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DEHN SURGERIES ON MIDDLE/HYPER DOUBLY SEIFERT TWISTED TORUS KNOTS

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Abstract. In this paper, we classify all twisted torus knots which are middle/hyper doubly Seifert. By the definition of middle/hyper doubly Seifert knots, these knots admit Dehn surgery yielding either Seifert-fibered spaces or graph manifolds at a surface slope. We show that middle/hyper doubly Seifert twisted torus knots admit the latter, that is, non-Seifert-fibered graph manifolds whose decomposing pieces consist of two Seifert-fibered spaces over the disk with two exceptional fibers.

1. Introduction

Let H be a genus two handlebody and k be a simple closed curve in the boundary of H. A 2-handle addition to H along k, denoted by $H[k]$, is defined to be a manifold obtained by adding a 2-handle to H along k. If $H[k]$ is a solid torus, then k is said to be *primitive* in H. If $H[k]$ is a Seifert-fibered space and not a solid torus, then k is said to be *Seifert* in H.

In [4], Dean introduced twisted torus knots K in S^3 which lie in a genus two Heegaard surface standardly embedded in S^3. The twisted torus knots are parameterized by p, q, r, m, and n, and are denoted by $K = K(p, q, r, m, n)$, where p, q and m, n come from a (p, q)-torus knot and an (m, n)-torus knot respectively with $1 \leq q < p$, and r means the number of parallel arcs of a (p, q)-torus knot with $0 < r \leq p + q$. The definition of twisted torus knots is given in Section 2.1.

Dean gave some criteria on the parameters p, q, r, m, and n of a twisted torus knot which make it primitive or Seifert. Using these criteria on twisted torus knots, he constructed primitive/Seifert knots which admit Dehn surgery yielding Seifert-fibered spaces over the sphere with three exceptional fibers. In other words, since a twisted torus knot lies in a genus two Heegaard surface

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of S^3, it is primitive in one genus two handlebody and Seifert in the other handlebody.

Especially Dean made three criteria on the parameters of a twisted torus knot such that it becomes Seifert in a genus two handlebody and a 2-handle addition is a Seifert-fibered space over the disk with two exceptional fibers. The three criteria give rise to the three types of Seifert curves; *hyper* Seifert-fibered, *middle* Seifert-fibered, *end* Seifert-fibered. As in the construction of primitive/Seifert knots, we can build Seifert/Seifert or doubly Seifert knots. Since there are three types of Seifert curves, we have the following 9 types of doubly Seifert twisted torus knots: hyper/hyper, hyper/middle, hyper/end, middle/hyper, middle/middle, middle/end, end/hyper, end/middle, end/end. For example, a twisted torus knot is called a *hyper/middle* doubly Seifert twisted torus knot if it is hyper Seifert in one handlebody and middle Seifert in the other handlebody. Note that by the definition of a twisted torus knot $K(p, q, r, m, n)$ and the condition $1 \leq q < p$, the A/B and B/A doubly Seifert twisted torus knots are distinct, where A, B are hyper, middle, or end and A$\neq$B.

In general, if a knot in a genus two Heegaard surface of S^3 is doubly Seifert, then by Lemma 2.1 in [4], Dehn surgery at a surface slope is either a Seifert-fibered space or a graph manifold. Here, a surface slope is defined to be the isotopy class of a component of the intersection between the boundary of a regular neighborhood of the knot and the Heegaard surface.

Throughout this paper, $S(a_1, \ldots, a_n)$ denotes a Seifert-fibered space over a surface S with n exceptional fibers of indexes $a_1, \ldots, a_n$.

In [6], [5], and [7], the author classified hyper/hyper, hyper/middle, and middle/middle doubly Seifert twisted torus knots respectively. By [6] there are 6 types of hyper/hyper doubly Seifert twisted torus knots. All of these knots admit $S^2(a, b, c, d)$ Dehn surgeries at a surface slope and turn out to be satellite knots. This result supports the positive answer of the conjecture that if a knot in S^3 admits $S^2(a, b, c, d)$ Dehn surgery, then the knot is not hyperbolic. By [5] and [7], there are 4 types of hyper/middle and 6 types of middle/middle doubly Seifert twisted torus knots respectively, and all of these knots admit Dehn surgeries yielding non-Seifert-fibered graph manifolds at a surface slope whose decomposing pieces are $D^2(a, b)$ and $D^2(c, d)$.

In this paper, we classify all middle/hyper twisted torus knots and show that, just as hyper/middle and middle/middle doubly Seifert twisted torus knots Dehn surgeries at a surface slope on these knots are graph manifolds whose decomposing pieces are $D^2(a, b)$ and $D^2(c, d)$. The result of this paper is as follows:

Theorem 1.1. *There are* 6 *types of middle/hyper doubly Seifert twisted torus knots in* S^3 *and all of these knots admit Dehn surgery yielding non-Seifert-fibered graph manifolds at a surface slope consisting of* $D^2(a, b)$ *and* $D^2(c, d)$*.*

Proof. This follows from Theorem 3.1 and Lemmas 4.1∼4.6. □

2. Twisted torus knots and R-R diagrams

Regarding twisted torus knots, their definition and properties were introduced by Dean in [4]. Especially some properties of twisted torus knots related to Dehn surgeries and primitive or Seifert curves were referred and cited in [5]. Regarding R-R diagrams, their definition and properties were given by Berge in [1] and R-R diagrams of twisted torus knots were given in [5]. Therefore similarly as in [7] we provide brief explanations about definitions, lemmas and propositions without details or proofs.

2.1. Twisted torus knots

Let V_1 and V_2 be two standardly embedded disjoint unlinked solid tori in S^3. Let $T(p,q)$ be a (p,q)-torus knot which lies in the boundary of V_1 and $rT(m,n)$ be the r parallel copies of $T(m,n)$ which lie in the boundary of V_2. Here we may assume that $0 < q < p$ and $m > 0$. Let D_1 be a disk in ∂V_1 so that $T(p,q)$ intersects D_1 in r disjoint parallel arcs, where $0 < r \leq p+q$, and D_2 a disk in ∂V_2 so that $rT(m,n)$ intersect D_2 in r disjoint parallel arcs, one for each component of $rT(m,n)$. By excising the disks D_1 and D_2 from their respective tori and gluing the punctured tori together along their boundaries so that the orientations of $T(p,q)$ and $rT(m,n)$ align correctly, we build a knot lying in the boundary of a genus two handlebody H which is obtained from V_1 and V_2 by gluing the disks D_1 and D_2. This knot is called a twisted torus knot, which is denoted by $K(p,q,r,m,n)$. Figure 1 shows a twisted torus knot $K(7,3,3,2,1)$.

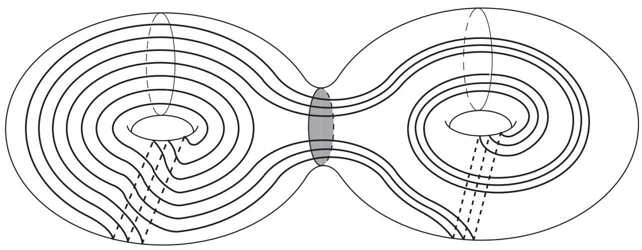

FIGURE 1. A twisted torus knot $K(7,3,3,2,1)$.

Let $H' = \overline{S^3 - H}$ and $\Sigma = \partial H = \partial H'$. Then $(H, H'; \Sigma)$ forms a genus two Heegaard splitting of S^3 and all twisted torus knots lie in the genus two Heegaard surface Σ bounding the two handlebodies H and H' of S^3 as constructed above.

Proposition 2.1. *The surface slope γ of a twisted torus knot $K(p,q,r,m,n)$ with respect to the Heegaard surface Σ is $pq + r^2mn$.*

Proof. This is Proposition 3.1 in [4]. ☐

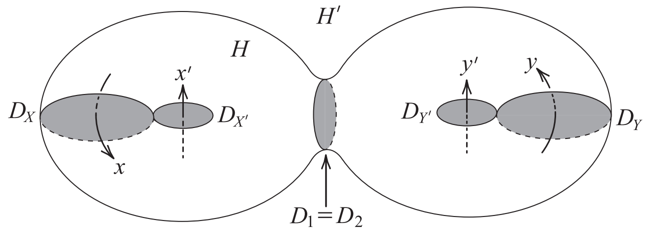

FIGURE 2. The generators of $\pi_1(H)$ and $\pi_1(H')$.

Let $G_{a,b} = \langle x, y \mid x^a y^b \rangle$ be a group presentation with two generators x, y and one relator $x^a y^b$. An element w in the free group $\langle x, y \rangle$ is said to be (a, b) *Seifert-fibered* if $\langle x, y \mid w \rangle$ is isomorphic to $G_{a,b}$. The following lemma indicates the geometric version of (a, b) Seifert-fiberedness.

Lemma 2.2. *Let k be a simple closed curve in the boundary of a genus two handlebody H. k is a Seifert curve in H with $H[k] = D^2(a, b)$ if and only if k in $\pi_1(H)$ is (a, b) Seifert-fibered.*

Proof. This is Lemma 2.2 in [4]. □

Let $\{x, y\}$ and $\{x', y'\}$ be the generating sets dual to the sets of the cutting disks[1] $\{D_X, D_Y\}$ and $\{D_{X'}, D_{Y'}\}$ of H and H' respectively as shown in Figure 2. Let $w_{p,q,r,m,n}$ and $w'_{p,q,r,m,n}$ be the conjugacy classes of a twisted torus knot $K(p, q, r, m, n)$ in $\pi_1(H) = \langle x, y\rangle$ and $\pi_1(H') = \langle x', y'\rangle$ respectively. Note that $w'_{p,q,r,m,n}$ is equal to $w_{q,p,r,n,m}$ with x replaced by x' and y replaced by y', and by the construction of a twisted torus knot, $w_{p,q,r,m,n}$ ($w'_{p,q,r,m,n}$, resp.) does not depend on the parameter n (m, resp.). Therefore we often omit n (m, resp.).

There are more properties in $w_{p,q,r,m,n}$. For g and h in a group G, we say g is *equivalent* to h, denoted by $g \equiv h$, if there is an automorphism of G carrying g to h.

Lemma 2.3. *The words $w_{p,q,r,m}$ have the following properties.*

(1) $w_{p,q,r,m} \equiv w_{p,q',r,m}$ *if* $q \equiv \pm q' \bmod p$.
(2) $w_{p,q,r,m} \equiv w_{p,q,r',m}$ *if* $r \equiv \pm r' \bmod p$.

Proof. This is Lemma 3.3 in [4]. □

The following lemma provides which values of the parameters p, q, r, m, and n produce a primitive curve of $K(p, q, r, m, n)$ with respect to H.

Lemma 2.4. $w_{p,q,r,m}$ *is primitive in* $\pi_1(H)$ *if and only if*

[1] If cutting a genus two handlebody H along two disks D_1 and D_2 yields a 3-ball, then $\{D_1, D_2\}$ is said to be a set of cutting disks of H. This definition is generalized to a genus n handlebody for $n > 2$.

(1) $p = 1$; *or*
(2) $m = 1$ *and* $r = \pm 1$ *or* $\pm q \mod p$.

Proof. This is Theorem 3.4 in [4]. □

For integers p and q, $\hat{q}^{-1}$ is defined to be the smallest positive integer congruent to $\pm q^{-1} \mod p$. For a real number x, $\tilde{x}$ denotes the smallest integer greater than or equal to x. The following proposition gives three criteria to determine which $w_{p,q,r,m,n}$ are Seifert-fibered in $\pi_1(H)$.

Proposition 2.5. *Let* $w = w_{p,q,r,m}$ *be a conjugacy class in* $\pi_1(H)$ *of a twisted torus knot* $K(p, q, r, m, n)$. *Let* q' *be an integer such that* $q \equiv \pm q' \mod p$ *with* $0 < q' < p/2$.

