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GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR OF A PLATE EQUATION WITH A CONSTANT DELAY TERM AND LOGARITHMIC NONLINEARITIES

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Abstract. In this paper, we investigate the viscoelastic plate equation with a constant delay term and logarithmic nonlinearities. Under some conditions, we will prove the global existence. Furthermore, we use weighted spaces to establish a general decay rate of solution.

1. Introduction

In this work, we consider the following Cauchy problem with logarithmic nonlinearity

$$
\begin{cases}
u_{tt}(x,t)+\phi(x)\left(\alpha\Delta^2 u-\displaystyle\int_0^t g(t-s)\Delta^2 u(x,s)ds\right) & \\
+\mu_1(t)u_t(x,t)+\mu_2(t)u_t(x,t-\tau)=u\ln|u|^k & \text{in } \mathbb{R}^n\times]0,+\infty[,\\
u(x,t)=0, & \text{on } \partial\mathbb{R}^n\times]0,+\infty[,\\
u(x,0)=u_0(x), u_t(x,0)=u_1(x), & \text{in } \mathbb{R}^n,\\
u_t(x,t-\tau)=f_0(x,t-\tau), & \text{in } \mathbb{R}^n\times]0,t[.
\end{cases}
\tag{1.1}
$$

Where $n\geq 1$, $\phi(x)>0$ and $(\phi(x))^{-1}=\rho(x)$ such that ρ is a function that will be defined later. The initial datum u_0,u_1,f_0 are given functions belonging to suitable spaces that will be specified later. μ_1,μ_2 are real functions and g is a positive non-increasing function defined on $\mathbb{R}^+$. Moreover $\tau>0$ represents the time delay term.

It is well known that the logarithmic nonlinearity is distinguished by several interesting physical properties. In recent years, there has been a growing interest in the viscoelastic wave equation, its properties and variants of the problem can be found for example in ([4, 6, 10, 11, 15]). The plate equation in $\mathbb{R}^n$ has been studied by many authors and some results have been proved (see for instance [1, 7, 9]) and the references therein.

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In the case of $k = 2$, T. Cazenave and A. Haraux, in [4] studied the following problem

$$\begin{cases} u_{tt}(x,t) + \Delta u - u \log |u|^2 + u_t + u|u|^2 = 0 & \text{in } \Omega\times]0,T[, \\ u(x,t) = 0, & \text{on } \partial\Omega\times]0,T[, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega. \end{cases}$$

They established the global existence of weak solutions to the problem by using Galerkin method, logarithmic Sobolev inequality and compactness theorem.

The author in [8] looked into a linear Cauchy viscoelastic problem with density. He obtained the exponential and polynomial rates by using the spaces weighted by density to compensate for the lack of Poincare's inequality.

In the case of delay term, Nicaise, Valein and Fridman in [14] proved an exponential stability result under the condition $\mu_1 < \sqrt{1-d}\mu_1$, where d is a constant such that $\tau'(t) \leq d < 1$. After that, S. Nicaise, C. Pignotti in [13] considered the following problem

$$\begin{cases} u_{tt}(x,t) - \Delta u = 0, & \text{in } \Omega \times [0,+\infty[, \\ u(x,t) = 0, & \text{on } \Gamma_D \times [0,+\infty[, \\ \dfrac{\partial u_t}{\partial \nu} = \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau(t)), & \text{on } \Gamma_N \times [0,+\infty[, \\ u(x,0) = u_0, u'(x,0) = u_1, & \text{in } \Omega, \\ u_t(x,t) = f_0(x,t-\tau(t)), & \text{on } \Gamma_N \times (0,\tau(0)). \end{cases} \tag{1.2}$$

Under suitable assumptions,they proved exponential stability of the solution.

A. Benaissa, A. Benguessoum and S. A. Messaoudi in [2] considered the wave equation with a weak internal constant delay term

$$u''(x,t) - \triangle u + \mu_1(t)u'(x,t) + \mu_2(t)u'(x,t-\tau) = 0 \text{ on } [0,+\infty[,$$

in a bounded domain, with $u'' = \frac{\partial^2 u}{\partial t}$ and $u' = \frac{\partial u}{\partial t}$. Under appropriate conditions on μ_1 and μ_2, they proved global existence of solutions by the Faedo-Galerkinś method and establish a decay rate estimate for the energy by using the multiplier method.

K. Bouhali and F. Ellaggoune in [3] studied in any spaces dimension, a general decay rate of solutions of viscoelastic wave equations with logarithmic nonlinearities. Furthermore, they established, under convenient hypotheses on g and the initial data, the existence of weak solution associated to this equation.

