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SOME RESULTS ON n-JORDAN HOMOMORPHISMS

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Abstract. With the motivation to extend the Zelasko's theorem on commutative algebras, it was shown in [2] that if $n \in \{3, 4\}$ is fixed, A, B are commutative algebras and $h : A \to B$ is an n-Jordan homomorphism, then h is an n-ring homomorphism. In this paper, we extend this result for all $n \geq 3$.

1. Introduction

Let $n \in \mathbb{N}$ and let A and B be rings (algebras). An additive mapping $h : A \to B$ is called an *n-Jordan homomorphism* if for all $a \in A$,

$$h(a^n) = h(a)^n.$$

Also, an additive mapping $h : A \to B$ is called an *n-ring homomorphism* if h is an n-multiplicative, that is, for all $a_1, a_2, \ldots, a_n \in A$,

$$h(a_1 a_2 \cdots a_n) = h(a_1)h(a_2) \cdots h(a_n).$$

If $h : A \to B$ is a linear n-ring homomorphism, then we say that h is an n-homomorphism. A 2-Jordan homomorphism is then just a Jordan homomorphism, in the usual sense, between algebras. Thus we may assume in the sequel that $n \geq 3$. Obviously, each homomorphism is an n-homomorphism for all $n \geq 2$, but the converse is not true, in general. For example, if φ is a homomorphism, then $h = -\varphi$ is a 3-homomorphism, which is not a homomorphism (see [1]). The concept of n-Jordan homomorphism was studied by Zelasko in [6] (see also [4]). In 2009, Eshaghi Gordji [2, Theorems 2.2 and 2.5] studied n-Jordan homomorphisms on Banach algebras for $n \in \{3, 4\}$, and presented a method to the proof of Zelasko's Theorem for $n = 3$. Eshaghi Gordji *et al.* [3] extended this problem for $n = 5$. In what follows, we provide an overall and simple approach to show that if A and B are commutative algebras and

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$h : A \to B$ is an n-Jordan homomorphism, then h is an n-ring homomorphism, for all $n \geq 3$ (Theorem 2.3). By proving this theorem, some of the important theorems such as theorem due to Park and Trout, which asserts that if A and B are two commutative algebras and $h : A \to B$ is a linear involution preserving n-Jordan homomorphism between commutative $C^\star$-algebras, then h is norm contractive, that is, $\|h\| \leq 1$ (Corollary 2.6), can be extended as a result.

2. n-Jordan homomorphisms

Obviously, each n-ring homomorphism is an n-Jordan homomorphism, the converse is not true in general, but under a certain condition, n-Jordan homomorphisms are n-ring homomorphisms. For the sake of completeness we first state the following results, which were appeared in [6] and [2, Theorem 2.2].

Theorem 2.1. *Suppose that A is a Banach algebra, which need not be commutative, and suppose that B is a semisimple commutative Banach algebra. Then each Jordan homomorphism $h : A \to B$ is a ring homomorphism.*

Theorem 2.2. *Let $n \in \{3, 4\}$ be fixed, A, B be commutative algebras and let $h : A \to B$ be an n-Jordan homomorphism. Then h is an n-ring homomorphism.*

Now we prove our main theorem, which is a generalization of Theorem 2.2.

Theorem 2.3. *Let A and B be commutative algebras, $n \geq 3$ an integer and let $h : A \to B$ be an n-Jordan homomorphism. Then h is an n-ring homomorphism.*

Proof. For $n \in \{3, 4\}$, the theorem was proved in [2, Theorem 2.2]. But we give another simple proof to find a method for the proof of the theorem for any $n \geq 3$. Let $x, y, z \in A$ be arbitrary. Recall that h is an additive mapping such that $h(a^3) = h(a)^3$ for all $a \in A$.

Define the mapping $\psi : A^3 \to B$ as follows:

$$\psi(x, y, z) = h(xyz) - h(x)h(y)h(z)$$

for all $x, y, z \in A$. Then we will show that $\psi(x, y, z) = 0$. Consider the mapping $\varphi_1 : A^2 \to B$ defined by

$$\varphi_1(x, y) = h((x + y)^3) - h(x + y)^3$$

for all $x, y \in A$. Then for all $x, y \in A$, $\varphi_1(x, y) = 0$. By direct calculation, we get

$$\varphi_1(x, y) = h(x^2y + xy^2) - h(x)^2h(y) - h(x)h(y)^2$$

for all $x, y \in A$. Now, define the mapping $\varphi_2 : A^3 \to B$ by

$$\varphi_2(x, y, z) = h((x + y + z)^3) - h(x + y + z)^3$$

for all $x, y, z \in A$. Then for all $x, y, z \in A$, $\varphi_2(x, y, z) = 0$. Also, by direct calculation, we get

$$\varphi_2(x, y, z) = \varphi_1(x, y) + \varphi_1(x, z) + \varphi_1(y, z) + \psi(x, y, z) \tag{1}$$

for all $x, y, z \in A$. But, since $\varphi_1(x, y) = 0, \varphi_1(x, z) = 0, \varphi_1(y, z) = 0$ for all $x, y, z \in A$,

$$\varphi_2(x, y, z) = 0 \tag{2}$$

for all $x, y, z \in A$. By (1) and (2), we obtain

$$\psi(x, y, z) = 0,$$

that is, $h(xyz) = h(x)h(y)h(z)$ for all $x, y, z \in A$. Hence h is a 3-ring homomorphism.

The proof for $n = 4$ is similar to $n = 3$.

