P}\left(\left|\int_{\Gamma_i} E_m(x,s)ds\vartheta_i\right| > n^{-\kappa}\right)\right. \\ & \left. + \mathrm{E}\left|\int_{\Gamma_i} E_m(x,s)ds\vartheta_i\right| I\left(\left|\int_{\Gamma_i} E_m(x,s)ds\vartheta_i\right| > n^{-\kappa}\right)\right\} \\ \leq & \sum_{i=1}^n n^{-\kappa} n^{t\kappa} \mathrm{E}\left|\int_{\Gamma_i} E_m(x,s)ds\vartheta_i\right|^t \\ \leq & C n^{-\kappa} n^{t\kappa} \sum_{i=1}^n \left|\int_{\Gamma_i} E_m(x,s)ds\right|^t \mathrm{E}|\vartheta_i|^t \\ \leq & C n^{\kappa(t-1)} \left(\max_{1\leq i\leq n}\left|\int_{\Gamma_i} E_m(x,s)ds\right|\right)^{t-1} \sum_{i=1}^n \left|\int_{\Gamma_i} E_m(x,s)ds\right| \\ \leq & C n^{-(t-1)(\frac{2}{3}-\kappa)} \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{4.5}$$

Firstly, to prove $\sum_{i=1}^{n} X_i^{(1)} \to 0$ a.s., we need only to show that

$$\sum_{n=1}^{\infty} \mathrm{P}\left(\left|\sum_{i=1}^{n} X_i^{(1)} - \mathrm{E}\left(\sum_{i=1}^{n} X_i^{(1)}\right)\right| > \varepsilon\right) < \infty. \tag{4.6}$$

It is obvious that for fixed $x \in c(g)$, $\left\{X_i^{(1)}, 1 \le i \le n\right\}$ is still an NOD sequence by the definition of $X_i^{(1)}$ and Lemma 3.1. So, by Lemma 3.4, it follows

$$\begin{aligned}
&\sum_{n=1}^{\infty} \mathrm{P}\left(\left|\sum_{i=1}^{n} X_i^{(1)} - \mathrm{E}\left(\sum_{i=1}^{n} X_i^{(1)}\right)\right| > \varepsilon\right) \\
&\le C\sum_{n=1}^{\infty}\left\{\sum_{i=1}^{n} \mathrm{E}\left|X_i^{(1)}\right|^r + \left(\sum_{i=1}^{n} \mathrm{E}\left|X_i^{(1)}\right|^2\right)^{r/2}\right\}.
\end{aligned} \tag{4.7}$$

Taking $r > \max\left\{t, \frac{\frac{5}{3}+t\kappa-\frac{2}{3}t}{\kappa}, \frac{2}{(2-t)\kappa+\frac{2}{3}(t-1)}, 3\right\}$, we have by Lemma 3.2 and (A4) that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathrm{E}\left|X_i^{(1)}\right|^r \\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\left\{n^{-r\kappa}\mathrm{P}\left(\left|\int_{\Gamma_i} E_m(x,s)ds\vartheta_i\right| > n^{-\kappa}\right)\right. \\
&\quad \left. + \mathrm{E}\left|\int_{\Gamma_i} E_m(x,s)ds\vartheta_i\right|^r I\left(\left|\int_{\Gamma_i} E_m(x,s)ds\vartheta_i\right| \le n^{-\kappa}\right)\right\} \\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} n^{-r\kappa}n^{t\kappa}\mathrm{E}\left|\int_{\Gamma_i} E_m(x,s)ds\vartheta_i\right|^t \\
&\le C\sum_{n=1}^{\infty} n^{-(r-t)\kappa}\left(\max_{1\le i\le n}\left|\int_{\Gamma_i} E_m(x,s)ds\right|\right)^{t-1}\sum_{i=1}^{n}\left|\int_{\Gamma_i} E_m(x,s)ds\right| \\
&\le C\sum_{n=1}^{\infty} n^{-((r-t)\kappa+\frac{2}{3}(t-1))} < \infty.
\end{aligned} \tag{4.8}$$