(1) *If* $m > 1$ *and* $r \equiv \pm 1$ *or* $\pm q' \mod p$, *then* w *is* (p, m) *Seifert-fibered.*
(2) *If* $m = 1$ *and* $r \equiv \pm \beta q' \mod p$, *where* $1 < \beta \leq p/q'$ *with* $p - \beta q' > 1$, *then* w *is* $(\beta, p - \beta q')$ *Seifert-fibered.*
(3) *If* $m = 1$ *and* $r \equiv \pm \bar{r} \mod p$, *where* $1 < \bar{r} \leq \widetilde{p/\hat{q'}^{-1}}$ *with* $p - \bar{r}\hat{q'}^{-1} > 1$, *then* w *is* $(\bar{r}, p - \bar{r}\hat{q'}^{-1})$ *Seifert-fibered.*

Proof. The parts (1), (2), and (3) are Propositions 3.6, 3.8, and 3.10 respectively in [4]. □

Dean in [4] conjectured that these three types describe all $w_{p,q,r,m}$ that are Seifert-fibered. The first type (1), the second type (2), and the third type (3) of Seifert-fibered $w_{p,q,r,m}$ (or $K(p, q, r, m, n)$) in Proposition 2.5 are called *hyper Seifert-fibered, middle Seifert-fibered*, and *end Seifert-fibered* in H respectively. With respect to the other handlebody H' we can apply Proposition 2.5 by switching p and q, and m and n to say that $w'_{p,q,r,n}$ ($w_{q,p,r,n}$ or $K(p, q, r, m, n)$) is *hyper Seifert-fibered, middle Seifert-fibered*, or *end Seifert-fibered* in H'.

Using the three types of Seifert curves of twisted torus knots, we can build doubly Seifert twisted torus knots. We have the following 9 types of doubly Seifert twisted torus knots: hyper/hyper, hyper/middle, hyper/end, middle/hyper, middle/middle, middle/end, end/hyper, end/middle, end/end. Note that by the definition of a twisted torus knot $K(p, q, r, m, n)$ and the condition $1 \leq q < p$, the A/B and B/A doubly Seifert twisted torus knots are distinct, where A, B are hyper, middle, or end, and A≠B.

Hyper/hyper, hyper/middle, and middle/middle doubly Seifert twisted torus knots were classified in [6], [5], and [7] respectively together with Dehn surgeries on those knots at a surface slope. Thus the goal of this paper is to classify all middle/hyper doubly Seifert twisted torus knots $K(p, q, r, m, n)$ by finding all possible values of the parameters p, q, r, m, and n. Furthermore we find a surface slope with which Dehn surgery is a non-Seifert-fibered graph manifold whose decomposing pieces are $D^2(a, b)$ and $D^2(c, d)$.

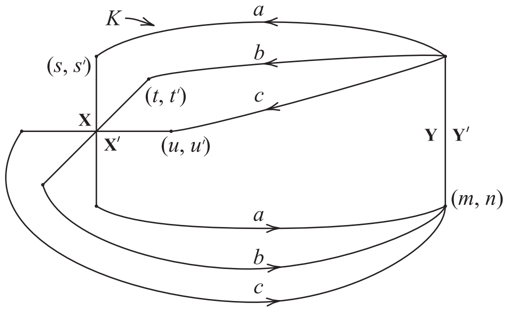

FIGURE 3. R-R diagram of $K = K(p, q, r, m, n)$ with respect to H and H', where $as + bt + cu = p, as' + bt' + cu' = q$, and $a + b + c = r$.

2.2. R-R diagrams

R-R diagrams were originally introduced by Osborne and Stevens in [8], and developed by Berge. R-R diagrams are a type of planar diagram related to Heegaard diagrams of simple closed curves in the boundary of a genus two handlebody and in particular useful for describing embeddings of simple closed curves in the boundary of a handlebody so that the embedded curves represent certain conjugacy classes in the fundamental group of the handlebody. For the definition and properties of R-R diagrams, see [1] or [5]. An R-R diagram of a twisted torus knot $K(p, q, r, m, n)$ has the form shown in Figure 3, where $K = K(p, q, r, m, n)$. The details of how to make an R-R diagram of a twisted torus knot $K(p, q, r, m, n)$ are given in [5].

The following theorem was originally given in [2] and the remarks after it were given in [5]. They are the key tool to find a regular fiber of $H[K]$ when $H[K] = D^2(a, b)$. Since the preprint [2] has not yet been posted, I add the proof of the following theorem, which is given in [2].

Theorem 2.6. *If k is a nonseparating simple closed curve in the boundary of a genus two handlebody H such that $H[k]$ is Seifert-fibered over D^2 with two exceptional fibers, then k has an R-R diagram of the form in Figure* 4a *with $n, s > 1$, or in Figure* 4b *with $n > 0, s > 1$, $a, b > 0$, and $\gcd(a, b) = 1$, up to homeomorphism of H.*

Conversely, if k has an R-R diagram of the form of Figure 4a *with $n, s > 1$, or Figure* 4b *with $n > 0, s > 1$, $a, b > 0$, and $\gcd(a, b) = 1$, then $H[k]$ is Seifert-fibered over D^2 with two exceptional fibers of indexes n and s, or indexes $n(a + b) + b$ and s respectively.*

In addition, the curves τ_1 and τ_2 in Figure 4a *and the curve τ in Figure* 4b *are regular fibers of $H[k]$.*

Proof. First assume that k is a nonseparating simple closed curve in the boundary of a genus two handlebody H such that $H[k]$ is Seifert-fibered over D^2 with

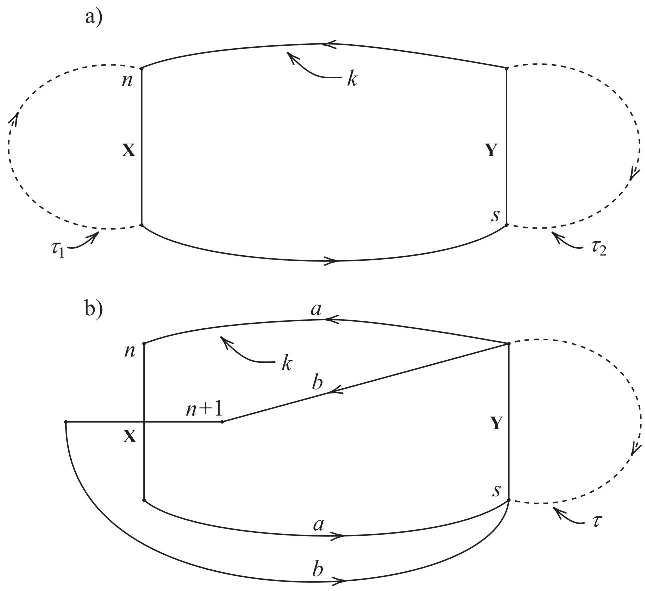

FIGURE 4. Two types of R-R diagrams of a Seifert curve k with $n, s > 1$ in Figure 4a, and $n > 0, s > 1$, $a, b > 1$, and $\gcd(a, b) = 1$ in Figure 4b, and regular fibers τ, τ_1, and τ_2 of $H[k]$.

two exceptional fibers. Since $H[k]$ is defined to be the manifold obtained by adding a 2-handle to H along k, $H[k]$ induces a genus two Heegaard decomposition of Seifert-fibered spaces over D^2 with two exceptional fibers. Theorem 2.7, which is a result of [3], shows that there are only three genus two Heegaard decompositions of a Seifert-fibered space over D^2 with two exceptional fibers up to homeomorphism. Also Theorem 2.8 verifies how to describe the three decompositions in terms of R-R diagrams, finishing the proof of the 'if' part.

Conversely, assume that k has an R-R diagram of the form in Figure 4a with $n, s > 1$. Figure 5 illustrates the situation when $n = 3$ and $s = 2$, where D_A and D_B are cutting disks of H underlying the A-handle and B-handle of the R-R diagram of k and two parallel copies of τ_2 are shown. Note that the two parallel copies of τ_2 are chosen to bound an essential separating annulus $\mathcal{A}$ in H as in Figure 5.

Cutting H apart along $\mathcal{A}$ yields a genus two handlebody W and a solid torus Z. Note that k lies in the boundary of the genus two handlebody W as a primitive curve, which implies that $W[k]$ is a solid torus. Therefore $H[k]$ is obtained by gluing the two solid tori $W[k]$ and Z together along $\mathcal{A}$. It follows that $H[k]$ is Seifert-fibered over D^2 with $\partial\mathcal{A}$ as regular fibers and with the cores

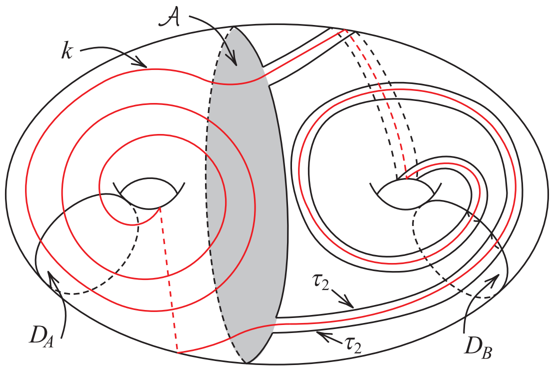

FIGURE 5. The curve k and two copies of τ_2 bounding an essential separating annulus $\mathcal{A}$ in H with $n = 3$ and $s = 2$.

of $W[k]$ and Z as exceptional fibers. This implies that τ_2 is a regular fiber of $H[k]$.

For the indexes of the two exceptional fibers of $H[k]$ it is clear that the annulus $\mathcal{A}$ wraps around the solid torus Z s times longitudinally, so the core of Z is an exceptional fiber of index s. The other index can be obtained by computing $\pi_1((W[k])[\tau_2])$ because $\pi_1((W[k])[\tau_2])$ indicates how many times τ_2 wraps around the core of the solid torus $W[k]$. Note that $(W[k])[\tau_2]$ is homeomorphic to $(W[\tau_2])[k]$, $W[\tau_2]$ is a solid torus, and k lies in the boundary of $W[\tau_2]$ and wraps around it n times longitudinally. We can also see that D_A in Figure 5 is a meridional disk of the solid torus $W[\tau_2]$ and k intersects D_A n times. Thus $\pi_1((W[k])[\tau_2]) = \pi_1((W[\tau_2])[k]) = \mathbb{Z}_n$, implying that the core of $W[k]$ is an exceptional fiber of index n.

Since the R-R diagram of k of the form in Figure 4a is symmetric, a similar argument can be applied to τ_1 instead of τ_2 to see that τ_1 is a regular fiber of $H[k]$.

Now we assume that k has an R-R diagram of the form in Figure 4b with $n > 0, s > 1$, $a, b > 0$, and $\gcd(a, b) = 1$. In this case, we can also construct an essential annulus $\mathcal{A}$ in H which is bounded by two copies of τ. As in the first case, cutting H apart along $\mathcal{A}$ yields a genus two handlebody W and a solid torus Z, where k also lies as a primitive curve in the boundary of W. Therefore $H[k]$ is Seifert-fibered over D^2 with two exceptional fibers such that τ is a regular fiber and the cores of $W[k]$ and Z are exceptional fibers. Since τ wraps around the core of Z s times, s is the index of the exceptional fiber corresponding to the core of Z.

For the index of the exceptional fiber corresponding to the core of $W[k]$, as in the first case, it is enough to compute $\pi_1((W[\tau])[k])$. Note that $W[\tau]$ is a solid torus such that the cutting disk D_A of H underlying the A-handle of the R-R diagram of k is a meridional disk of $W[\tau]$. Since we can see from Figure 4b that k intersects D_A $na+(n+1)b$ times, $n(a+b)+b$ is the second index, as desired. □

To finish the proof of Theorem 2.6, we let $S(\nu/p, \omega/q)$ denote an orientable Seifert-fibered space over the disk D^2 which has two exceptional fibers of types ν/p and ω/q with $0<\nu<p$ and $0<\omega<q$.