The content of this paper is organized as follows. In Section 2, we provide assumptions and lemmas that will be used later. In Section 3, we state and prove the existence result. In Section 4, we prove a result of polynomial stability of the solution.

2. Preliminaries

We first recall some basic definitions and abstract results on weighted spaces. We define the function spaces of our problem and its norm as follows

$$\mathcal{D}^{2,2}(\mathbb{R}^n) = \left\{ f \in L^{2n/n-4}(\mathbb{R}^n) \ / \ \Delta_x f \in L^2(\mathbb{R}^n) \right\},$$

and $\mathcal{D}^{2,2}(\mathbb{R}^n)$ can be embedded continuously in $L^{2n/n-4}(\mathbb{R}^n)$. The space $L^2_\rho(\mathbb{R}^n)$ to be the closure of $C_0^\infty(\mathbb{R}^n)$

$$\|f\|_{L^q_\rho} = \left(\int_{\mathbb{R}^n} \rho |f|^q dx \right)^{\frac{1}{q}}.$$

The space $L^2_\rho(\mathbb{R}^n)$ is a separable Hilbert space.

In the following, we will give sufficient conditions and assumptions that guarantee the global existence of the problem (1.1).

(H_1) : g is a positive bounded function satisfying

$$\alpha - \int_0^\infty g(s)ds = l > 0, \quad \alpha > 0, \tag{2.1}$$

and there exists a positive non-increasing function $H \in C^2(\mathbb{R}^+)$ such that, for $t \geq 0$, we have

$$g'(t) \leq -H(t)g(t), \quad H(0) = 0, \tag{2.2}$$

where H is linear or strictly increasing and strictly convex function on $(0, r], r < 1$.

(H_2) : According to results in [6]

(1) we can deduce that there exists $t_1 > 0$ large enough such that $\forall t \geq t_1$, we have $\lim\limits_{t\to\infty} g(s) = 0$ so $\lim\limits_{t\to\infty} g'(s) = 0$ and $g(t_1) > 0$.
Then
$$\max\{g(s), -g'(s)\} < \min\{r, H(s), H_0(s)\},$$
where $H_0(t) = H(D(t))$, and D is a positive C^1 function, with $D(0) = 0$, for which H_0 is strictly increasing and strictly convex function on $(0; r]$ and
$$\int_0^\infty g(s)H_0(-g'(s))ds < \infty.$$

(2) For $0 \leq t \leq t_1$ we have $g(0) \leq g(t) \leq g(t_1)$, ($g$ is non-increasing).
Since H is a positive continuous function, then
$$g'(t) < H(g(t)) \leq -kg(t), \quad k > 0.$$

(3) Let ${H_0}^*$ be the convex conjugate of H_0 in the sense of Young (see [1]), then
$${H_0}^* = s(H_0')^{-1}(s) - H_0((H_0')^{-1}(s)), \quad s \in (0, (H_0'(r)),$$
and satisfies the following Young's inequality
$$AB \leq {H_0}^*(A) + H_0(B), \quad A \in (0, (H_0'(r)), \quad B \in (0, r].$$

(H_3) : For the functions μ_1, μ_2 we assume

(1) μ_1 is a positive function of class C^1 satisfying:

$$\Big|\frac{\mu_1'(t)}{\mu_1(t)}\Big| \leq M, \qquad M > 0. \tag{2.3}$$

(2) μ_2 is a real function of class C^1 such that

$$\mu_2(t) \leq \beta\mu_1(t), \quad 0 < \beta < 1$$
$$\mu_2'(t) \leq \tilde{M}\mu_1(t), \quad \tilde{M} > 0.$$

(H_4) : The function $\rho : \mathbb{R}^n \to \mathbb{R}^n_+$ satisfies $\rho(x) \in C^{0,\gamma}(\mathbb{R}^n)$ with $\gamma \in (0,1)$ and $\rho \in L^s(\mathbb{R}^n)$, where $s = \frac{2n}{2n-qn+4q}$.

We also need the following technical lemmas in the course of our investigation.

Let λ_1 be the first eigenvalue of the spectral Dirichlet problem

$$\Delta u = \lambda_1 u \quad \text{in } \mathbb{R}^n, \qquad u = \frac{\partial u}{\partial \eta} = 0 \quad \text{in } \partial\mathbb{R}^n,$$
$$\|\nabla u\|_2 \leq \omega \|\Delta u\|_2,$$

where $\omega = \frac{1}{\sqrt{\lambda_1}}$.