While $\frac{3}{2} < t \le 2$, thus

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} \mathrm{E}\left|X_i^{(1)}\right|^2\right)^{r/2} \\
&\le C\sum_{n=1}^{\infty}\left\{\sum_{i=1}^{n} n^{-2\kappa}\mathrm{P}\left(\left|\int_{\Gamma_i} E_m(x,s)ds\vartheta_i\right| > n^{-\kappa}\right)\right. \\
&\quad \left. + \mathrm{E}\left|\int_{A_{ni}} E_m(x,s)ds\vartheta_i\right|^2 I\left(\left|\int_{\Gamma_i} E_m(x,s)ds\vartheta_i\right| \le n^{-\kappa}\right)\right\}^{r/2}
\end{aligned}$$

$$\leq C\sum_{n=1}^{\infty}\left\{\sum_{i=1}^{n} n^{-\kappa(2-t)}\mathrm{E}\left|\int_{\Gamma_i} E_m(x,s)ds\vartheta_i\right|^t\right\}^{r/2}$$

$$\leq C\sum_{n=1}^{\infty} n^{-((2-t)\kappa+\frac{2}{3}(t-1))r/2} < \infty. \tag{4.9}$$

On the other hand, while $t > 2$, by $\sup_i \mathrm{E}\vartheta_i^2 < \infty$, thus

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\mathrm{E}\left|X_i^{(1)}\right|^2\right)^{r/2}\\
&\leq C\sum_{n=1}^{\infty}\left\{\sum_{i=1}^{n}\left(n^{-2\kappa}\mathrm{P}\left(\left|\int_{\Gamma_i} E_m(x,s)ds\vartheta_i\right| > n^{-\kappa}\right)\right.\right.\\
&\qquad\qquad\left.\left.+\left|\int_{\Gamma_i} E_m(x,s)ds\right|^2\right)\right\}^{r/2}\\
&\leq C\sum_{n=1}^{\infty} n^{-((2-t)\kappa+\frac{2}{3}(t-1))r/2} + C\sum_{n=1}^{\infty} n^{-r/3}\\
&< \infty. \qquad (4.10)
\end{aligned}$$

From (4.8)-(4.10) and combining (4.7), we can obtain (4.6), which together with (4.5) yields that

$$\sum_{i=1}^{n} X_i^{(1)} \to 0 \quad \text{a.s.}. \tag{4.11}$$

Next, note that $0 \leq X_i^{(2)} < \frac{\varepsilon}{N}$, $\left|\sum_{i=1}^{n} X_i^{(2)}\right| = \sum_{i=1}^{n} X_i^{(2)} > \varepsilon$ means that there are at least N i' s such that $X_i^{(2)} \neq 0$, then we can derive

$$\begin{aligned}
&\mathrm{P}\left(\left|\sum_{i=1}^{n} X_i^{(2)}\right| > \varepsilon\right)\\
&\leq \mathrm{P}\left(\text{there are at least } N\ i'\text{ s such that} X_i^{(2)} \neq 0\right)\\
&\leq \sum_{1\leq i_1<\cdots<i_N\leq n}\mathrm{P}\left(X_{i_1}^{(2)} \neq 0, \ldots, X_{i_N}^{(2)} \neq 0\right)\\
&\leq \sum_{1\leq i_1<\cdots<i_N\leq n}\mathrm{P}\left(\int_{\Gamma_{i_1}} E_m(x,s)ds\vartheta_{i_1} > n^{-\kappa}, \ldots, \int_{\Gamma_{i_N}} E_m(x,s)ds\vartheta_{i_N} > n^{-\kappa}\right)\\
&\leq M\sum_{1\leq i_1<\cdots<i_N\leq n}\mathrm{P}\left(\int_{\Gamma_{i_1}} E_m(x,s)ds\vartheta_{i_1} > n^{-\kappa}\right)\cdots\mathrm{P}\left(\int_{\Gamma_{i_N}} E_m(x,s)ds\vartheta_{i_N} > n^{-\kappa}\right)
\end{aligned}$$

