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ON THE IMPROVED REGULARITY CRITERION OF THE SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

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Abstract. This note deals with the question of the regularity of (Leray) weak solutions of the Navier-Stokes equations in terms of the pressure. This criterion improves on the existing results.

1. Introduction

This paper studies the regularity criterion for weak solutions of the Navier-Stokes equations in $\mathbb{R}^3 \times [0,T)$:

$$\left\{ \begin{array}{l} \partial_t u + (u\cdot\nabla)\,u - \Delta u + \nabla\pi = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \end{array} \right. \tag{1.1}$$

where $u = u(x,t)$ is the velocity field, $\pi = \pi\,(x,t)$ is the scalar pressure of the fluid, while u_0 is a given initial velocity field satisfying $\nabla\cdot u_0 = 0$ in the sense of distributions.

For $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla\cdot u_0 = 0$, a global weak solution of (1.1) satisfying

$$u \in L^\infty\left(\mathbb{R}_+; L^2(\mathbb{R}^3)\right) \cap L^2\left(\mathbb{R}_+; H^1(\mathbb{R}^3)\right)$$

was constructed by Leray [15] and Hopf [14] in 1951. It is proved that the weak solution is strong (and unique) locally if the initial datum $u_0 \in H^1(\mathbb{R}^3)$ in addition, and the strong solution exists globally for small initial datum (see [16, 17]). However, the regularity of their weak solutions is one of the most outstanding open problems in mathematical fluid mechanics and has been extensively investigated and many interesting results have been established (see e.g. [1–11, 20–22, 24] and reference therein).

In [13], He and Gala proved the regularity of weak solutions under the condition

$$\int_0^T \|\pi(\cdot,t)\|^2_{\dot{B}^{-1}_{\infty,\infty}}\, dt < \infty. \tag{1.2}$$

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Here and thereafter, $\dot{B}^{-1}_{\infty,\infty}$ stands for the homogeneous Besov space, (for the definition see e.g. [13] and [23]). Later, Guo and Gala [23] refined the condition (1.2) to

$$\int_0^T \frac{\|\pi(\cdot,t)\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{1+\ln\left(e+\|\pi(\cdot,t)\|_{\dot{B}^{-1}_{\infty,\infty}}\right)}\,dt<\infty. \tag{1.3}$$

Motivated by the paper of Guo and Gala [23], the purpose of the present work is to refine (1.3) as follows.

Theorem 1.1. *Suppose that $u(x,t)$ is a weak solution of* (1.1) *in* $(0,T)$ *with* $u_0\in H^1(\mathbb{R}^3)$ *and* $\nabla\cdot u_0=0$ *in the sense of distributions. If the pressure π satisfies the following condition*:

$$\int_0^T \frac{\|\pi(\cdot,t)\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\left(e+\ln\left(e+\|\pi(\cdot,t)\|_{\dot{B}^{-1}_{\infty,\infty}}\right)\right)\ln\left(e+\ln\left(e+\|\pi(\cdot,t)\|_{\dot{B}^{-1}_{\infty,\infty}}\right)\right)}\,dt<\infty, \tag{1.4}$$

then u is a regular solution in $\mathbb{R}^3\times(0,T)$.

This result provides a new information concerning the question of the regularity of weak solutions of the Navier-Stokes equations and extend those of [13] and [23]. In particular, the double-logarithm estimate (1.4) is sharper than any other results [4, 23].

Let us recall the definition of weak solution to (1.1).

Definition 1.1 (Weak solutions)**.** Let $u_0\in L^2(\mathbb{R}^3)$ with $\nabla\cdot u_0=0$. A measurable function defined on $\mathbb{R}^3\times(0,T)$ is called a weak solution of (1.1) on $(0,T)$ with initial data u_0, if u satisfies the following properties:

a) Leray-Hopf class:
$$u\in L^\infty\left((0,T)\,;L^2(\mathbb{R}^3)\right)\cap L^2\left((0,T)\,;H^1(\mathbb{R}^3)\right);$$
b) $\partial_t u+(u\cdot\nabla)\,u-\Delta u+\nabla\pi=0$ in $\mathcal{D}'\left(\mathbb{R}^3\times[0,T)\right)$;
c) $\nabla\cdot u=0$ in $\mathcal{D}'\left(\mathbb{R}^3\times[0,T)\right)$;
d) the energy inequality:
$$\|(u(\cdot,t)\|^2_{L^2}+2\int_0^t\|\nabla u(\cdot,\tau)\|^2_{L^2}d\tau\le\|u_0\|^2_{L^2} \tag{1.5}$$
for all $0\le t\le T$.