Lemma 2.1 ([5]). *Assume that the function ρ satisfies the assumption (H_4). Then for any $u \in \mathcal{D}^{2,2}(\mathbb{R}^n)$ we have*

$$\|u\|_{L^2_\rho(\mathbb{R}^n)} \leq C_0 \|\Delta u\|_{L^2(\mathbb{R}^n)},$$

where $C_0 = \|\rho\|_{L^s_\rho}$, with $s = \frac{2n}{2n-qn+4q}$ and $2 \leq q \leq \frac{2n}{n-4}$.

Lemma 2.2 ([11]). *For any $g \in C^1$ and $\varphi \in H^1_0(0,T)$, we have*

$$\begin{aligned} -2\int_0^t \int_{\mathbb{R}^n} g(t-s)\varphi\ \varphi_t dx ds &= g(t)\|\varphi\|_2^2 - (g' \circ \varphi)(t) \\ &\quad + \frac{d}{dt}\Big((g \circ \varphi)(t) - \int_0^t g(s)ds\|\varphi\|_2^2\Big), \end{aligned} \tag{2.4}$$

where

$$(g \circ \varphi)(t) = \int_0^t g(t-s) \int_{\mathbb{R}^n} |(\varphi(s) - \varphi(t))|^2 dx ds.$$

Lemma 2.3 ([12]). *Let $u \in \mathcal{D}^{2,2}(\mathbb{R}^n)$ and $c_1, c_2 > 0$ be two numbers. Then*

$$2\int_{\mathbb{R}^n} \rho(x)|u|^2 \ln\Big(\frac{|u|}{\|u\|^2_{L^2_\rho}}\Big)dx + n(1+c_1)\|u\|^2_{L^2_\rho} \leq c_2 \frac{\|\rho\|^2_{L^2}}{\pi}\|\nabla u\|_2^2. \tag{2.5}$$

If u is a solution of the problem (1.1) and $v \in \mathcal{D}^{2,2}(\mathbb{R}^n)$, then

$$\begin{aligned} \int_{\mathbb{R}^n} \rho(x)|u|\ln|u|^k v dx = &\int_{\mathbb{R}^n} \rho(x) u_{tt} v dx - \int_{\mathbb{R}^n} \rho(x)\Delta u \Delta v dx \\ &+ \int_{\mathbb{R}^n} \alpha \Delta u \Delta v dx - \int_{\mathbb{R}^n}\int_0^t g(t-s)\Delta u(x,s)\Delta v ds dx \end{aligned} \tag{2.6}$$

$$+\mu_1\int_{\mathbb{R}^n}(t)\rho(x)u_t v dx+\mu_2(t)\int_{\mathbb{R}^n}\rho(x)u_t(x,t-\tau)v dx.$$

Lemma 2.4 ([11])**.** *Let $u\in\mathcal{D}^{2,2}(\mathbb{R}^n)$. Then we have*

$$\begin{aligned}&\Big(\int_0^t g(t-s)(u(s)-u(t))ds\Big)^2\\ &\le\Big(\int_0^t|g(s)|^{2(1-\theta)}ds\Big)\Big(\int_0^t|g(t-s)|^{2\theta}|(u(s)-u(t))|^2ds\Big).\end{aligned}\tag{2.7}$$

Like in [14] we introduce the auxiliary unknown

$$z(x,\gamma,t)=u_t(x,t-\tau\gamma),\ x\in\mathbb{R}^n,\ \gamma\in(0,1),\ t>0.$$

Then, we have

$$\tau z_t(x,\gamma,t)+z_\gamma(x,\gamma,t)=0.$$

Therefore the problem (1.1) takes the form

$$\begin{cases}u_{tt}(x,t)+\phi(x)\left(\alpha\Delta^2u-\displaystyle\int_0^t g(t-s)\Delta^2u(x,s)ds\right)\\ +\mu_1(t)u_t(x,t)+\mu_2(t)z(x,1,t)=u\ln|u|^k, & \text{in } \mathbb{R}^n\times]0,+\infty[,\\ \tau z_t(x,\gamma,t)+z_\rho(x,\gamma,t)=0, & \text{in } \mathbb{R}^n\times]0,+\infty[,\\ u(x,t)=0, & \text{on } \partial\mathbb{R}^n\times]0,+\infty[,\\ u(x,0)=u_0(x),u_t(x,0)=u_1(x), & \text{in } \mathbb{R}^n,\\ z(x,\gamma,0)=f_0(x,-\tau\gamma), & \text{in } \mathbb{R}^n\times]0,t[.\end{cases}\tag{2.8}$$