$$\leq M\left(\sum_{i=1}^{n}\mathrm{P}\left(\int_{\Gamma_i}E_m(x,s)ds\vartheta_i>n^{-\kappa}\right)\right)^N$$
$$\leq M\left(\sum_{i=1}^{n}\mathrm{P}\left(\left|\int_{\Gamma_i}E_m(x,s)ds\vartheta_i\right|>n^{-\kappa}\right)\right)^N$$
$$\leq C\left(\sum_{i=1}^{n}n^{t\kappa}\mathrm{E}\left|\int_{\Gamma_i}E_m(x,s)ds\vartheta_i\right|^t\right)^N$$
$$\leq Cn^{-((\frac{2}{3}-\kappa)t-\frac{2}{3})N},$$

obviously, we have by letting $N\geq 1$ that $\left((\frac{2}{3}-\kappa)t-\frac{2}{3}\right)N>1$. Thus, we obtain

$$\sum_{n=1}^{\infty}\mathrm{P}\left(\left|\sum_{i=1}^{n}X_i^{(2)}\right|>\varepsilon\right)<\infty,$$

which gives

$$\sum_{i=1}^{n}X_i^{(2)}\to 0\ \text{ a.s..}\tag{4.12}$$

Note that $-\frac{\varepsilon}{N}<X_i^{(3)}\leq 0$, $\left|\sum_{i=1}^{n}X_i^{(3)}\right|=-\sum_{i=1}^{n}X_i^{(3)}>\varepsilon$ means that there are at least N i' s such that $X_i^{(3)}\neq 0$. And similarly to the proof of $\sum_{i=1}^{n}X_i^{(2)}\to 0$ a.s., one may prove that

$$\sum_{i=1}^{n}X_i^{(3)}\to 0\ \text{ a.s..}\tag{4.13}$$

Finally, notice that

$$\begin{aligned}\sum_{i=1}^{n}X_i^{(4)}\leq&\sum_{i=1}^{n}\left|\int_{\Gamma_i}E_m(x,s)ds\vartheta_i\right|I\left(\left|\int_{\Gamma_i}E_m(x,s)ds\vartheta_i\right|\geq\frac{\varepsilon}{N}\right)\\&+n^{-\kappa}\sum_{i=1}^{n}I\left(\left|\int_{\Gamma_i}E_m(x,s)ds\vartheta_i\right|\geq\frac{\varepsilon}{N}\right)\\&\leq n^{-\frac{2}{3}}\sum_{i=1}^{n}|\vartheta_i|I\left(|\vartheta_i|\geq Ci^{\frac{2}{3}}\right)+n^{-\kappa}\sum_{i=1}^{n}I\left(|\vartheta_i|\geq Ci^{\frac{2}{3}}\right).\end{aligned}\tag{4.14}$$

Writing $T_n\doteq\sum_{j=1}^{n}j^{-\frac{2}{3}}|\vartheta_j|I\left(|\vartheta_j|\geq Cj^{\frac{2}{3}}\right)$, the $\{T_n\}$ almost sure convergence is proved by subsequence method, and using natural sequence with order t sum inequality, then for $m\geq n\geq 1$ we have

$$\mathrm{E}|T_m-T_n|=\sum_{j=n+1}^{m}j^{-\frac{2}{3}}\mathrm{E}|\vartheta_j|I\left(|\vartheta_j|\geq Cj^{\frac{2}{3}}\right)\ll\sum_{j=n+1}^{m}j^{\frac{2}{3}(1-t)-\frac{2}{3}}$$

$$\ll \sum_{j=n+1}^{\infty} j^{-\frac{2}{3}t} \ll n^{-\frac{2}{3}t+1} \to 0 \quad \text{as} \quad n \to \infty.$$

Here, $\{T_n\}$ is a Cauchy sequence in L_1, thus there exists random variable T, satisfying $\mathrm{E}|T| < \infty$ and $\mathrm{E}|T_n - T| \to 0$. Then it follows that for $\forall\ \epsilon > 0$,

$$\begin{aligned} \mathrm{P}(|T_{2^k} - T| > \epsilon) &\ll \mathrm{E}|T_{2^k} - T_n| + \mathrm{E}|T_n - T| \\ &\ll \limsup_n \mathrm{E}|T_{2^k} - T_n| \\ &\ll \sum_{j=2^k+1}^{\infty} j^{-\frac{2}{3}t} \ll 2^{-k(\frac{2}{3}t-1)}. \end{aligned}$$