Also let $W_{m,n}(x,y)$ be the unique primitive word up to conjugacy in the free group $F(x,y)$ which has (m,n) as its abelianization. Then, if v and w are words in x and y, $W_{m,n}(v,w)$ is the word obtained from $W_{m,n}(x,y)$ by substituting v for x and w for y in $W_{m,n}(x,y)$.

Then the following theorem, which is Theorem 5.4 in [3], completely describes the genus two Heegaard diagrams of $S(\nu/p, \omega/q)$. For the notations in Theorem 2.7, see Sections 2, 4, and 5 in [3].

Theorem 2.7. *The manifold $S(\nu/p, \omega/q)$ admits three genus two Heegaard decompositions HD_0, HD_S, and HD_T, represented by the following Heegaard diagrams*:

$$\begin{aligned} HD_0 &\leftrightarrow (s^p t^{-q}; \lambda, \mu), \\ HD_S &\leftrightarrow (W_{p,\nu}(u^{-1}, t^q); -, \mu), \\ HD_T &\leftrightarrow (W_{q,\omega}(v^{-1}, s^p); \lambda, -). \end{aligned}$$

Here $\nu\lambda \equiv 1 \mod p$ and $\omega\mu \equiv 1 \mod q$. Any Heegaard decomposition of genus two of $S(\nu/p, \omega/q)$ is homeomorphic to one of these. Moreover,

(1) *HD_0 is homeomorphic to HD_T (or HD_S) if and only if $\omega \equiv \pm 1 \mod q$ (or $\nu \equiv \pm 1 \mod p$, respectively).*
(2) *If $\omega \equiv \pm 1 \mod q$ and $\nu \equiv \pm 1 \mod p$, then HD_0, HD_S, and HD_T are all homeomorphic.*
(3) *HD_S and HD_T are homeomorphic if and only if either case (2) occurs or $\nu/p \equiv \pm\omega/q \mod 1$ (that is, $p=q$ and $\nu \equiv \omega \mod p$).*

Theorem 2.8. *The R-R diagrams of k in Figures 6, 7, and 8 correspond to the Heegaard diagrams HD_0, HD_S, and HD_T of Theorem 2.7 respectively.*

Proof. First, consider the curve k in the R-R diagram of Figure 6. Then it follows immediately that k represents $s^p t^{-q}$ in $\pi_1(H)$. As demonstrated in the second part of the proof of Theorem 2.6, we see that $H[k]$ is Seifert-fibered over D^2 with two exceptional fibers of indexes p and q such that the cores of the A-handle and B-handle of H correspond to the exceptional fibers of $H[k]$, and the curves τ_1 and τ_2 in Figure 6 are regular fibers of $H[k]$.

To compute the type of the exceptional fibers, we denote the cores of the A- and B-handles by C_A and C_B respectively, and regular neighborhoods in H of C_A and C_B by $N(C_A)$ and $N(C_B)$ respectively. We let M_A and M_B be meridional disks of $N(C_A)$ and $N(C_B)$. Now we consider curves γ_s and γ_t as shown in Figure 6. Then we can regard the curves τ_1 and γ_s (τ_2 and γ_t, resp.) as lying on $\partial N(C_A)$ ($\partial N(C_B)$, resp.). Furthermore, since γ_s and γ_t intersect

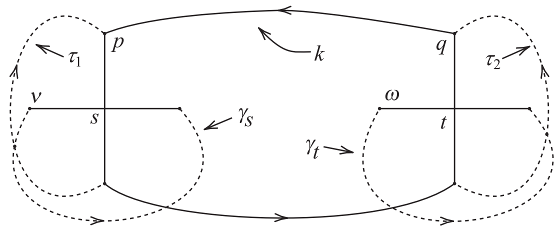

FIGURE 6. Genus two Heegaard decomposition HD_0 of $S(\nu/p, \omega/q)$: Suppose ν, ω, p, and q are positive integers such that $0 < \nu < p$, $0 < \omega < q$, $\gcd(\nu, p) = \gcd(\omega, q) = 1$, and H is a genus two handlebody. Then the manifold $H[k]$, obtained by adding a 2-handle to ∂H along a simple closed curve k in ∂H that has an R-R diagram with the form of this figure, is a Seifert-fibered space over D^2 with exceptional fibers of types ν/p and ω/q.

τ_1 and τ_2 once respectively, γ_s and γ_t can be regarded as the boundary circles of the section of the fiber bundle.

Observe from Figure 6 that τ_1 and γ_s intersect transversely ∂M_A p and ν times respectively. Therefore $\partial M_A = (\tau_1^{\nu}\gamma_s^{p})^{\pm 1}$ in $\pi_1(\partial N(C_A))$. Similarly, $\partial M_A = (\tau_2^{\omega}\gamma_t^{q})^{\pm 1}$ in $\pi_1(\partial N(C_B))$. This implies that C_A and C_B are the exceptional fibers of types ν/p and ω/q in the Seifert-fibration of $H[k]$.

Thus we can conclude that the R-R diagram in Figure 6 corresponds to the Heegaard diagram HD_0.

Second, consider the curve k in the R-R diagram of Figure 7. We see that k represents $W_{p,\nu}(u^{-1}, t^q)$ in $\pi_1(H)$. As discussed in the second part of the proof of Theorem 2.6, two parallel copies of τ bound an annulus $\mathcal{A}$ which separates H into a genus two handlebody W and a solid torus Z. Also $H[k]$ can be obtained by gluing the two solid tori $W[k]$ and Z along $\mathcal{A}$, implying that $H[k]$ is Seifert-fibered over D^2 with two exceptional fibers of indexes $n(a+b)+a$ and q such that the cores of $W[k]$ and Z correspond to the exceptional fibers of $H[k]$, and the curve τ in Figure 7 is a regular fiber of $H[k]$. Taking a curve γ_t as shown in Figure 7, we can apply the argument of the case of HD_0 to show that the exceptional fiber corresponding to the core of Z has the type ω/q.

For the type of the exceptional fiber corresponding to the core of $W[k]$, we need to consider the curve k and the regular fiber τ in the boundary of the genus two handlebody W. The corresponding R-R diagram of k and τ in W appears as in Figure 9. Let M be a meridional disk of the solid torus $W[k]$. Since the exceptional fiber corresponding to the core of $W[k]$ has the index $p = n(a+b)+a$, in order to compute the type of the exceptional fiber, it suffices to compute the intersection number of M with γ_u, where γ_u is a curve intersecting τ once as shown in Figure 9. This can be obtained by computing

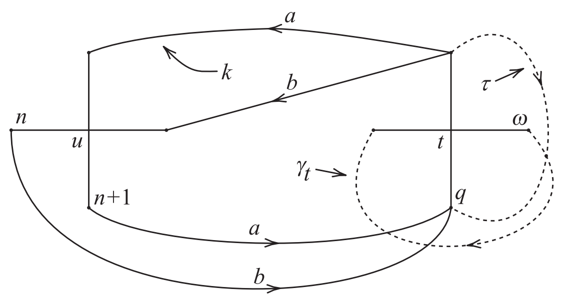

FIGURE 7. Genus two Heegaard decomposition HD_S of $S(\nu/p, \omega/q)$: Suppose ν, ω, p, and q are positive integers such that $1 < \nu < p$, $0 < \omega < q$, and $\gcd(\nu, p) = \gcd(\omega, q) = 1$. In addition, suppose a, b, and n are positive integers such that $a + b = \nu$, $n\nu + a = p$. Then the manifold $H[k]$, obtained by adding a 2-handle to ∂H along a simple closed curve k in ∂H that has an R-R diagram with the form of this figure, is a Seifert-fibered space over D^2 with exceptional fibers of types ν/p and ω/q.

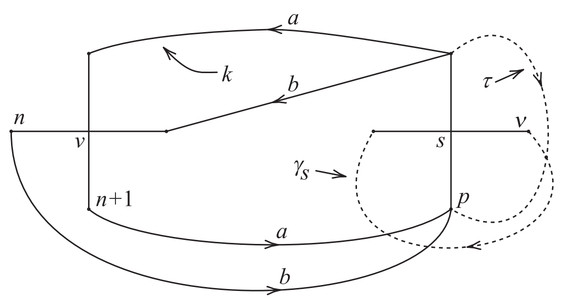

FIGURE 8. Genus two Heegaard decomposition HD_T of $S(\nu/p, \omega/q)$: Suppose ν, ω, p, and q are positive integers such that $0 < \nu < p$, $1 < \omega < q$, and $\gcd(\nu, p) = \gcd(\omega, q) = 1$. In addition, suppose a, b, and n are positive integers such that $a + b = \omega$, $n\omega + a = q$. Then the manifold $H[k]$, obtained by adding a 2-handle to ∂H along a simple closed curve k in ∂H that has an R-R diagram with the form of this figure, is a Seifert-fibered space over D^2 with exceptional fibers of types ν/p and ω/q.

$\pi_1((W[\gamma_u])[k])$. It follows from Figure 9 that $W[\gamma_u]$ is a solid torus and k intersects a meridional disk of $W[\gamma_u]$ $a + b$ times. Since $\nu = a + b$, the type of the exceptional fiber corresponding to the core of $W[k]$ is ν/p.

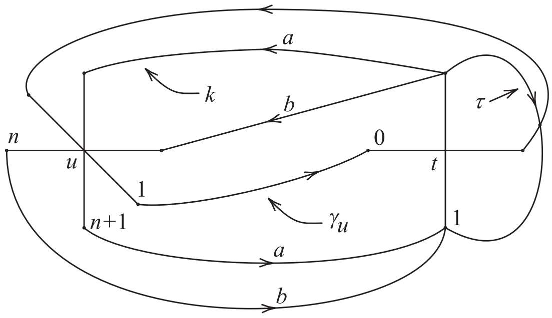

FIGURE 9. R-R diagram of k, τ, and γ_u in W.

We can conclude that the R-R diagram in Figure 7 corresponds to the Heegaard diagram HD_S.

Lastly, we consider the curve k in the R-R diagram of Figure 8. However the R-R diagram of Figure 8 is similar to that of Figure 7 so that we can apply the similar argument as that of Figure 7 to show that the R-R diagram in Figure 8 corresponds to the Heegaard diagram HD_T. □

Remark 2.9. (1) Algebraically in $\pi_1(H) = \langle x, y \rangle$ the Seifert curve k in Figure 4a represents $x^n y^s$, while k in Figure 4b is the product of $x^n y^s$ and $x^{n+1} y^s$ with $|x^n y^s| = a$ and $|x^{n+1} y^s| = b$. Here $|x^n y^s|$ denotes the total number of appearances of $x^n y^s$ in the word of k in $\pi_1(H)$, etc.

(2) Algebraically the regular fibers τ_1 and τ_2 in Figure 4a of $H[k]$ represent x^n and y^s respectively, while the regular fiber τ in Figure 4b represents y^s in $\pi_1(H)$ with $n, s > 1$. In other words, the regular fibers correspond to the generator in the word of k which has only one exponent.

(3) If a curve disjoint from k in Figure 4a represents x^n (y^s, resp.), then this curve is isotopic to the curve τ_1 (τ_2, resp.) and thus can be a regular fiber of $H[k]$. Similarly if a curve disjoint from k in Figure 4b represents y^s, then this curve is isotopic to the curve τ and thus can be a regular fiber of $H[k]$.

3. Finding the parameters p, q, r, m, and n

In this section we find all possible values of the parameters p, q, r, m, and n for which $K(p, q, r, m, n)$ is middle Seifert-fibered in H and hyper Seifert-fibered in H'.