First we define the energy of solution by

$$\begin{aligned}E(t)=&\frac12\|u_t\|^2_{L^2_\rho}+\Big(\frac{\alpha}{2}-\frac12\int_0^t g(s)ds\Big)\|\Delta u\|_2^2\\ &+\frac12(g\circ\Delta u)(t)-\frac{k}{2}\int_{\mathbb{R}^n}\rho(x)u^2\ln|u|dx+\frac{k}{4}\|u\|^2_{L^2_\rho}\\ &+\frac12\xi(t)\int_{\mathbb{R}^n}\int_0^1\rho(x)z^2(x,\gamma,t)d\gamma dx.\end{aligned}\tag{2.9}$$

Where ξ is non-increasing function such that

$$\tau\beta<\overline{\xi}<\tau(2-\beta),\quad t>0,\quad \xi(t)=\overline{\xi}\mu_1(t).\tag{2.10}$$

Lemma 2.5. *Let (u,z) be a solution of the problem* (2.8)*. Then, the energy functional defined by* (2.9) *satisfies*

$$\begin{aligned}E'(t)\le&\frac12(g'\circ\Delta u)-\frac12g(t)\|\Delta u\|_2^2-\Big(\mu_1(t)-\frac{\xi(t)}{2\tau}-\frac{\mu_2(t)}{2}\Big)\|u_t\|^2_{L^2_\rho}\\ &-\Big(\frac{\xi(t)}{2\tau}-\frac{\mu_2(t)}{2}\Big)\|z(x,1,t)\|^2_{L^2_\rho}\le0.\end{aligned}\tag{2.11}$$

Proof. Multiplying the first equation in (2.8) by $\rho(x)u_t$, integrating over $\mathbb{R}^n$ and using Green's identity, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|u_t\|^2_{L^2_\rho} + \alpha\|\Delta u\|_2^2 + \frac{k}{2}\|u\|^2_{L^2_\rho} - \int_{\mathbb{R}^n}\rho(x)u^2\ln|u|^k dx\Big) \\ &+ \mu_1(t)\|u_t\|^2_{L^2_\rho} + \mu_2(t)\int_{\mathbb{R}^n}\rho(x)u_t z(x,1,t)dx \\ &- \int_0^t g(t-s)\int_{\mathbb{R}^n}\Delta u(x,s)\Delta u_t(x,t)dxds = 0. \end{aligned} \tag{2.12}$$

We simplify the last term by using Lemma 2.2, we get

$$\begin{aligned} &-\int_0^t g(t-s)\int_{\mathbb{R}^n}\Delta u(x,s)\Delta u_t(x,t)dxds \\ &= \frac{1}{2}\frac{d}{dt}(g\circ\Delta u) - \frac{1}{2}(g'\circ\Delta u) + \frac{1}{2}g(t)\|\Delta u\|_2^2 - \frac{1}{2}\frac{d}{dt}\int_0^t g(s)ds\|\Delta u\|_2^2. \end{aligned} \tag{2.13}$$

Replacing (2.13) in (2.12) we arrive at

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|u_t\|^2_{L^2_\rho} + \left(\alpha - \int_0^t g(s)ds\right)\|\Delta u\|_2^2\right) \\ &+ \frac{1}{2}\frac{d}{dt}\left(\frac{k}{2}\|u\|^2_{L^2_\rho} - \int_{\mathbb{R}^n}\rho(x)u^2\ln|u|^k + g\circ\Delta u\right) + \mu_1(t)\|u_t\|^2_{L^2_\rho} \\ &+ \mu_2(t)\int_{\mathbb{R}^n}\rho(x)u_t z(x,1,t)dx - \frac{1}{2}(g'\circ\Delta u) + \frac{1}{2}g(t)\|\Delta u\|_2^2 = 0. \end{aligned} \tag{2.14}$$

Multiplying the second equation in (2.8) by $\frac{1}{\tau}\xi(t)\rho(x)z$, where $\xi(t)$ satisfying (2.10) and integrating over $\mathbb{R}^n \times (0,1)$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\xi(t)\int_{\mathbb{R}^n}\int_0^1\rho(x)z^2(x,\gamma,t)d\gamma dx - \frac{\xi'(t)}{2}\int_{\mathbb{R}^n}\int_0^1\rho(x)z^2(x,\gamma,t)d\gamma dx \\ &+ \frac{\xi(t)}{2\tau}\|z^2(x,1,t)\|^2_{L^2_\rho} - \frac{\xi(t)}{2\tau}\|u_t\|^2_{L^2_\rho} = 0. \end{aligned} \tag{2.15}$$