It is easily seen that

$$\sum_{k=1}^{\infty} \mathrm{P}(|T_{2^k} - T| > \epsilon) < \infty.$$

This implies that

$$T_{2^k} \to T.$$

In addition,

$$\begin{aligned} \mathrm{P}\left(\max_{2^{k-1}<n\le 2^k} |T_n - T_{2^{k-1}}| > \epsilon\right) &\ll \mathrm{P}(|T_{2^k} - T_{2^{k-1}}| > \epsilon) \\ &\ll \sum_{j=2^{k-1}+1}^{2^k} j^{-\frac{2}{3}t} \ll 2^{-k(\frac{2}{3}t-1)}, \end{aligned}$$

and similarly,

$$\sum_{k=1}^{\infty} \mathrm{P}\left(\max_{2^{k-1}<n\le 2^k} |T_n - T_{2^{k-1}}| > \epsilon\right) < \infty,$$

so we have that for $k \to \infty$,

$$\max_{2^{k-1}<n\le 2^k} |T_n - T_{2^{k-1}}| \to 0.$$

Consequently, T_n and T_{2^k} have the same limit in the sense of almost sure convergence, and thus $T_n \to T$ a.s., which shows that

$$\sum_{n=1}^{\infty} n^{-\frac{2}{3}} |\vartheta_n| I\left(|\vartheta_n| \ge C n^{\frac{2}{3}}\right) < \infty \quad \text{a.s..}$$

Therefore, applying Kronecker's lemma, one gets

$$n^{-\frac{2}{3}} \sum_{i=1}^{n} |\vartheta_i| I\left(|\vartheta_i| \ge C i^{\frac{2}{3}}\right) \to 0 \quad \text{a.s..} \tag{4.15}$$

Similarly, we can obtain

$$\sum_{n=1}^{\infty} n^{-\kappa} I\left(|\vartheta_n| \ge C n^{\frac{2}{3}}\right) < \infty \quad \text{a.s..}$$

Thus, applying Kronecker's lemma again, one has

$$n^{-\kappa}\sum_{i=1}^{n} I\left(|\vartheta_i|\geq Ci^{\frac{2}{3}}\right)\to 0 \quad \text{a.s..} \tag{4.16}$$

From (4.15)-(4.16), and combining (4.14), it follows

$$\sum_{i=1}^{n} X_i^{(4)}\to 0 \quad \text{a.s..} \tag{4.17}$$

Hence (4.3) follows by (4.11)-(4.13), (4.17) and (4.4) immediately. This completes the proof of the theorem. □

Proof of Theorem 2.2. The proof is similar to that of Theorem 2.1, it suffices to show that

$$\sum_{i=1}^{n}\int_{\Gamma_i} E_m(x,s)ds\vartheta_i\to 0 \quad \text{completely, as} \quad n\to\infty. \tag{4.18}$$

In fact, to prove (4.18) holds for any $x\in c(g)$, we need only to prove that for all $\varepsilon>0$,

$$\sum_{n=1}^{\infty}\mathrm{P}\left(\left|\sum_{i=1}^{n}\int_{\Gamma_i} E_m(x,s)ds\vartheta_i\right|>\varepsilon\right)<\infty. \tag{4.19}$$

Without loss of generality, we still assume that $\int_{\Gamma_i} E_m(x,s)ds>0$. For any $\varepsilon>0$, we choose positive integer N (to be specified later) and small positive constant ϖ such that $\varpi<\frac{2}{9}$. Denote for $1\leq i\leq n$ that

$$\begin{aligned}
X_i^{(1)} \doteq\ & \vartheta_i I\left(\left|\int_{\Gamma_i} E_m(x,s)ds\vartheta_i\right|\leq n^{-\varpi}\right)\\
&-\frac{1}{\int_{\Gamma_i} E_m(x,s)ds}n^{-\varpi} I\left(\int_{\Gamma_i} E_m(x,s)ds\vartheta_i<-n^{-\varpi}\right)\\
&+\frac{1}{\int_{\Gamma_i} E_m(x,s)ds}n^{-\varpi} I\left(\int_{\Gamma_i} E_m(x,s)ds\vartheta_i>-n^{-\varpi}\right),
\end{aligned}$$