Theorem 3.1. *Let K be a twisted torus knot $K(p, q, r, m, n)$ lying in a genus two Heegaard splitting $(H, H'; \Sigma)$ of S^3 with $0 < q < p$, $gcd(p, q) = 1$, and $0 < r \leq p + q$. K is a middle/hyper doubly Seifert twisted torus knot if and only if the parameter set (p, q, r, m, n) belongs to one of the six classes in Table 1. Table 2 describes $H[K]$ and $H'[K]$ explicitly.*

Table 1. All possible values of parameters p, q, r, m, and n for middle/hyper doubly Seifert twisted torus knot $K(p, q, r, m, n)$.

	(p, q, r, m, n)	satisfying
I	$((\alpha+\beta)q+\bar{p}, q, \alpha q+\bar{p}, 1, n)$	$\beta > 1$, $0 < \bar{p} < q$, $\alpha \geq 0$,
II	$(\beta q+\bar{p}, 2\bar{p}-\epsilon, \beta q+2\bar{p}, 1, n)$	$\beta > 1$, $\epsilon = \pm 1$, $\bar{p} > 1$
III	$((\beta+1)\bar{p}+\epsilon, \beta\bar{p}+\epsilon, \beta\bar{p}, 1, n)$	$\beta > 1$, $\epsilon = \pm 1$, $\bar{p}+\epsilon > 1$
IV	$((\beta+2)\bar{p}+\epsilon, (\beta+1)\bar{p}+\epsilon, 2(\beta+1)\bar{p}+\epsilon, 1, n)$	$\beta > 1$, $\epsilon = \pm 1$, $2\bar{p}+\epsilon > 1$
V	$((\alpha+1)\bar{p}+\bar{q}, \alpha\bar{p}+\bar{q}, (\alpha-1)\bar{p}+\bar{q}, 1, n)$	$0 \leq \bar{q} < \bar{p}$, $\alpha > 0$
VI	$((\alpha+1)\bar{p}+\bar{q}, \alpha\bar{p}+\bar{q}, (2\alpha-1)\bar{p}+2\bar{q}, 1, n)$	$0 \leq \bar{q} < \bar{p}$, $\alpha > 1$

Table 2. $H[K]$ and $H'[K]$ when K is middle Seifert-fibered in H and hyper Seifert-fibered in H'.

	$H[K]$	$H'[K]$
I	$D^2(\beta, \alpha q+\bar{p})$	$D^2(q, n)$
II	$D^2(\beta, \bar{p})$	$D^2(2\bar{p}-\epsilon, n)$
III	$D^2(\beta, \bar{p}+\epsilon)$	$D^2(\beta\bar{p}+\epsilon, n)$
IV	$D^2(\beta, 2\bar{p}+\epsilon)$	$D^2((\beta+1)\bar{p}+\epsilon, n)$
V	$D^2(2, (\alpha-1)\bar{p}+\bar{q})$	$D^2(\alpha\bar{p}+\bar{q}, n)$
VI	$D^2(3, (\alpha-2)\bar{p}+\bar{q})$	$D^2(\alpha\bar{p}+\bar{q}, n)$

Proof. If the parameter set (p, q, r, m, n) of K belongs to one of the classes in Table 1, then it is easy to check that (p, q, r, m, n) satisfies the conditions in (1) and (2) in Proposition 2.5. In other words, K is middle Seifert-fibered in H and hyper Seifert-fibered in H'.

Now we prove the "only if" part. Assume that K is middle Seifert-fibered in H and hyper Seifert-fibered in H'. By Proposition 2.5 together with Lemma 2.3 for H' we have following conditions:

(1) the "middle" condition: $m = 1, r \equiv \pm\beta q'$ mod p, where $q' \equiv \pm q$ mod p, $0 < 2q' < p$, $1 < \beta < p/q'$, and $p - \beta q' > 1$, and K is $(\beta, p-\beta q')$ Seifert-fibered in H.
(2) the "hyper" condition: $|n| > 1$, $q > 1$, and $r \equiv \pm 1$ or $\pm p$ mod q, and K is (q, n) Seifert-fibered in H'.

The inequalities $\beta > 1$, $p - \beta q > 1$, $|n| > 1$, and $q > 1$ in the conditions come from Seifert-fiberedness of K in H and H'. If $r = 1$ or $r = p + q$, then by Lemma 2.4, K is primitive in H. Therefore we may assume that $1 < q < p$ and $1 < r < p + q$.

In the "middle" condition, the value q' depends on the values p and q. Thus we divide the argument into two cases: Case 1: $p > 2q$, Case 2: $p < 2q$.

Case 1: Suppose $p > 2q$.

Then $q' = q$ in the "middle" condition on H and $r \equiv \pm\beta q \mod p$, where $1 < \beta < p/q$. Thus since $r < p + q$, the possible values of r are

$$r = \beta q, p - \beta q, \text{ or } 2p - \beta q. \tag{3.2}$$

On H', by the "hyper" condition, $r \equiv \pm 1$ or $\pm p \mod q$. Thus the possible values of r are

$$r = kq + \epsilon, kq - p, \text{ or } p - kq, \tag{3.3}$$

where $\epsilon = \pm 1$.

We need to find the parameters p, q, and r by equating both values of r in Equations 3.2 and 3.3. However $r = \beta q$ in Equation 3.2 does not satisfy Equation 3.3. Therefore there are six subcases to consider:
(1) $r = p - \beta q = kq + \epsilon$, (2) $r = p - \beta q = kq - p$, (3) $r = p - \beta q = p - kq$,
(4) $r = 2p - \beta q = kq + \epsilon$, (5) $r = 2p - \beta q = kq - p$, (6) $r = 2p - \beta q = p - kq$.

Subcase (1): $r = p - \beta q = kq + \epsilon$. Then $p = (k + \beta)q + \epsilon$ and $r = kq + \epsilon$. This belongs to the solution (I) in Table 1 with $\alpha = k$ and $\bar{p} = 1$ if $\epsilon = 1$, and with $\alpha = k - 1$ and $\bar{p} = q - 1$ if $\epsilon = -1$.

Subcase (2): $r = p - \beta q = kq - p$. Then $2p = (k + \beta)q$. Since $\gcd(p, q) = 1$, $q = 2$, $p = k+\beta$, and $r = k-\beta$. Also $\gcd(p, q) = 1$ implies that $p = k+\beta$ is odd and so is $r = k-\beta$. This is part of the solution (I) with $\alpha = (k-\beta-1)/2, q = 2$, and $\bar{p} = 1$.

Subcase (3): $r = p - \beta q = p - kq$. Then $k = \beta$. If we let $r = \alpha q + \bar{p}$, where $0 < \bar{p} < q$ and $\alpha \geq 0$, then $p = (\alpha + \beta)q + \bar{p}$. This yields the solution (I) in Table 1. Here $\bar{p} \neq 0$, otherwise $p = (\alpha + \beta)q$ and due to $\gcd(p, q) = 1$, $q = 1$, a contradiction.

Subcase (4): $r = 2p-\beta q = kq+\epsilon$. Then $2p = (k+\beta)q+\epsilon$. Since $r = 2p-\beta q$ and $r < p + q$, $p - \beta q < q$. By the "middle" condition $p - \beta q > 1$. Therefore, we obtain the inequality $1 < p - \beta q < q$, which implies $\beta q < p < (\beta + 1)q$ and equivalently $2\beta q < 2p < 2(\beta + 1)q$. But the equation $2p = (k + \beta)q + \epsilon$ induces that

$$2\beta q < (k + \beta)q + \epsilon < 2(\beta + 1)q.$$

If $\epsilon = 1$, then $\beta \leq k < \beta + 2$ and thus $k = \beta$ or $k = \beta + 1$. If $\epsilon = -1$, then $\beta < k \leq \beta+2$ and thus $k = \beta+1$ or $k = \beta+2$. However, if $k = \beta$ or $k = \beta+2$, then $k+\beta$ is even. This is a contradiction to the equation $2p = (k+\beta)q+\epsilon$. It follows that $k = \beta+1$ and $2p = (2\beta+1)q+\epsilon$. The equation $2p = (2\beta+1)q+\epsilon$ gives rise to the determinant of the matrix as follows:

$$\begin{vmatrix} 2 & 2\beta+1 \\ q & p \end{vmatrix} = \epsilon.$$

Since $\begin{vmatrix} 2 & 2\beta+1 \\ -\epsilon & -\epsilon\beta \end{vmatrix} = \epsilon$, $q = 2\bar{p} - \epsilon$, $p = (2\beta+1)\bar{p} - \epsilon\beta = (2\bar{p} - \epsilon)\beta + \bar{p} = \beta q + \bar{p}$, and $r = 2p - \beta q = \beta q + 2\bar{p}$ for some $\bar{p} > 0$. But since $p - \beta q > 1$, $\bar{p} > 1$. This yields the solution (II) in Table 1.

Subcase (5): $r = 2p - \beta q = kq - p$. Then $3p = (k + \beta)q$. Since $\gcd(p, q) = 1$ and $q > 1$, $q = 3$ and $p = k + \beta$. From the equation $r = 2p - \beta q$ and the inequalities $r < p + q$ and $p - \beta q > 1$, we obtain

$$1 < p - \beta q < q \Leftrightarrow 1 < k - 2\beta < 3.$$

Thus $k = 2\beta + 2$ and then $p = 3\beta + 2, q = 3$, and $r = 3\beta + 4$. This belongs to the solution (II) with $\bar{p} = 2$ and $\epsilon = 1$.

Subcase (6): $r = 2p - \beta q = p - kq$. Then $p = (\beta - k)q$, which is a contradiction to $\gcd(p, q) = 1$ and $q > 1$.

Case 2: Suppose $p < 2q$.

We let $p = q + \bar{p}$. Then $2\bar{p} < p$. Thus in the "middle" condition, $q' = \bar{p}$ and $r \equiv \pm\beta\bar{p} \bmod p$, $1 < \beta < p/\bar{p}$, and $p - \beta\bar{p} > 1$. The possible values of r are

$$r = \beta\bar{p},\ p - \beta\bar{p},\ 2p - \beta\bar{p},\ \text{or } p + \beta\bar{p}. \tag{3.4}$$

In the "hyper" condition, $r \equiv \pm 1$ or $r \equiv \pm p \bmod q$ and $q > 1, n > 1$. The equation $r \equiv \pm p \bmod q$ implies that

$$r = kq + \epsilon,\ kq - p,\ \text{or } p - kq,$$

where $\epsilon = \pm 1$. However since $1 < r < p + q$, if $r = p - kq$, then either $k = 0$ or $k = 1$. If $k = 0$, then $r = p$. By the four possible r in Equation 3.4, $r = p = \beta\bar{p}, p - \beta\bar{p}, 2p - \beta\bar{p}$, or $p + \beta\bar{p}$. If $p = \beta\bar{p}$ or $2p - \beta\bar{p}$, then $p - \beta\bar{p} = 0$, a contradiction to $p - \beta\bar{p} > 1$, and if $p = p - \beta\bar{p}$ or $p + \beta\bar{p}$, then $\beta\bar{p} = 0$, a contradiction as well. If $k = 1$, then $r = p - q$ and thus K is primitive in H by Lemma 2.4. Therefore the case where $r = p - kq$ is excluded. Furthermore if $r = kq + \epsilon$, then due to $p < 2q$ and $1 < r < p + q$, $k = 1$ or 2. If $r = kq - p$, then by the same reason, $k = 2$ or 3. Therefore the possible values of r in the "hyper" condition are

$$r = kq + \epsilon \text{ with } k = 1, 2,\ \text{or } kq - p \text{ with } k = 2, 3. \tag{3.5}$$

Note that since $\gcd(p, q) = 1$ and $p = q + \bar{p}$, $\gcd(q, \bar{p}) = 1$. Now we break into 4 subcases according to the possible r in Equation 3.4:

(1) $r = \beta\bar{p}$, (2) $r = p - \beta\bar{p}$, (3) $r = 2p - \beta\bar{p}$, (4) $r = p + \beta\bar{p}$.