Combination of (2.14) and (2.15), by recalling at the definition of $E(t)$, we deduce that

$$\begin{aligned} &E'(t) + \mu_1(t)\|u_t\|^2_{L^2_\rho} + \mu_2(t)\int_{\mathbb{R}^n}\rho(x)u_t z(x,1,t)dx - \frac{1}{2}(g'\circ\Delta u) \\ &+ \frac{1}{2}g(t)\|\Delta u\|_2^2 - \frac{\xi'(t)}{2}\int_{\mathbb{R}^n}\int_0^1\rho(x)z^2(x,\gamma,t)d\gamma dx \\ &+ \frac{\xi(t)}{2\tau}\|z^2(x,1,t)\|^2_{L^2_\rho} - \frac{\xi(t)}{2\tau}\|u_t\|^2_{L^2_\rho} = 0, \end{aligned} \tag{2.16}$$

then

$$E'(t) = -\Big(\mu_1(t) - \frac{\xi(t)}{2\tau}\Big)\|u_t\|^2_{L^2_\rho} + \frac{1}{2}(g'\circ\Delta u) - \frac{1}{2}g(t)\|\Delta u\|_2^2 \tag{2.17}$$

$$
\begin{aligned}
&- \mu_2(t)\int_{\mathbb{R}^n}\rho(x)u_t z(x,1,t)dx - \frac{\xi(t)}{2\tau}\|z^2(x,1,t)\|^2_{L^2_\rho}\\
&+ \frac{\xi'(t)}{2}\int_{\mathbb{R}^n}\int_0^1 \rho(x)z^2(x,\gamma,t)d\gamma dx.
\end{aligned}
$$

Due to Young's inequality and using the assumptions for $\xi(t)$ and g, we obtain

$$
\begin{aligned}
E'(t) \le{}& \frac{1}{2}(g'\circ\Delta u) - \frac{1}{2}g(t)\|\Delta u\|_2^2 - \Big(\mu_1(t) - \frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2}\Big)\|u_t\|^2_{L^2_\rho}\\
&- \Big(\frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2}\Big)\|z(x,1,t)\|^2_{L^2_\rho} \le 0,
\end{aligned}
\tag{2.18}
$$

where

$$
C_1 = \mu_1(t) - \frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2} > 0, \quad C_2 = \frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2} > 0. \qquad \square
$$

3. Global existence

According to logarithmic Sobolev inequality and by using Galerkin's method combined with compact theorem, similar to the proof in [6], we have the following result

Theorem 3.1 (Local existence). *Let $(u_0, u_1, f_0) \in \mathcal{D}^{2,2}(\mathbb{R}^n)\times L^2_\rho(\mathbb{R}^n)\times L^2(\mathbb{R}^n\times(0,1))$ be given. Assume that g satisfies (H_1) and μ_1, μ_2 satisfy (H_3). Then the problem* (2.8) *admits a unique local solution (u,z) satisfying*

$$
\begin{aligned}
u &\in C([0,T);\mathcal{D}^{2,2}(\mathbb{R}^n)),\\
u' &\in C([0,T);L^2_\rho(\mathbb{R}^n)),\\
z &\in C([0,T);L^2(\mathbb{R}^n\times(0,1)).
\end{aligned}
$$

Now, we introduce the two functionals as follow

$$
\begin{aligned}
J(t) ={}& \Big(\frac{\alpha}{2} - \frac{1}{2}\int_0^t g(s)ds\Big)\|\Delta u\|_2^2 + \frac{1}{2}(g\circ\Delta u)(t)\\
&- \frac{k}{2}\int_{\mathbb{R}^n}\rho(x)u^2\ln|u|dx + \frac{1}{2}\xi(t)\int_0^1 \|z^2(x,\gamma,t)\|^2_{L^2_\rho}d\gamma dx + \frac{k}{4}\|u\|^2_{L^2_\rho},
\end{aligned}
\tag{3.1}
$$

and

$$
I(t) = 2J(t) - \frac{k}{2}\|u\|^2_{L^2_\rho}.
$$

As in [15] to establish the corresponding method of potential wells which is related to the logarithmic nonlinear term, we introduce the stable set as follows

$$
W = \Big\{u\in\mathcal{D}^{2,2}(\mathbb{R}^n - \{0\}); I(t) > 0,\ J(t) < d\Big\}\cup\{0\},
$$

where d is the mountain pass level defined by

$$
d = \inf\{\sup J(\mu u)\},
$$

with $\mu \geq 0, u \in \mathcal{D}^{2,2}(\mathbb{R}^n - \{0\})$. Also, by introducing the set called "Nehari manifold"

$$\mathcal{N} = \Big\{u \in \mathcal{D}^{2,2}(\mathbb{R}^n) - \{0\}, I(t) = 0\Big\}.$$

Similar to results in [16], it is readily seen that the potential depth d is characterized by

$$d = \inf_{u \in \mathcal{N}} J(t).$$

This characterization of d shows that

$$dist(0, \mathcal{N}) = \min_{u \in \mathcal{N}} \|u\|_{\mathcal{D}^{2,2}(\mathbb{R}^n)}.$$

By the fact of $E'(t) < 0$, we will prove the invariance of the set W. That is if for some $t_0 > 0$ if $u(t_0) \in W$, then $u(t) \in W$, $\forall t \geq t_0$. Now we give the existence Lemma of the potential depth (see Lemma 2.4 in [5]).