Now, fix $n \in \mathbb{N}$. Recall that h is additive and $h(a^n) = h(a)^n$ for all $a \in A$. Let $x_1, x_2, \ldots, x_n \in A$ be arbitrary. Define the mapping ψ by

$$\psi(x_1, x_2, \ldots, x_n) = h(x_1 x_2 \cdots x_n) - h(x_1)h(x_2) \cdots h(x_n)$$

for all $x_1, x_2, \ldots, x_n \in A$. Then we will show that $\psi(x_1, x_2, \ldots, x_n) = 0$. Consider the mapping $\varphi_1 : A^2 \to B$ defined by

$$\varphi_1(x_1, x_2) = h((x_1 + x_2)^n) - h(x_1 + x_2)^n$$

for all $x_1, x_2 \in A$. Then for all $x_1, x_2 \in A$, $\varphi_1(x_1, x_2) = 0$. Also, by direct calculation, we get

$$\begin{aligned}\varphi_1(x_1, x_2) &= h(nx_1^{n-1}x_2 + \cdots + nx_1x_2^{n-1}) \\ &\quad - (nh(x_1)^{n-1}h(x_2) + \cdots + h(x_1)h(x_2)^{n-1}).\end{aligned}$$

Now, define the mapping $\varphi_2 : A^3 \to B$ by

$$\varphi_2(x_1, x_2, x_3) = h((x_1 + x_2 + x_3)^n) - h(x_1 + x_2 + x_3)^n$$

for all $x_1, x_2, x_3 \in A$. By direct calculation, we get

$$\varphi_2(x_1, x_2, x_3) = \varphi_1(x_1, x_2) + \varphi_1(x_1, x_3) + \varphi_1(x_2, x_3) + \cdots .$$

Indeed, with the repetition of this method, we have

$$\begin{aligned}\varphi_{n-1}(x_1, x_2, \ldots, x_n) = &\sum_{i,j=1, i<j}^{n} \varphi_1(x_i, x_j) \\ &+ \sum_{i,j,k=1, i<j<k}^{n} \varphi_2(x_i, x_j, x_k) + \cdots + n!\psi(x_1, x_2, \ldots, x_n),\end{aligned}$$

and since

$$\begin{aligned}\varphi_1(x_i, x_j) &= 0, \;\; i < j, \\ \varphi_2(x_i, x_j, x_k) &= 0, \;\; i < j < k, \\ \varphi_3(x_i, x_j, x_k, x_l) &= 0, \;\; i < j < k < l, \\ &\vdots\end{aligned}$$

we have $\psi(x_1, x_2, \ldots, x_n) = 0$ and then

$$h(x_1 x_2 \cdots x_n) = h(x_1)h(x_2)\cdots h(x_n),$$

that is, h is an n-ring homomorphism, as desired. □

By Theorem 2.3 and [5, Theorem 3.2], we deduce the following result, which is more general than [2, Corollary 2.3].

Corollary 2.4. *Let $h : A \to B$ be a linear involution preserving n-Jordan homomorphism between commutative $C^\star$-algebras. If $n \geq 3$ is odd, then h is norm contractive (that is, $\|h\| \leq 1$).*

Also, by Theorem 2.3 and [5, Theorem 2.3], we have the following corollary.

Corollary 2.5. *Let $h : A \to B$ be a linear involution preserving n-Jordan homomorphism between commutative $C^\star$-algebras. If $n \geq 4$ is even, then h is completely positive and h is bounded.*

By Theorem 2.3, Corollary 2.5 and [5, Theorem 2.5], we have the following result, which is more general than Corollary 2.4.

Corollary 2.6. *Let $h : A \to B$ be a linear involution preserving n-Jordan homomorphism between commutative $C^\star$-algebras. Then h is norm contractive (that is, $\|h\| \leq 1$).*

The following corollaries are generalizations of [3, Theorems 2.1 and 2.2].

Corollary 2.7. *Let $n \in \mathbb{N}$ be fixed. Suppose A, B are commutative Banach algebras. Let δ and ε be nonnegative real numbers and let p, q be a real numbers such that $(p-1)(q-1) > 0$, $q \geq 0$ or $(p-1)(q-1) > 0$, $q < 0$ and $f(0) = 0$. Assume that $f : A \to B$ satisfies the system of functional inequalities*

$$\|f(a+b) - f(a) - f(b)\| \leq \varepsilon(\|a\|^p + \|b\|^p), \tag{3}$$

$$\|f(a^n) - f(a)^n\| \leq \delta\|a\|^{nq} \tag{4}$$

for all $a, b \in A$. Then there exists a unique n-ring homomorphism $h : A \to B$ such that

$$\|f(a) - h(a)\| \leq \frac{2\varepsilon}{|2 - 2^p|}\|a\|^p$$

for all $a \in A$.

Proof. It follows from Theorem 2.3 and [3, Theorems 2.1 and 2.2]. □

Corollary 2.8. *Let $n \in \mathbb{N}$ be fixed. Suppose A, B are commutative C^*-algebras. Let δ and ε be nonnegative real numbers and let p, q be a real numbers such that $(p-1)(q-1) > 0$, $q \geq 0$ or $(p-1)(q-1) > 0$, $q < 0$ and $f(0) = 0$ such that the inequalities (3) and (4) are valid and $f(a^*) = f(a)^*$. Then there exists a unique norm contractive involutive n-ring homomorphism $h : A \to B$ such that*

$$\|f(a) - h(a)\| \leq \frac{2\varepsilon}{|2 - 2^p|}\|a\|^p$$

for all $a \in A$.

Proof. It follows from Theorem 2.3 and [3, Theorem 2.1]. □

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