Subcase (1): $r = \beta\bar{p}$. Since r satisfies Equation 3.5, either $r = \beta\bar{p} = kq + \epsilon$ with $k = 1, 2$ or $r = \beta\bar{p} = kq - p$ with $k = 2, 3$. Assume first that $r = \beta\bar{p} = kq + \epsilon$ with $k = 1, 2$. If $k = 2$, then $\beta\bar{p} = 2q + \epsilon$. It follows from the equation $p - \beta\bar{p} > 1$ that $p > \beta\bar{p} + 1 = 2q + 1 + \epsilon$, which is a contradiction to the inequality $p < 2q$. Therefore $k = 1$, $r = \beta\bar{p}$, $q = \beta\bar{p} + \epsilon$, and $p = q + \bar{p} = (\beta + 1)\bar{p} + \epsilon$. This gives rise to the solution (III) in Table 1.

Now assume that $r = \beta\bar{p} = kq - p$ with $k = 2, 3$. By replacing p by $q + \bar{p}$ in the equation $\beta\bar{p} = kq - p$, we obtain $(\beta + 1)\bar{p} = (k - 1)q$. If $k = 2$, then $\bar{p} = 1$ because of $\gcd(q, \bar{p}) = 1$. Therefore $q = \beta + 1, p = \beta + 2$, and $r = \beta$. This goes into the solution (III) with $\bar{p} = 1$ and $\epsilon = 1$. If $k = 3$, then $(\beta + 1)\bar{p} = 2q$. Since

$\gcd(q, \bar{p}) = 1$, either $\bar{p} = 1$ and $2q = \beta + 1$, or $\bar{p} = 2$ and $q = \beta + 1$. Both cases are impossible by $p - \beta\bar{p} > 1$ or an equivalent inequality $q + (1 - \beta)\bar{p} > 1$.

Subcase (2): $r = p - \beta\bar{p}$. First assume that $r = p - \beta\bar{p} = kq + \epsilon$ with $k = 1, 2$. If $k = 2$, then $p = 2q + \beta\bar{p} + \epsilon$ and thus $p > 2q$, a contradiction to $p < 2q$. Therefore $k = 1$ and $p - \beta\bar{p} = q + \epsilon$. This equation with p replaced by $q + \bar{p}$ induces $(1 - \beta)\bar{p} = \epsilon$. Since $\beta > 1$, $\epsilon = -1$ and $(1 - \beta)\bar{p} = -1$, which implies that $\beta = 2, \bar{p} = 1$. Therefore $p = q + 1$ and $r = q - 1$. This belongs to the solution (III) in Table 1 with $\bar{p} = 1$ and $\epsilon = 1$.

Second assume that $r = p - \beta\bar{p} = kq - p$ with $k = 2, 3$. From $p = q + \bar{p}$, we see that $(2 - \beta)\bar{p} = (k - 2)q$. If $k = 3$, then $(2 - \beta)\bar{p} = q$, which is impossible because $\beta > 1$ and $q > 1$. Hence $k = 2$ and $\beta = 2$. Also $p = q + \bar{p}$ and $r = p - 2\bar{p} = q - \bar{p}$. Let $q = \alpha\bar{p} + \bar{q}$, where $0 \leq \bar{q} < \bar{p}$. Then $p = (\alpha + 1)\bar{p} + \bar{q}$ and $r = (\alpha - 1)\bar{p} + \bar{q}$. This yields the solution (V) in Table 1.

Subcase (3): $r = 2p - \beta\bar{p}$. First assume that $r = 2p - \beta\bar{p} = kq + \epsilon$ with $k = 1, 2$. Since $p = q + \bar{p}$, we obtain $(2 - \beta)\bar{p} = (k - 2)q + \epsilon$. If $k = 1$, then $q = (\beta - 2)\bar{p} + \epsilon$ and $p = (\beta - 1)\bar{p} + \epsilon$. This implies that $p - \beta\bar{p} = -\bar{p} + \epsilon \leq 0$, a contradiction. Thus $k = 2$ and the equation $(2 - \beta)\bar{p} = (k - 2)q + \epsilon$ becomes $(2 - \beta)\bar{p} = \epsilon$. Consequently, $\epsilon = -1$ and $(2 - \beta)\bar{p} = -1$. So $\beta = 3$ and $\bar{p} = 1$. It follows that $p = q + 1$ and $r = 2q - 1$. This belongs to the solution (VI) in Table 1 with $\bar{p} = 1$ and $\bar{q} = 0$.

Second assume that $r = 2p - \beta\bar{p} = kq - p$ with $k = 2, 3$. Then $(3 - \beta)\bar{p} = (k - 3)q$. If $k = 2$, then $q = (\beta - 3)\bar{p}$, which implies that $\bar{p} = 1, q = \beta - 3$, and $p = \beta - 2$. This is a contradiction to the inequality $p - \beta\bar{p} > 1$. Therefore $k = 3$ and from the equation $(3 - \beta)\bar{p} = (k - 3)q$, $\beta = 3$. Consequently $p = q + \bar{p}$ and $r = 2q - \bar{p}$. Let $q = \alpha\bar{p} + \bar{q}$, where $0 \leq \bar{q} < \bar{p}$. Then $p = (\alpha + 1)\bar{p} + \bar{q}$ and $r = (2\alpha - 1)\bar{p} + \bar{q}$. Since $p - \beta\bar{p} > 1$, $\alpha > 1$. This yields the solution (VI) in Table 1.

Subcase (4): $r = p + \beta\bar{p}$. Assume that $r = p + \beta\bar{p} = kq + \epsilon$ with $k = 1, 2$. If $k = 1$, then $p + \beta\bar{p} = q + \epsilon$, which is a contradiction to $q < p$. If $k = 2$, then the equations $p + \beta\bar{p} = kq + \epsilon$ and $p = q + \bar{p}$ induce $(1 + \beta)\bar{p} = q + \epsilon$. Thus $q = (\beta + 1)\bar{p} + \epsilon, p = (\beta + 2)\bar{p} + \epsilon$, and $r = (2\beta + 2)\bar{p} + \epsilon$. This produces the solution (IV) in Table 1.

Lastly we assume that $r = p + \beta\bar{p} = kq - p$ with $k = 2, 3$. Then $(\beta + 2)\bar{p} = (k - 2)q$. Thus $k = 3, \bar{p} = 1$, and $q = \beta + 2$. Also $p = \beta + 3$ and $r = 2\beta + 3$. This belongs to the solution (IV) in Table 1 with $\bar{p} = 1$ and $\epsilon = 1$. □

4. Dehn surgeries on the middle/hyper doubly Seifert twisted torus knots at a surface slope

In Section 3, we have shown that there are six infinite families of middle/hyper doubly Seifert twisted torus knots. By the definition of middle/hyper doubly Seifert twisted torus knots, it follows that the Dehn surgeries at a surface slope are either $S^2(a, b, c, d)$ or a graph manifold consisting of $D^2(a, b)$ and

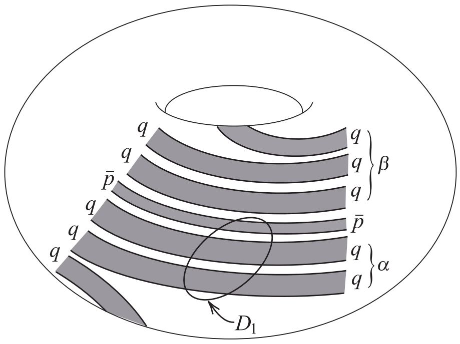

FIGURE 10. A torus knot $T(p, q)$ and a disk D_1 containing r parallel arcs of $T(p, q)$, where $p = (\alpha+\beta)q+\bar{p}$ and $r = \alpha q+\bar{p}$.

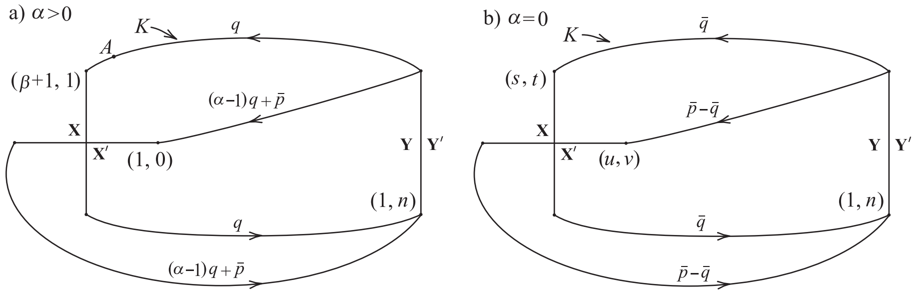

FIGURE 11. R-R diagrams of K when a) $\alpha > 0$ and b) $\alpha = 0$, where $(s, t) = ((\rho + 1)\beta + 1, \rho + 1)$ and $(u, v) = (\rho\beta + 1, \rho)$.

$D^2(c, d)$. In this section we investigate Dehn surgeries on these knots at a surface slope by finding regular fibers of $H[K] = D^2(a, b)$ and $H'[K] = D^2(c, d)$.

Lemma 4.1. *Let $K = K(p, q, r, m, n)$ be a twisted torus knot lying in a genus two Heegaard surface Σ of S^3 such that K is of type* I *in Table* 1*, i.e., $(p, q, r, m, n) = ((\alpha+\beta)q+\bar{p}, q, \alpha q+\bar{p}, 1, n)$ with $\beta > 1$, $0 < \bar{p} < q$, $\alpha \geq 0$. Then for the surface slope γ, $K(\gamma)$ is a non-Seifert-fibered graph manifold consisting of $D^2(\beta, \alpha q + \bar{p})$ and $D^2(q, n)$.*

Proof. It follows from Table 2 in Theorem 3.1 that $H[K] = D^2(\beta, \alpha q + \bar{p})$ and $H'[K] = (q, n)$. Since $K(\gamma) \cong H[K] \cup_\partial H'[K]$ due to Lemma 2.1 in [4], in order to show that $K(\gamma)$ is a non-Seifert-fibered graph manifold we need only to prove that two regular fibers of $H[K]$ and $H'[K]$ intersect at least once transversely in Σ.

Figure 10 shows a torus knot $T(p, q)$ and a disk D_1 containing r parallel arcs of $T(p, q)$ in V_1, where $p = (\alpha+\beta)q+\bar{p}$ and $r = \alpha q+\bar{p}$. Then the corresponding R-R diagram of K depends on the value α. If $\alpha > 0$, then the R-R diagram

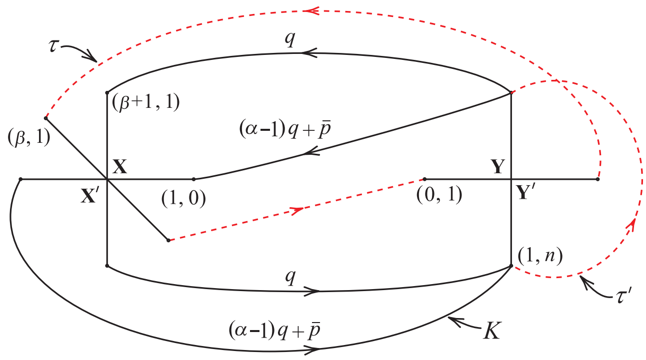

FIGURE 12. Regular fibers τ and τ' of $H[K]$ and $H'[K]$ respectively.

is given in Figure 11a. If $\alpha = 0$, then we let $q = \rho\bar{p} + \bar{q}$, where $\rho > 0$ and $0 < \bar{q} < \bar{p}$. Here $\bar{q} \neq 0$, otherwise $\bar{p} = 1$ and thus $H[K] = D^2(\beta, \alpha q + \bar{p})$ is a solid torus by Lemma 2.4, a contradiction. The R-R diagram in this case is shown in Figure 11b, where $(s,t) = ((\rho+1)\beta+1, \rho+1)$ and $(u,v) = (\rho\beta+1, \rho)$.