By a strong solution we mean a weak solution of the Navier–Stokes equation such that
$$u\in L^\infty\left((0,T)\,;H^1(\mathbb{R}^3)\right)\cap L^2\left((0,T)\,;H^2(\mathbb{R}^3)\right).$$
It is well known that strong solutions are regular (we say classical) and unique in the class of weak solutions.

2. Proof of Theorem 1.1

We now give the proof of our main result.

Proof. Before going to the proof, we recall the following inequality established in [12] (see also [23]):

$$\|f\|_{L^4}^2 \leq C \|f\|_{\dot{B}^{-1}_{\infty,\infty}} \|\nabla f\|_{L^2}. \tag{2.1}$$

Testing $(1.1)_1$ by $|u|^2 u$, using $(1.1)_2$ and (2.1), we see by Hölder's inequality that

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\|u(\cdot,t)\|_{L^4}^4 + \int_{\mathbb{R}^3} |\nabla u|^2 |u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \left|\nabla |u|^2\right|^2 dx \\
=\ & -\int_{\mathbb{R}^3} \nabla\pi \cdot |u|^2 u dx = \int_{\mathbb{R}^3} \pi u \cdot \nabla |u|^2 dx \\
\leq\ & \int_{\mathbb{R}^3} |\pi|\, |u| \left|\nabla |u|^2\right| dx \\
\leq\ & \frac{1}{4}\int_{\mathbb{R}^3} \left|\nabla |u|^2\right|^2 dx + C\int_{\mathbb{R}^3} |\pi|^2 |u|^2 dx \\
\leq\ & \frac{1}{4}\int_{\mathbb{R}^3} \left|\nabla |u|^2\right|^2 dx + C\|\pi\|_{\dot{B}^{-1}_{\infty,\infty}} \|\nabla\pi\|_{L^2} \|u\|_{L^4}^2 \\
\leq\ & \frac{1}{4}\int_{\mathbb{R}^3} \left|\nabla |u|^2\right|^2 dx + \frac{1}{2}\| |\nabla u|\, |u| \|_{L^2}^2 + C\|\pi\|_{\dot{B}^{-1}_{\infty,\infty}}^2 \|u\|_{L^4}^4,
\end{aligned}$$

which yields

$$\|u(\cdot,t)\|_{L^4} \leq \|u_0\|_{L^4} \exp\left(C\int_0^t \|\pi(\cdot,\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}^2 d\tau \right). \tag{2.2}$$

Here, we have used the elementary inequality (see e.g. [18, 19])

$$\|\nabla\pi\|_{L^2} \leq C\|u\cdot\nabla u\|_{L^2}.$$

Next, testing $(1.1)_1$ by $-\Delta u$, using $(1.1)_2$, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla u(\cdot,t)\|_{L^2}^2 + \|\Delta u(\cdot,t)\|_{L^2}^2 &= \int_{\mathbb{R}^3} (u\cdot\nabla u)\cdot\Delta u dx \\
&\leq \|u\|_{L^4} \|\nabla u\|_{L^4} \|\Delta u\|_{L^2} \\
&\leq C\|u\|_{L^4}^{\frac{6}{5}} \|\Delta u\|_{L^2}^{\frac{9}{5}} \\
&\leq \frac{1}{2}\|\Delta u\|_{L^2}^2 + C\|u\|_{L^4}^{12},
\end{aligned}$$

where we have used the Gagliardo-Nirenberg inequality:

$$\|\nabla u\|_{L^4} \leq C\|u\|_{L^4}^{\frac{1}{5}} \|\Delta u\|_{L^2}^{\frac{4}{5}}.$$

Integrating the above inequality over $(0,t)$, we have

$$\|\nabla u(\cdot,t)\|_{L^2}^2+\int_0^t\|\Delta u(\cdot,\tau)\|_{L^2}^2\,d\tau\le\|\nabla u_0\|_{L^2}^2+C\int_0^t\|u(\cdot,\tau)\|_{L^4}^{12}\,d\tau. \tag{2.3}$$

On the other hand, by a Sobolev embedding theorem $\dot{H}^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$, (2.3) and (2.2), we obtain that

$$\begin{aligned}
e+\|\pi(\cdot,t)\|_{L^3}&\le e+C\|u(\cdot,t)\|_{L^6}^2\\
&\le e+C\|\nabla u(\cdot,t)\|_{L^2}^2\\
&\le e+C\|\nabla u_0\|_{L^2}^2+C\int_0^t\|u(\cdot,\tau)\|_{L^4}^{12}\,d\tau\\
&\le e+C\|\nabla u_0\|_{L^2}^2+C(e+t)\sup_{0\le\tau\le t}\|u(\cdot,\tau)\|_{L^4}^{12}\\
&\le C\left(e+\|\nabla u_0\|_{L^2}^2\right)(e+t)\sup_{0\le\tau\le t}\|u(\cdot,\tau)\|_{L^4}^{12}\\
&\le C_0(e+t)\exp\left(C\int_0^t\|\pi(\cdot,\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}^2\,d\tau\right),
\end{aligned} \tag{2.4}$$