Lemma 3.2. *d is positive constant.*

Lemma 3.3. *Let $u \in \mathcal{D}^{2,2}(\mathbb{R}^n)$, and $\eta = \exp(\frac{n}{2}(1+c_1))$. If $\|u\|^2_{L^2(\rho)} < \eta$, then $I(t) > 0$. If $I(t) = 0$, $\|u\|^2_{L^2(\rho)} \neq 0$, then $\|u\|^2_{L^2(\rho)} > \eta$.*

Proof. By Lemma 2.3 we have

$$\begin{aligned} J(t) = & \Big(\alpha - \int_0^t g(s)ds\Big)\|\Delta u\|_2^2 + (g \circ \Delta u)(t) \\ & - k\int_{\mathbb{R}^n} \rho(x) u^2 \ln|u|^2 dx + \xi(t)\int_{\mathbb{R}^n}\int_0^1 z^2(x,\gamma,t)d\gamma dx \qquad (3.2) \\ \geq & \ (l - k\omega c_2 \frac{\|\rho\|^2_{L^2}}{2\pi})\|\Delta u\|_2^2 + k(\frac{n}{2}(1+c_1) - \ln\|u\|^2_{L^2_\rho})\|u\|^2_{L^2_\rho} + \frac{k}{4}\|u\|^2_{L^2_\rho}. \end{aligned}$$

Choosing $c_2 < \frac{\pi}{\omega k \|\rho\|^2_{L^2_\rho}}$, then

$$I(t) \geq k(\frac{n}{2}(1+c_1) - \ln\|u\|^2_{L^2_\rho})\|u\|^2_{L^2_\rho}. \qquad (3.3)$$

Therefore if $\|u\|^2_{L^2_\rho} < \eta$, then $I(t) > 0$.

If $I(t) = 0$, $\|u\|^2_{L^2_\rho} \neq 0$ we have $\|u\|^2_{L^2_\rho} > \eta$. □

Now, we are in position to state the theorem of global existence.

Theorem 3.4 (Global Existence)**.** *Let $u_0 \in \mathcal{D}^{2,2}(\mathbb{R}^n), u_1(x) \in L^2_\rho(\mathbb{R}^n)$ and $0 < E(0) < d, I(0) > 0$. Then, under hypothesis (H_1) and conditions of the function ρ, the problem (2.8) has a global solution in time.*

Proof. From the definition of energy for the weak solution and since E is increasing, we have

$$\frac{1}{2}\|u_t\|^2_{L^2_\rho} + J(t) \leq \frac{1}{2}\|u_1\|^2_{L^2_\rho} + J(0), \quad \forall t \in [0, T_{\max}),$$

where $T_{\max}$ is the maximal existence time of weak solution of u. Then, by the definition of the stable set and using Lemma 3.3, we have $u \in W, \forall t \in [0, T_{\max})$. □

4. Asymptotic behavior

We apply the multiplier techniques and we introduce an appropriate Lyapunov functional to obtain the asymptotic behavior. For this purpose, we introduce the so called Lyapunov functional L defined by

$$L(t) = \xi_1 E(t) + \psi(t) + \xi_2\varphi(t) + \varepsilon_1\phi(t), \quad \xi_1 > 1,\ \xi_2 > 1,\ \varepsilon_1 > 0, \tag{4.1}$$

where

$$\begin{aligned} \psi(t) &= \int_{\mathbb{R}^n} \rho(x) u_t u dx, \\ \varphi(t) &= -\int_{\mathbb{R}^n} \rho(x) u_t \int_0^t g(t-s)(u(t)-u(s))dsdx, \\ \phi(t) &= -\xi(t)\tau \int_{\mathbb{R}^n} \rho(x) \int_0^1 e^{-2\tau\gamma} u_t^2(x, t-\tau\gamma)d\gamma dx. \end{aligned} \tag{4.2}$$

Now we present some Lemmas to get the asymptotic behavior of solutions.