Using these R-R diagrams, we describe regular fibers of $H[K]$ and $H'[K]$. Since K is hyper Seifert-fibered in H', $|n| > 1$. Also by the R-R diagram of K n is the only exponent of the generator y'. Therefore by Theorem 2.6 and Remark 2.9 a curve which represents y'^n and is disjoint from K is a regular fiber of $H'[K]$. Consider the curve τ' in Figure 12. Then it represents y'^n and is disjoint from K. Therefore τ' is a regular fiber of $H'[K]$.

To describe a regular fiber of $H[K]$, we now record the curve K algebraically with respect to H whose fundamental group is $\pi_1(H) = \langle x, y \rangle$.

First assume that $\alpha > 0$. In the R-R diagram in Figure 11a, we start with q parallel arc entering into the $(\beta+1, 1)$-connection in the (X, X')-handle. In other words, we read off a word of K from the point A in Figure 11a lying on q parallel edges entering into the $(\beta+1, 1)$-connection in the (X, X')-handle.

The q parallel edges trace out

$$x^{\beta+1}y(xy)^{\alpha-1}\cdots$$

and then they split into two subsets of parallel edges, one of which has $q - \bar{p}$ parallel edges and the other has $\bar{p}$ parallel edges. After splitting the $q - \bar{p}$ parallel edges come back the starting point A while the $\bar{p}$ parallel edges trace out xy before they come back to A. This implies that K is the product of two subwords

$$x^{\beta+1}y(xy)^{\alpha-1} \text{ and } x^{\beta+1}y(xy)^{\alpha}$$

with $|x^{\beta+1}y(xy)^{\alpha-1}| = q - \bar{p}$ and $|x^{\beta+1}y(xy)^{\alpha}| = \bar{p}$.

We perform a change of cutting disks of the handlebody H underlying the R-R diagram, which induces an automorphism of $\pi_1(H)$ that takes $y \mapsto x^{-1}y$ and leaves x fixed. Then by this change of cutting disks, the two subwords

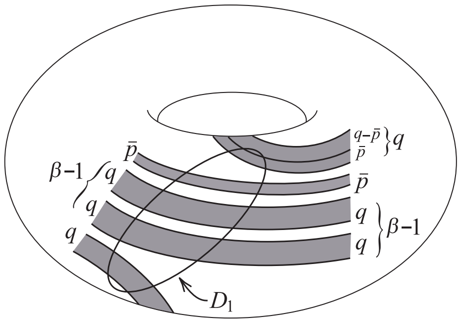

FIGURE 13. A torus knot $T(p,q)$ and a disk D_1 containing r parallel arcs of $T(p,q)$, where $p = \beta q + \bar{p}$ and $r = \beta q + 2\bar{p}$.

are carried into $x^\beta y^\alpha$ and $x^\beta y^{\alpha+1}$ respectively, where only x^β appears in the word of K. Thus from Theorem 2.6 and Remark 2.9 a curve representing x^β is a regular fiber of $H[K]$. Consider the curve τ in the R-R diagram of K in Figure 12. Then τ is disjoint from K and represents x^β and is sent to x^β itself after performing the automorphism $y \mapsto x^{-1}y$. Therefore τ is a regular fiber of $H[K]$ and intersects a regular fiber τ' of $H'[K]$.

Now assume that $\alpha = 0$. By applying a similar argument as above, we see that regular fibers of $H[K]$ and $H'[K]$ lie as in Figure 12 and intersect once transversely. This completes the proof. □

Lemma 4.2. *Let $K = K(p,q,r,m,n)$ be a twisted torus knot lying in a genus two Heegaard surface Σ of S^3 such that K is of type* II *in Table 1, i.e., $(p,q,r,m,n) = (\beta q + \bar{p}, 2\bar{p} - \epsilon, \beta q + 2\bar{p}, 1, n)$ with $\beta > 1$, $\epsilon = \pm 1$, $\bar{p} > 1$. Then for the surface slope γ, $K(\gamma)$ is a non-Seifert-fibered graph manifold consisting of $D^2(\beta, \bar{p})$ and $D^2(2\bar{p} - \epsilon, n)$.*

Proof. Figure 13 shows a torus knot $T(p,q)$ and a disk D_1 containing r parallel arcs of $T(p,q)$ in V_1, where $p = \beta q + \bar{p}$ and $r = \beta q + 2\bar{p}$. The corresponding R-R diagram of K is depicted in Figure 14.

We start with the point A in Figure 14 lying on $\bar{p}$ parallel edges entering into the $(0,1)$-connection in the (X, X')-handle to record the curve K algebraically. Since $q = 2\bar{p} + \epsilon$, where $\epsilon = \pm 1$, we divide into two cases: $\epsilon = 1$ and $\epsilon = -1$.

First assume that $\epsilon = 1$. Then $q = 2\bar{p} - 1$ and the $\bar{p}$ parallel edges trace out

$$x^0 y(xy)^\beta \cdots$$

and then they split into two subsets of parallel edges, one of which has one edge and the other has $\bar{p} - 1$ parallel edges. The one edge traces out xy while the $\bar{p} - 1$ parallel edges trace out $xy(xy)^{\beta-1}xy$ before they come back to the starting point A. One can check this when $\beta = 3$, which is given in Figure 15. Consequently K is the product of two subwords

$$x^0 y(xy)^\beta xy = xy^2(xy)^\beta \text{ and}$$

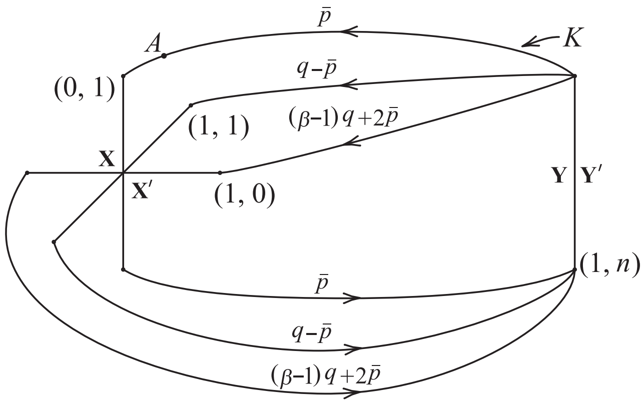

FIGURE 14. R-R diagram of K.

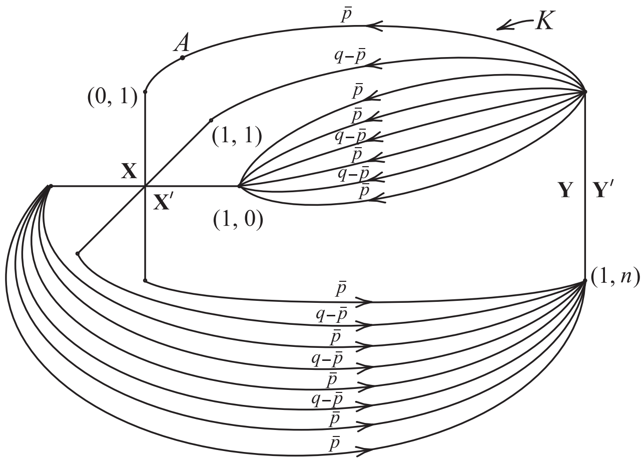

FIGURE 15. R-R diagram of K when $\beta = 3$.

$$x^0 y(xy)^\beta xy(xy)^{\beta-1}xy = xy^2(xy)^{2\beta}$$

with $|xy^2(xy)^\beta| = 1$ and $|xy^2(xy)^{2\beta}| = \bar{p} - 1$.

We perform a change of cutting disks of the handlebody H inducing an automorphism $x \mapsto xy^{-1}$ of $\pi_1(H)$. Then by this automorphism, $xy^2(xy)^\beta$ and $xy^2(xy)^{2\beta}$ are sent to $yx^{\beta+1}$ and $yx^{2\beta+1}$ respectively.

We perform another change of cutting disks of H inducing an automorphism $y \mapsto yx^{-\beta-1}$ to send $yx^{\beta+1}$ and $yx^{2\beta+1}$ to y and yx^β respectively. It follows that only x^β appears and thus a curve representing x^β is a regular fiber of $H[K]$.

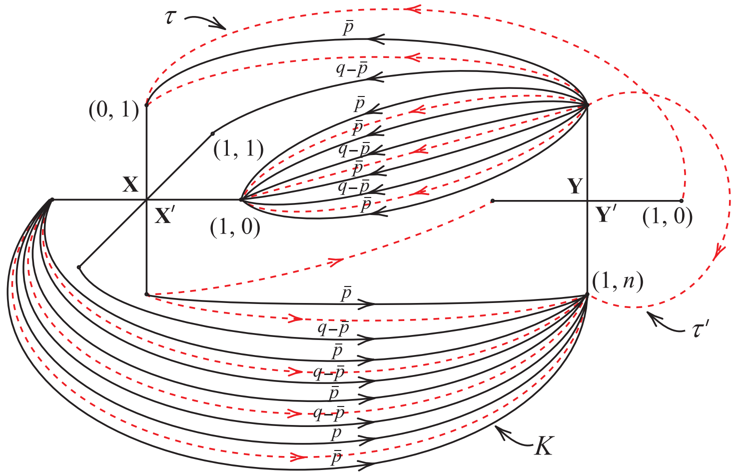

FIGURE 16. Regular fibers τ and τ' of $H[K]$ and $H'[K]$ respectively when $\beta = 3$.

Now assume that $\epsilon = -1$. Then $q = 2\bar{p} + 1$ and from the R-R diagram the $\bar{p}$ parallel edges trace out

$$x^0 y(xy)^\beta xy(xy)^{\beta-1}\cdots$$

and then they split into two subsets of parallel edges, one of which has $\bar{p}-1$ parallel edges and the other has one edge. The $\bar{p}-1$ parallel edges trace out xy while the one edge traces out $xy(xy)^{\beta-1}xy$ before they come back to the starting point A. So K is the product of two subwords

$$x^0 y(xy)^\beta xy(xy)^{\beta-1}xy = xy^2(xy)^{2\beta} \text{ and}$$
$$x^0 y(xy)^\beta xy(xy)^{\beta-1}xy(xy)^{\beta-1}xy = xy^2(xy)^{3\beta}$$

with $|xy^2(xy)^{2\beta}| = \bar{p}-1$ and $|xy^2(xy)^{3\beta}| = 1$.

By performing two automorphisms $x \mapsto xy^{-1}$ and $y \mapsto yx^{-2\beta-1}$ consecutively, we see that $xy^2(xy)^{2\beta}$ and $xy^2(xy)^{3\beta}$ are carried to y and yx^β respectively. Therefore a curve representing x^β is a regular fiber of $H[K]$.

We have demonstrated that in both cases of $\epsilon = 1$ and $\epsilon = -1$, a curve representing x^β is a regular fiber of $H[K]$. If we consider a curve τ as shown in Figure 16, where $\beta = 3$, then τ is disjoint from K and represents

$$x^0 y(xy)^{\beta-1}xyx^0y^{-1} = (xy)^\beta$$

in $\pi_1(H)$, which is sent to x^β after performing the two automorphisms $x \mapsto xy^{-1}$ and $y \mapsto yx^{-\beta-1}$ ($y \mapsto yx^{-2\beta-1}$, resp.) consecutively when $\epsilon = 1$ ($\epsilon = -1$, resp.) as performed to K. This implies that the curve τ is a regular fiber of $H[K]$.

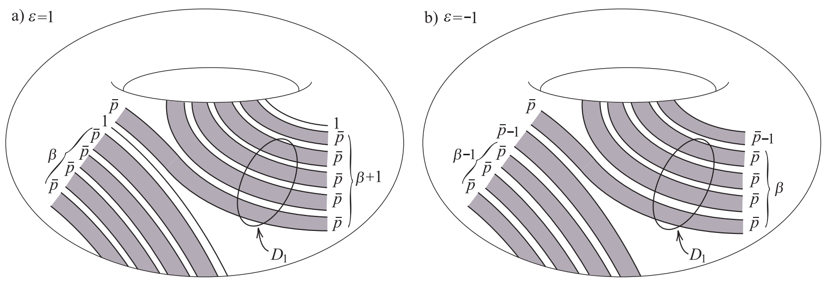

FIGURE 17. A torus knot $T(p,q)$ and a disk D_1 containing r parallel arcs of $T(p,q)$, where $p=(\beta+1)\bar{p}+\epsilon, q=\beta\bar{p}+\epsilon$, and $r=\beta\bar{p}$.