$$X_i^{(2)} \doteq \left(\vartheta_i-\frac{1}{\int_{\Gamma_i} E_m(x,s)ds}n^{-\varpi}\right) I\left(n^{-\varpi}<\int_{\Gamma_i} E_m(x,s)ds\vartheta_i\leq\frac{\varepsilon}{N}\right),$$

$$X_i^{(3)} \doteq \left(\vartheta_i+\frac{1}{\int_{\Gamma_i} E_m(x,s)ds}n^{-\varpi}\right) I\left(-\frac{\varepsilon}{N}\leq\int_{\Gamma_i} E_m(x,s)ds\vartheta_i<-n^{-\varpi}\right),$$

$$\begin{aligned}
X_i^{(4)} \doteq\ & \left(\vartheta_i+\frac{1}{\int_{\Gamma_i} E_m(x,s)ds}n^{-\varpi}\right) I\left(\int_{\Gamma_i} E_m(x,s)ds\vartheta_i<-\frac{\varepsilon}{N}\right)\\
&+\left(\vartheta_i-\frac{1}{\int_{\Gamma_i} E_m(x,s)ds}n^{-\varpi}\right) I\left(\int_{\Gamma_i} E_m(x,s)ds\vartheta_i>\frac{\varepsilon}{N}\right).
\end{aligned}$$

It is easy to check that $X_i^{(1)}+X_i^{(2)}+X_i^{(3)}+X_i^{(4)}=\vartheta_i$, this yields that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\mathrm{P}\left(\left|\sum_{i=1}^{n}\int_{\Gamma_i}E_m(x,s)ds\vartheta_i\right|>4\varepsilon\right)\\
\le&\sum_{n=1}^{\infty}\mathrm{P}\left(\left|\sum_{i=1}^{n}\int_{\Gamma_i}E_m(x,s)dsX_i^{(1)}\right|>\varepsilon\right)\\
&+\sum_{n=1}^{\infty}\mathrm{P}\left(\left|\sum_{i=1}^{n}\int_{\Gamma_i}E_m(x,s)dsX_i^{(2)}\right|>\varepsilon\right)\\
&+\sum_{n=1}^{\infty}\mathrm{P}\left(\left|\sum_{i=1}^{n}\int_{\Gamma_i}E_m(x,s)dsX_i^{(3)}\right|>\varepsilon\right)\\
&+\sum_{n=1}^{\infty}\mathrm{P}\left(\left|\sum_{i=1}^{n}\int_{\Gamma_i}E_m(x,s)dsX_i^{(4)}\right|>\varepsilon\right)\\
\doteq&\, T_1+T_2+T_3+T_4.
\end{aligned}$$

Hence, in order to prove (4.19), it suffices to show that $T_1<\infty$, $T_2<\infty$, $T_3<\infty$ and $T_4<\infty$.

Firstly, since $\max_{1\le i\le n}\left|\int_{\Gamma_i}E_m(x,s)ds\left(X_i^{(1)}-\mathrm{E}X_i^{(1)}\right)\right|\le 2n^{-\varpi}$ for every $n\ge1$, so that

$$\mathrm{P}\left(\max_{1\le i\le n}\left|\int_{\Gamma_i}E_m(x,s)ds\left(X_i^{(1)}-\mathrm{E}X_i^{(1)}\right)\right|\le 2n^{-\varpi}\right)=1 \text{ for every } n\ge1. \tag{4.20}$$

Base on $\left|X_i^{(1)}\right|\le|\vartheta_i|$, so it follows from Lemmas 3.2, 3.6 and (A4) that

$$\begin{aligned}
B_n^2\doteq&\sum_{i=1}^{n}\mathrm{E}\left(\int_{\Gamma_i}E_m(x,s)ds\left(X_i^{(1)}-\mathrm{E}X_i^{(1)}\right)\right)^2\\
\le&\, C\sum_{i=1}^{n}\left(\int_{\Gamma_i}E_m(x,s)ds\right)^2\mathrm{E}X^2=o\left((\log n)^{-1}\right).
\end{aligned}\tag{4.21}$$