Lemma 4.1. *Suppose that (H_1)-(H_4) hold and let $(u_0, u_1) \in \mathcal{D}^{2,2}(\mathbb{R}^n) \times L^2_\rho(\mathbb{R}^n)$ be given. If (u, z) is the solution of (2.8), then the derivative of the functional ψ satisfies the following inequality for $\delta > 0$.*

$$\begin{aligned} \psi'(t) \le{}& (1 + \frac{1}{2}\mu_1(t))\|u_t\|^2_{L^2_\rho} + (\alpha - l)c_\delta(g \circ \Delta u)(t) \\ &+ (\delta - l)\|\Delta u\|_2^2 + \frac{1}{2}\mu_2(t)\|z(x,1,t)\|^2_{L^2_\rho} \\ &+ \|\rho\|^2_{L^2_\rho}\Big(\frac{k\omega c_2}{2\pi} + k\ln\|u\|^2_{L^2_\rho} - \frac{kn}{2}(1+c_1) + \frac{1}{2}[\mu_1(t) + \mu_2(t)]\Big)\|\Delta u\|_2^2. \end{aligned} \tag{4.3}$$

Proof. By using the first equation in (2.8), we have

$$\psi'(t) = \int_{\mathbb{R}^n} \rho(x) u_{tt} u dx + \int_{\mathbb{R}^n} \rho(x)|u_t|^2 dx, \tag{4.4}$$

$$\begin{aligned} \psi'(t) ={}& \int_{\mathbb{R}^n} \rho(x)|u_t|^2 dx - \alpha \int_{\mathbb{R}^n} \Delta u^2 dx \\ &+ \int_{\mathbb{R}^n} \Delta u_t \int_0^t g(t-s)\Delta u(x,s)dsdx - \mu_1(t)\int_{\mathbb{R}^n} \rho(x) u_t u(x,t)dx \\ &- \mu_2(t)\int_{\mathbb{R}^n} \rho(x) u z(x,1,t)dx + \int_{\mathbb{R}^n} \rho(x) u^2 \ln|u|^k dx. \end{aligned} \tag{4.5}$$

We now estimate the right hand side of (4.5) and applying Lemma 2.4, we have the estimates as follows

$$\int_{\mathbb{R}^n} \Delta u \int_0^t g(t-s)\Delta u(s)dsdx$$

$$
\begin{aligned}
(4.6)\qquad &= \int_0^t g(s)ds\|\Delta u\|_2^2 + \int_{\mathbb{R}^n}\Delta u\int_0^t g(t-s)(\Delta u(s)-\Delta u(t))dsdx\\
&\le (\delta+\alpha-l)\|\Delta u\|_2^2 + c_\delta\int_{\mathbb{R}^n}\Big(\int_0^t g(t-s)|\Delta u(s)-\Delta u(t)|ds\Big)^2dx\\
&\le (\delta+\alpha-l)\|\Delta u\|_2^2 + (\alpha-l)c_\delta(g\circ\Delta u)(t).
\end{aligned}
$$

By using Young's inequality, Sobolev's inequality and Lemma 2.3 we have

$$
\begin{aligned}
(4.7)\qquad & -\mu_1(t)\int_{\mathbb{R}^n}\rho(x)u_tu(x,t)dx-\mu_2(t)\int_{\mathbb{R}^n}\rho(x)uz(x,1,t)dx\\
&\le \frac{1}{2}\Big(\mu_1(t)+\mu_2(t)\Big)\|\rho\|^2_{L^2_\rho}\|\Delta u\|_2^2+\frac{1}{2}\mu_1(t)\|u_t\|^2_{L^2_\rho}+\frac{1}{2}\mu_2(t)\|z(x,1,t)\|^2_{L^2_\rho},
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\rho(x)u^2\ln|u|^kdx = k\int_{\mathbb{R}^n}\rho(x)u^2\Big(\ln\frac{u}{\|u\|^2_{L^2_\rho}}+\ln\|u\|^2_{L^2_\rho}\Big)dx\\
&\le \frac{k\omega c_2}{2\pi}\|\rho\|^2_{L^2_\rho}\|\Delta u\|_2^2 + k\Big[\ln\|u\|^2_{L^2_\rho}-\frac{n}{2}(1+c_1)\Big]\|u\|^2_{L^2_\rho}\\
&\le \Big(\frac{k\omega c_2}{2\pi}\|\rho\|^2_{L^2_\rho}\|\Delta u\|_2^2 + k\Big[\ln\|u\|^2_{L^2_\rho}-\frac{n}{2}(1+c_1)\Big]\Big)\|\rho\|^2_{L^2_\rho}\|\Delta u\|_2^2.
\end{aligned}
$$