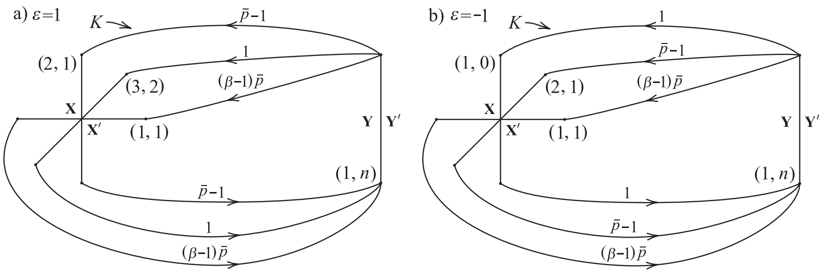

FIGURE 18. R-R diagram of K.

As in the proof of Lemma 4.1, the curve τ' in Figure 16 is a regular fiber of $H'[K]$. Thus the two regular fibers intersect once transversely as desired. □

Lemma 4.3. *Let $K=K(p,q,r,m,n)$ be a twisted torus knot lying in a genus two Heegaard surface Σ of S^3 such that K is of type* III *in Table* 1*, i.e., $(p,q,r,m,n)=((\beta+1)\bar{p}+\epsilon,\beta\bar{p}+\epsilon,\beta\bar{p},1,n)$ with $\beta>1$, $\epsilon=\pm1$, $\bar{p}+\epsilon>1$. Then for the surface slope γ, $K(\gamma)$ is a non-Seifert-fibered graph manifold consisting of $D^2(\beta,\bar{p}+\epsilon)$ and $D^2(\beta\bar{p}+\epsilon,n)$.*

Proof. Figures 17 and 18 show a torus knot $T(p,q)$ and a disk D_1 containing r parallel arcs of $T(p,q)$ and R-R diagram of K depending on ϵ, where $p=(\beta+1)\bar{p}+\epsilon, q=\beta\bar{p}+\epsilon$, and $r=\beta\bar{p}$ respectively.

First we assume that $\epsilon=1$. We read off a word of K from the one edge entering into the $(3,2)$-connection in the (X,X')-handle. It is easy to see from the R-R diagram that the word of K is

$$x^3y(xy)^{\beta-1}(x^2y(xy)^{\beta-1})^{\bar{p}-1}.$$

The automorphism $y\mapsto x^{-1}y$ maps $x^3y(xy)^{\beta-1}(x^2y(xy)^{\beta-1})^{\bar{p}-1}$ to

$$x^2y^\beta(xy^\beta)^{\bar{p}-1},$$

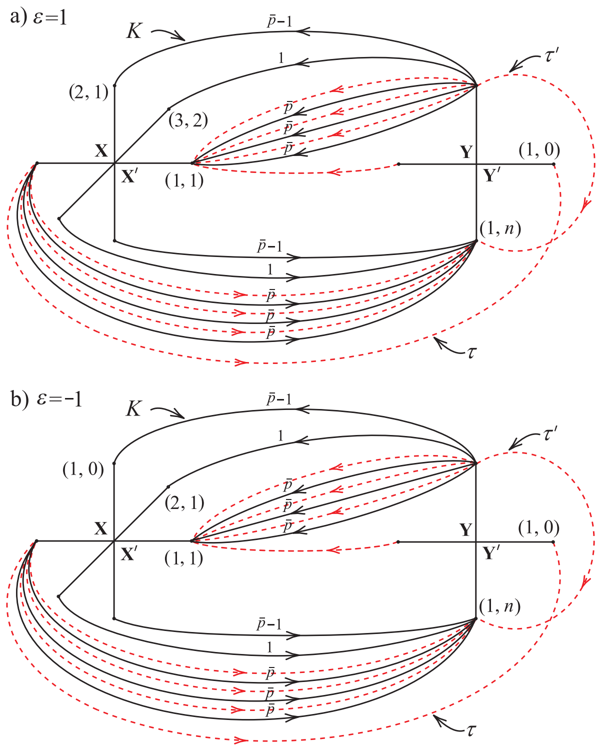

FIGURE 19. Regular fibers τ and τ' of $H[K]$ and $H'[K]$ respectively when $\beta = 4$.

in which only β appears in the exponent of y. Thus a curve representing y^β can be a regular fiber of $H[K]$.

Consider a curve τ as shown in Figure 19a which illustrates the R-R diagram of K when $\beta = 4$. Then the word of τ is $(xy)^\beta$ and becomes y^β after the automorphism $y \mapsto x^{-1}y$. Also τ is disjoint from K. Therefore the curve τ is a regular fiber and intersects a regular fiber τ' of $H'[K]$ once transversely as in Figure 19a.

Now we assume that $\epsilon = -1$. Note that since $\bar{p} + \epsilon > 1$, $\bar{p} > 2$. Applying a similar argument in the R-R diagram of Figure 18b as in the case that $\epsilon = 1$, we see that K represents

$$xy(xy)^{\beta-1}(x^2y(xy)^{\beta-1})^{\bar{p}-1} = (xy)^\beta(x^2y(xy)^{\beta-1})^{\bar{p}-1},$$

which is sent to $x^\beta(xy^{-1}x^\beta)^{\bar{p}-1} = y^{-1}x^{2\beta+1}(y^{-1}x^{\beta+1})^{\bar{p}-2}$ by the automorphism $x \mapsto xy^{-1}$. Then the automorphism $y^{-1} \mapsto y^{-1}x^{-\beta-1}$ maps

$$y^{-1}x^{2\beta+1}(y^{-1}x^{\beta+1})^{\bar{p}-2}$$

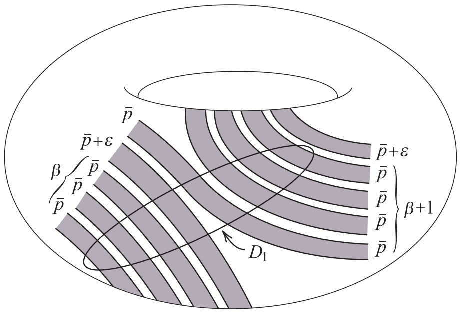

FIGURE 20. A torus knot $T(p,q)$ and a disk D_1 containing r parallel arcs of $T(p,q)$, where $p=(\beta+2)\bar{p}+\epsilon, q=(\beta+1)\bar{p}+\epsilon$, and $r=2(\beta+1)\bar{p}+\epsilon$.

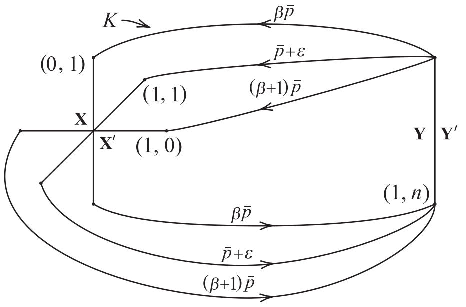

FIGURE 21. R-R diagram of K.

to $y^{-\bar{p}+1}x^{\beta}$. Therefore a curve representing x^{β} can be a regular fiber of $H[K]$. The curve τ in Figure 19b which has $(xy)^{\beta}$ in $\pi_1(H)$ is a regular fiber of $H[K]$ and also intersects a regular fiber τ' of $H'[K]$ once transversely as in Figure 19b. □

Lemma 4.4. *Let $K=K(p,q,r,m,n)$ be a twisted torus knot lying in a genus two Heegaard surface Σ of S^3 such that K is of type* IV *in Table* 1*, i.e.,* $(p,q,r,m,n)=((\beta+2)\bar{p}+\epsilon,(\beta+1)\bar{p}+\epsilon,2(\beta+1)\bar{p}+\epsilon,1,n)$ *with* $\beta>1$, $\epsilon=\pm1$, $2\bar{p}+\epsilon>1$. *Then for the surface slope γ, $K(\gamma)$ is a non-Seifert-fibered graph manifold consisting of $D^2(\beta,2\bar{p}+\epsilon)$ and $D^2((\beta+1)\bar{p}+\epsilon,n)$.*

Proof. A torus knot $T(p,q)$ and a disk D_1 containing r parallel arcs of $T(p,q)$ in V_1 are illustrated in Figure 20, where $p=(\beta+2)\bar{p}+\epsilon, q=(\beta+1)\bar{p}+\epsilon$, and $r=2(\beta+1)\bar{p}+\epsilon$. The corresponding R-R diagram of K is depicted in Figure 21.

We start with the $\bar{p}$ parallel edges which are innermost in the $\beta\bar{p}$ parallel edges entering into the $(0,1)$-connection in the (X,X')-handle. Then as handled in the previous types, we can show that when $\epsilon=1$, K is the product of two subwords

$$x^0yxyxyxy(x^0yxy)^{\beta-1}=(xy)^2(xy^2)^{\beta} \text{ and}$$
$$x^0yxyxyxyxy(x^0yxy)^{\beta-1}=(xy)^3(xy^2)^{\beta}$$

with $|(xy)^2(xy^2)^{\beta}|=\bar{p}-1$ and $|(xy)^3(xy^2)^{\beta}|=1$, and when $\epsilon=-1$, K is the product of two subwords

$$x^0yxyxy(x^0yxy)^{\beta-1}=xy(xy^2)^{\beta} \text{ and}$$
$$x^0yxyxyxy(x^0yxy)^{\beta-1}=(xy)^2(xy^2)^{\beta}$$

with $|xy(xy^2)^{\beta}|=1$ and $|(xy)^2(xy^2)^{\beta}|=\bar{p}-1$.

This can be confirmed from Figure 22, which shows the R-R diagram of K when $\beta=2$. After applying two automorphisms $y\mapsto x^{-1}y$ and $x^{-1}\mapsto x^{-1}y^{-2}$ consecutively, when $\epsilon=1$, they are sent to $y^2x^{-\beta}$ and $y^3x^{-\beta}$ respectively, and when $\epsilon=-1$, they are sent to $yx^{-\beta}$ and $y^2x^{-\beta}$ respectively. Therefore in both cases, a curve representing $x^{-\beta}$ is a regular fiber of $H[K]$. The curve τ in the original R-R diagram of K shown in Figure 22, where $\beta=2$, represents $(x^0yxy)^{\beta}$ in $\pi_1(H)$, which is sent to $x^{-\beta}$ after performing $y\mapsto x^{-1}y$ and $x^{-1}\mapsto x^{-1}y^{-2}$ consecutively as performed to K. Thus τ is a regular fiber of $H[K]$ and intersects a regular fiber τ' of $H'[K]$ once transversely as in Figure 22. □

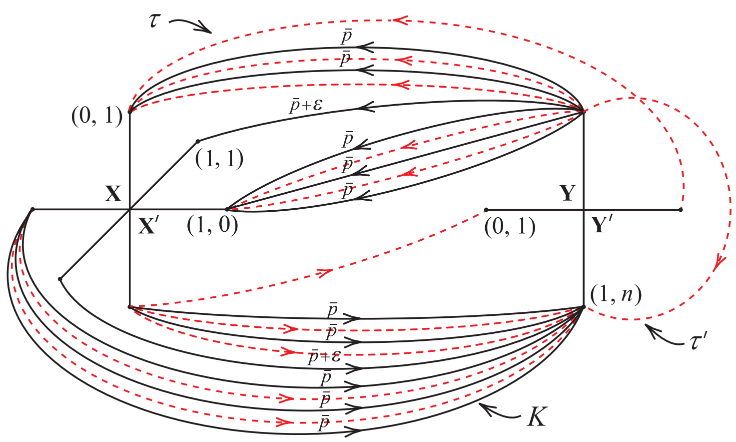

FIGURE 22. Regular fibers τ and τ' of $H[K]$ and $H'[K]$ respectively when $\beta=2$.