In particular, $\left\{\int_{\Gamma_i}E_m(x,s)ds\left(X_i^{(1)}-\mathrm{E}X_i^{(1)}\right),1\le i\le n\right\}$ are still NOD by Lemma 3.1. Using Lemma 3.5 with $x=\varepsilon$ and $y=2n^{-\varpi}$, from (4.20)–(4.21), we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}\mathrm{P}\left(\left|\sum_{i=1}^{n}\int_{\Gamma_i}E_m(x,s)ds\left(X_i^{(1)}-\mathrm{E}X_i^{(1)}\right)\right|>\varepsilon\right)\\
\le&\, 2\sum_{n=1}^{\infty}\mathrm{P}\left(\max_{1\le i\le n}\left|\int_{\Gamma_i}E_m(x,s)ds\left(X_i^{(1)}-\mathrm{E}X_i^{(1)}\right)\right|>2n^{-\varpi}\right)\\
&+C\sum_{n=1}^{\infty}\exp\left\{-\frac{\varepsilon^2}{2\left(2\varepsilon n^{-\varpi}+o\left((\log n)^{-1}\right)\right)}\right\}
\end{aligned}$$

$$\leq C\sum_{n=1}^{\infty}\exp\{-2\log n\}<\infty.$$

Thus, to prove $T_1<\infty$, we need only to prove

$$\left|\sum_{i=1}^{n}\int_{\Gamma_i}E_m(x,s)ds\mathrm{E}X_i^{(1)}\right|\to 0 \quad \text{as} \quad n\to\infty. \tag{4.22}$$

Note that $\mathrm{E}\vartheta_i=0$ and $\varpi<\frac{2}{9}$, it follows by Markov's inequality, Lemmas 3.2, 3.6 and (A4) that

$$\begin{aligned}
&\left|\sum_{i=1}^{n}\int_{\Gamma_i}E_m(x,s)ds\mathrm{E}X_i^{(1)}\right|\\
&\leq n^{2\varpi}\sum_{i=1}^{n}\mathrm{E}\left|\int_{\Gamma_i}E_m(x,s)ds\vartheta_i\right|^3+n^{2\varpi}\sum_{i=1}^{n}\mathrm{E}\left|\int_{\Gamma_i}E_m(x,s)ds\vartheta_i\right|^3\\
&\leq Cn^{2\varpi}\left(\max_{1\leq i\leq n}\left|\int_{\Gamma_i}E_m(x,s)ds\right|\right)\sum_{i=1}^{n}\left(\int_{\Gamma_i}E_m(x,s)ds\right)^2\\
&\leq Cn^{2\varpi-4/3}\to 0 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Hence, (4.22) holds, this leads to $T_1<\infty$.

Next, we will show that $T_2<\infty$. Note that $0<\int_{\Gamma_i}E_m(x,s)dsX_i^{(2)}\leq\frac{\varepsilon}{N}$, we can find

$$\left|\sum_{i=1}^{n}\int_{\Gamma_i}E_m(x,s)dsX_i^{(2)}\right|=\sum_{i=1}^{n}\int_{\Gamma_i}E_m(x,s)dsX_i^{(2)}>\varepsilon$$

means that there are at least N integers such that $\int_{\Gamma_i}E_m(x,s)dsX_i^{(2)}\neq 0$. Thus, we have by Lemmas 3.2, 3.6 and (A4) that

$$\begin{aligned}
&\mathrm{P}\left(\left|\sum_{i=1}^{n}\int_{\Gamma_i}E_m(x,s)dsX_i^{(2)}\right|>\varepsilon\right)\\
&\leq\sum_{1\leq i_1<i_2<\cdots<i_N\leq n}\mathrm{P}\left(\int_{\Gamma_{i_1}}E_m(x,s)dsX_{i_1}^{(2)}\neq 0,\right.\\
&\qquad\left.\int_{\Gamma_{i_2}}E_m(x,s)dsX_{i_2}^{(2)}\neq 0,\ldots,\int_{\Gamma_{i_N}}E_m(x,s)dsX_{i_N}^{(2)}\neq 0\right)\\
&\leq\sum_{1\leq i_1<i_2<\cdots<i_N\leq n}\mathrm{P}\left(\int_{\Gamma_{i_1}}E_m(x,s)ds\vartheta_{i_1}>n^{-\varpi}\right)\\
&\qquad\times\mathrm{P}\left(\int_{\Gamma_{i_2}}E_m(x,s)ds\vartheta_{i_2}>n^{-\varpi}\right)\cdots\mathrm{P}\left(\int_{\Gamma_{i_N}}E_m(x,s)ds\vartheta_{i_N}>n^{-\varpi}\right)
\end{aligned}$$