where the constant $C_0=C(e,\|\nabla u_0\|_{L^2},\|u_0\|_{L^4})$. Here, we have used the elementary inequality (see e.g. [18,19]):

$$\|\pi\|_{L^q}\le C\|u\|_{L^{2q}}^2\quad\text{for all }\ 1<q<\infty.$$

Using the fact that $L^3(\mathbb{R}^3)\subset\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$, it follows that

$$e+\|\pi(\cdot,t)\|_{\dot{B}^{-1}_{\infty,\infty}}\le C(e+t)\exp\left(C\int_0^t\|\pi(\cdot,\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}^2\,d\tau\right). \tag{2.5}$$

Now, taking "log" on both sides of (2.5), we can conclude that

$$\ln(e+\|\pi(\cdot,t)\|_{\dot{B}^{-1}_{\infty,\infty}})\le\ln(C(e+t))+C\int_0^t\|\pi(\cdot,\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}^2\,d\tau. \tag{2.6}$$

For simplicity, let

$$\mathcal{Z}(t)=\ln(e+\|\pi(\cdot,t)\|_{\dot{B}^{-1}_{\infty,\infty}}),$$

$$\mathcal{E}(t)=\ln(C(e+t))+C\int_0^t\|\pi(\cdot,\tau)\|_{\dot{B}^{-1}_{\infty,\infty}}^2\,d\tau, \tag{2.7}$$

with $\mathcal{E}(0)=\ln(Ce)$. Then, the above inequality (2.6) implies that

$$0<\mathcal{Z}(t)\le\mathcal{E}(t)$$

and we easily get

$$(e+\mathcal{Z}(t))\ln(e+\mathcal{Z}(t))\le(e+\mathcal{E}(t))\ln(e+\mathcal{E}(t)).$$

On the other hand, we have

$$\begin{aligned}\frac{d}{dt}\ln(e+\mathcal{E}(t)) &= \frac{1}{e+\mathcal{E}(t)}\left(\frac{1}{e+t}+C\left\|\pi(\cdot,t)\right\|^2_{\dot{B}^{-1}_{\infty,\infty}}\right)\\ &\leq \frac{1}{e^2}+C\frac{\left\|\pi(\cdot,t)\right\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{e+\mathcal{E}(t)}\\ &= \frac{1}{e^2}+C\frac{\left\|\pi(\cdot,t)\right\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{(e+\mathcal{E}(t))\ln(e+\mathcal{E}(t))}\ln(e+\mathcal{E}(t))\\ &\leq \frac{1}{e^2}+C\frac{\left\|\pi(\cdot,t)\right\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{(e+\mathcal{Z}(t))\ln(e+\mathcal{Z}(t))}\ln(e+\mathcal{E}(t)).\end{aligned}$$

Applying the Gronwall inequality to $\ln(e+\mathcal{E}(t))$, we find

$$\begin{aligned}&\ln(e+\mathcal{E}(t))\\ \leq\ &\ln(e+\mathcal{E}(0))\exp\left(\frac{T}{e^2}+C\int_0^t\frac{\left\|\pi(\cdot,\tau)\right\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{(e+\mathcal{Z}(\tau))\ln(e+\mathcal{Z}(\tau))}d\tau\right),\end{aligned}$$

which yields

$$e+\mathcal{E}(t)\leq(e+\mathcal{E}(0))^{\exp\left(\frac{T}{e^2}+C\int_0^t\frac{\left\|\pi(\cdot,\tau)\right\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{(e+\mathcal{Z}(\tau))\ln(e+\mathcal{Z}(\tau))}d\tau\right)}$$

and from (2.7), we deduce that

$$\int_0^t\left\|\pi(\cdot,\tau)\right\|^2_{\dot{B}^{-1}_{\infty,\infty}}d\tau\leq(e+\mathcal{E}(0))^{\exp\left(\frac{T}{e^2}+\frac{1}{C}\int_0^t\frac{\left\|\pi(\cdot,\tau)\right\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{(e+\mathcal{Z}(\tau))\ln(e+\mathcal{Z}(\tau))}d\tau\right)}<\infty. \tag{2.8}$$

Hence by virtue of (1.5), (2.2), (2.3) and (2.8), we conclude that

$$u\in L^\infty\left((0,T)\,;H^1(\mathbb{R}^3)\right)\cap L^2\left((0,T)\,;H^2(\mathbb{R}^3)\right),$$

which completes the proof of Theorem 1.1. □

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