Lemma 4.5. *Let $K = K(p, q, r, m, n)$ be a twisted torus knot lying in a genus two Heegaard surface Σ of S^3 such that K is of type* V *in Table* 1*, i.e., $(p, q, r, m, n) = ((\alpha+1)\bar{p}+\bar{q}, \alpha\bar{p}+\bar{q}, (\alpha-1)\bar{p}+\bar{q}, 1, n)$ with $0 \le \bar{q} < \bar{p}$, $\alpha > 0$. Then for the surface slope γ, $K(\gamma)$ is a non-Seifert-fibered graph manifold consisting of $D^2(2, (\alpha-1)\bar{p}+\bar{q})$ and $D^2(\alpha\bar{p}+\bar{q}, n)$.*

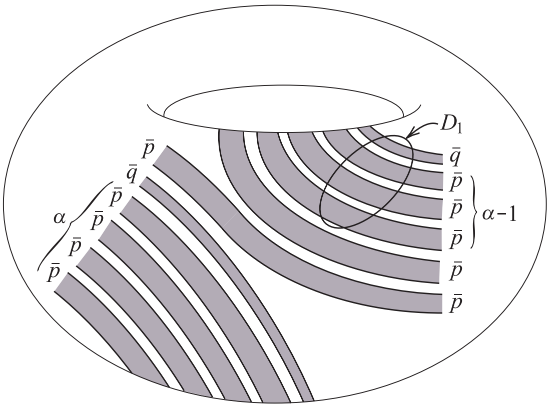

FIGURE 23. A torus knot $T(p, q)$ and a disk D_1 containing r parallel arcs of $T(p, q)$, where $p = (\alpha+1)\bar{p}+\bar{q}, q = \alpha\bar{p}+\bar{q}$, and $r = (\alpha-1)\bar{p}+\bar{q}$.

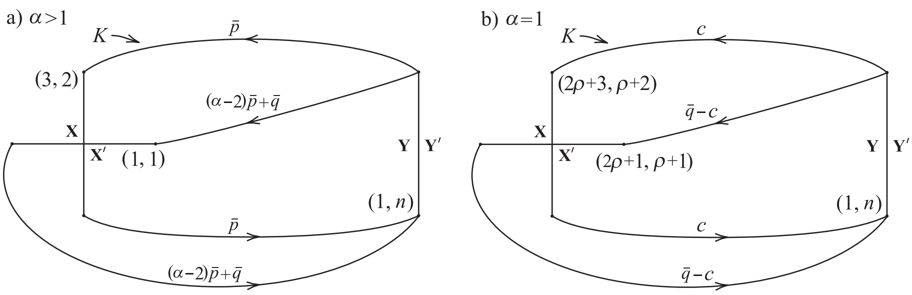

FIGURE 24. R-R diagram of K when a) $\alpha > 1$ and b) $\alpha = 1$.

Proof. Figure 23 shows a torus knot $T(p, q)$, where $p = (\alpha+1)\bar{p}+\bar{q}$ and $q = \alpha\bar{p}+\bar{q}$, and a disk D_1 containing $r = (\alpha-1)\bar{p}+\bar{q}$ parallel arcs of $T(p, q)$ in V_1. The corresponding R-R diagram depends on the value α. If $\alpha > 1$, then the R-R diagram is shown in Figure 24a. If $\alpha = 1$, we let $\bar{p} = \rho\bar{q}+c$, where $0 < c < \bar{p}$. Here $c \ne 0$, otherwise $\bar{q} = 1$ and $H[K] = D^2(2, (\alpha-1)\bar{p}+\bar{q})$ is a solid torus, a contradiction. The R-R diagram of K has the form shown in Figure 24b. Observe that the two R-R diagrams have the same form. Therefore

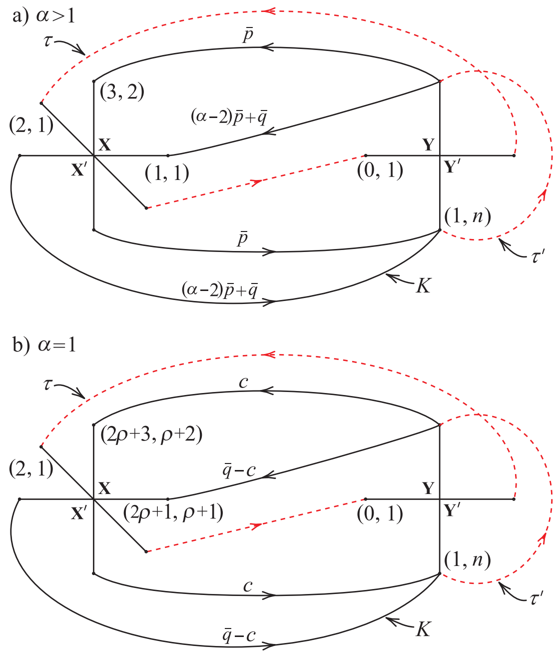

FIGURE 25. Regular fibers τ and τ' of $H[K]$ and $H'[K]$ respectively.

finding regular fibers when $\alpha = 1$ can be achieved in the same manner as when $\alpha > 1$.

Assume that $\alpha > 1$. As before, we record the word of K with respect to H from the $\bar{p}$ parallel arcs entering into the $(3, 2)$-connection in the (X, X')-handle. Then it follows that K is the product of two subwords

$$x^3(yx)^{\alpha-2}y \text{ and } x^3(yx)^{\alpha-2}yxy$$

with $|x^3(yx)^{\alpha-2}y| = \bar{p} - \bar{q}$ and $|x^3(yx)^{\alpha-2}yxy| = \bar{q}$.

An automorphism $y \mapsto yx^{-1}$ sends them into $x^2y^{\alpha-1}$ and x^2y^{α} respectively. Thus a curve representing x^2 is a regular fiber of $H[K]$. The curve τ in the original R-R diagram of K shown in Figure 25a represents x^2 in $\pi_1(H)$, which is sent to x^2 after performing the automorphism $y \mapsto yx^{-1}$. Therefore τ is a regular fiber of $H[K]$ and intersects a regular fiber τ' of $H'[K]$ once as shown in Figure 25.

For the case that $\alpha = 1$, we apply a similar argument. Then regular fibers τ and τ' appear in the R-R diagram of K as in Figure 25b, which shows that they intersect once transversely. □

Lemma 4.6. *Let $K = K(p, q, r, m, n)$ be a twisted torus knot lying in a genus two Heegaard surface Σ of S^3 such that K is of type* VI *in Table* 1*, i.e.,*

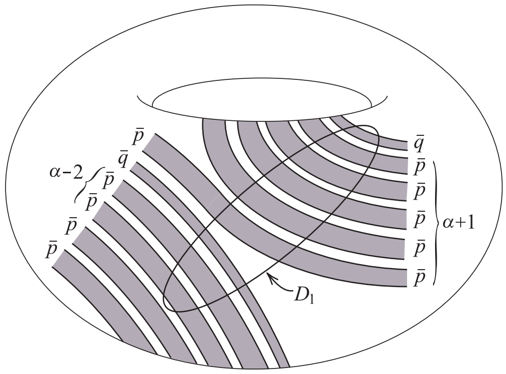

FIGURE 26. A torus knot $T(p,q)$ and a disk D_1 containing r parallel arcs of $T(p,q)$, where $p = (\alpha+1)\bar{p}+\bar{q}, q = \alpha\bar{p}+\bar{q}$, and $r = (2\alpha-1)\bar{p}+2\bar{q}$.

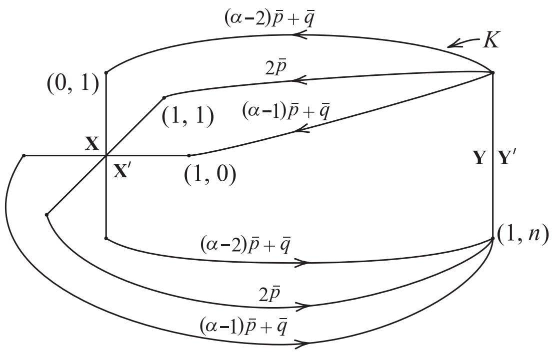

FIGURE 27. R-R diagram of K.

$(p,q,r,m,n) = ((\alpha+1)\bar{p}+\bar{q}, \alpha\bar{p}+\bar{q}, (2\alpha-1)\bar{p}+2\bar{q}, 1, n)$ *with* $0 \le \bar{q} < \bar{p}$, $\alpha > 1$. *Then for the surface slope* γ, $K(\gamma)$ *is a non-Seifert-fibered graph manifold consisting of* $D^2(3, (\alpha-2)\bar{p}+\bar{q})$ *and* $D^2(\alpha\bar{p}+\bar{q}, n)$.

Proof. A torus knot $T(p,q)$, where $p = (\alpha+1)\bar{p}+\bar{q}$ and $q = \alpha\bar{p}+\bar{q}$, and a disk D_1 containing $r = (2\alpha-1)\bar{p}+2\bar{q}$ are depicted in Figure 26. Also its corresponding R-R diagram of K is given in Figure 27.

As done in Type IV, we start with the $\bar{p}$ parallel arcs which are innermost in the $2\bar{p}$ parallel arcs entering into the $(1,1)$-connection in the (X,X')-handle. Then K is the product of two subwords

$$xyxy(x^0yxy)^{\alpha-2}(xy) = (xy^2)^{\alpha-2}(xy)^3 \text{ and}$$
$$xyxy(x^0yxy)^{\alpha-2}x^0yxy(xy) = (xy^2)^{\alpha-1}(xy)^3$$

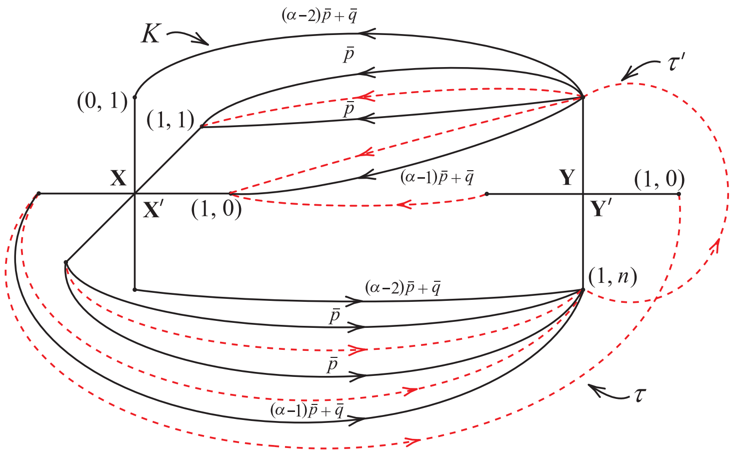

FIGURE 28. Regular fibers τ and τ' of $H[K]$ and $H'[K]$ respectively.

with $|(xy^2)^{\alpha-2}(xy)^3| = \bar{p} - \bar{q}$ and $|(xy^2)^{\alpha-1}(xy)^3| = \bar{q}$.

They are carried into $y^{\alpha-2}x^3$ and $y^{\alpha-1}x^3$ respectively by two automorphisms $x \mapsto xy^{-1}$ and $y \mapsto x^{-1}y$ in a row. Therefore a curve representing x^3 is a regular fiber of $H[K]$. The curve τ in the R-R diagram of K of the form shown in Figure 28 has a word $(xy)^3$, which is sent to x^3 by two automorphisms $x \mapsto xy^{-1}$ and $y \mapsto x^{-1}y$. This implies that the curve τ is a regular fiber of $H[K]$ and intersects a regular fiber τ' of $H'[K]$ once transversely as shown in Figure 28. □

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