$$
\begin{aligned}
&\le \left(\sum_{i=1}^{n} \mathrm{P}\left(\int_{\Gamma_i} E_m(x,s)ds\vartheta_i > n^{-\varpi}\right)\right)^N \\
&\le \left(C\sum_{i=1}^{n} \mathrm{P}\left(\left|\int_{\Gamma_i} E_m(x,s)dsX\right| > n^{-\varpi}\right)\right)^N \\
&\le C\left(n^{3\varpi}\sum_{i=1}^{n} \mathrm{E}\left|\int_{\Gamma_i} E_m(x,s)dsX\right|^3\right)^N \\
&\le C\left(n^{3\varpi}\left(\max_{1\le i\le n}\left|\int_{\Gamma_i} E_m(x,s)ds\right|\right)\sum_{i=1}^{n}\left(\int_{\Gamma_i} E_m(x,s)ds\right)^2\right)^N \\
&\le Cn^{-(4/3-3\varpi)N}.
\end{aligned}
$$

Since $\varpi < \frac{2}{9}$, choosing some large integer N such that $(\frac{4}{3} - 3\varpi)N > 1$. Therefore, $T_2 < \infty$.

Noting that $-\frac{\varepsilon}{N} \le \int_{\Gamma_i} E_m(x,s)dsX_i^{(3)} < 0$, we can find

$$
\left|\sum_{i=1}^{n}\int_{\Gamma_i} E_m(x,s)dsX_i^{(3)}\right| = -\sum_{i=1}^{n}\int_{\Gamma_i} E_m(x,s)dsX_i^{(3)} > \varepsilon
$$

means that there are at least N integers such that $\int_{\Gamma_i} E_m(x,s)dsX_i^{(3)} \neq 0$. Therefore, similar to the proof of $T_2 < \infty$, we can get $T_3 < \infty$.

Finally, we will show that $T_4 < \infty$. we have by Lemma 3.6 and $\mathrm{E}|X|^3 < \infty$ that

$$
\begin{aligned}
T_4 = &\sum_{n=1}^{\infty} \mathrm{P}\left(\left|\sum_{i=1}^{n}\int_{\Gamma_i} E_m(x,s)dsX_i^{(4)}\right| > \varepsilon\right) \\
\le &\sum_{n=1}^{\infty} \mathrm{P}\left(\bigcup_{i=1}^{n}\left(\left|\int_{\Gamma_i} E_m(x,s)ds\vartheta_i\right| > \frac{\varepsilon}{N}\right)\right) \\
\le &\, C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathrm{P}\left(\left|\int_{\Gamma_i} E_m(x,s)ds\vartheta_i\right| > \frac{\varepsilon}{N}\right) \\
\le &\, C\sum_{n=1}^{\infty} n\mathrm{P}\left(|X| > Cn^{2/3}\right) \\
= &\, C\sum_{n=1}^{\infty} n\sum_{k=n}^{\infty} \mathrm{P}\left(k^{2/3} < |X| \le (k+1)^{2/3}\right) \\
= &\, C\sum_{k=1}^{\infty} \mathrm{P}\left(k^{2/3} < |X| \le (k+1)^{2/3}\right)\sum_{n=1}^{k} n \\
\le &\, C\sum_{k=1}^{\infty} k^2\mathrm{P}\left(k^{2/3} < |X| \le (k+1)^{2/3}\right)
\end{aligned}
$$

$$\leq C\sum_{k=1}^{\infty}\mathrm{E}|X|^3 I\left(k^{2/3} < |X| \leq (k+1)^{2/3}\right)$$
$$\leq C\mathrm{E}|X|^3 < \infty.$$

This completes the proof of the theorem. □

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