By combining the last inequalities, we arrive at

$$
\begin{aligned}
&\psi'(t)\le (1+\frac{1}{2}\mu_1(t))\|u_t\|^2_{L^2_\rho}+(\alpha-l)c_\delta(g\circ\Delta u)(t)\\
(4.8)\qquad &+(\delta-l)\|\Delta u\|_2^2+\frac{1}{2}\mu_2(t)\|z(x,1,t)\|^2_{L^2_\rho}\\
&+\|\rho\|^2_{L^2_\rho}\Big(\frac{k\omega c_2}{2\pi}+k\ln\|u\|^2_{L^2_\rho}-\frac{kn}{2}(1+c_1)+\frac{1}{2}[\mu_1(t)+\mu_2(t)]\Big)\|\Delta u\|_2^2.
\end{aligned}
$$

□

Lemma 4.2. *Suppose that (H_3) is fulfilled and let $(u_0,u_1)\in\mathcal{D}^{2,2}(\mathbb{R}^n)\times L^2_\rho(\mathbb{R}^n)$ be given. If (u,z) is the solution of (2.8), then the derivative of the functional φ satisfies the following inequality for some $\delta>0$.*

$$
\begin{aligned}
\varphi'(t)\le{}& \Big[\delta l + k\Big(\frac{\delta c_2}{2\pi}+\ln\|u\|^2_{L^2_\rho}-\frac{n(1+c_1)}{2}\Big)\|\rho\|^2_{L^2_\rho}\Big]\|\Delta u\|_2^2\\
&+\Big(1+c_\delta l+(1+\frac{kc_\delta\omega c_2}{2\pi})\|\rho\|^2_{L^2_\rho}\Big)(\alpha-l)(g\circ\Delta u)\\
&-c_\delta\|\rho\|^2_{L^2_\rho}(g'\circ\Delta u)+\Big(\delta+\frac{1}{2}\mu_1(t)-\int_0^t g(s)\Big)\|u_t\|^2_{L^2_\rho}\\
&+\frac{1}{2}\mu_2(t)\|z(x,1,t)\|^2_{L^2_\rho}.
\end{aligned}\tag{4.9}
$$

Proof. Taking the derivative of φ, we obtain easily

$$
\varphi'(t) = -\int_{\mathbb{R}^n}\rho(x)u_{tt}\int_0^t g(t-s)(u(t)-u(s))dsdx
$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^n}\rho(x)u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx-\int_0^t g(s)ds\|u_t\|^2_{L^2_\rho} \qquad (4.10)\\
&=\alpha\int_{\mathbb{R}^n}\Delta u\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))dsdx\\
&-\int_{\mathbb{R}^n}\Big(\int_0^t g(t-s)\Delta u(s)ds\Big)\Big(\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))ds\Big)dx\\
&+\mu_1(t)\int_{\mathbb{R}^n}\rho(x)u_t\int_0^t g(t-s)(u(t)-u(s))dsdx\\
&+\mu_2(t)\int_{\mathbb{R}^n}\rho(x)z(x,\rho,t)\int_0^t g(t-s)(u(t)-u(s))dsdx\\
&-\int_{\mathbb{R}^n}\rho(x)u\ln|u|^k\int_0^t g(t-s)(u(t)-u(s))dsdx\\
&-\int_{\mathbb{R}^n}\rho(x)u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx-\int_0^t g(s)ds\|u_t\|^2_{L^2_\rho},
\end{aligned}
$$

then

$$
\begin{aligned}
\varphi'(t)=&\Big(\alpha-\int_0^t g(s)ds\Big)\int_{\mathbb{R}^n}\Delta u\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))dsdx\\
&+\int_{\mathbb{R}^n}\Big(\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))ds\Big)^2dx \qquad (4.11)\\
&+\mu_1(t)\int_{\mathbb{R}^n}\rho(x)u_t\int_0^t g(t-s)(u(t)-u(s))dsdx\\
&+\mu_2(t)\int_{\mathbb{R}^n}\rho(x)z(x,\rho,t)\int_0^t g(t-s)(u(t)-u(s))dsdx\\
&-\int_{\mathbb{R}^n}\rho(x)u\ln|u|^k\int_0^t g(t-s)(u(t)-u(s))dsdx\\
&-\int_{\mathbb{R}^n}\rho(x)u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx-\int_0^t g(s)ds\|u_t\|^2_{L^2_\rho}.
\end{aligned}
$$

By Holder's and Young's inequalities and Sobolev Poincare's inequality, we estimate

$$
\begin{aligned}
&\Big(\alpha-\int_0^t g(s)ds\Big)\int_{\mathbb{R}^n}\Delta u\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))dsdx \qquad (4.12)\\
&\quad+\int_{\mathbb{R}^n}\Big(\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))ds\Big)^2dx\\
&\le l\delta\|\Delta u\|_2^2+(c_\delta l+1)(\alpha-l)(g\circ\Delta u).
\end{aligned}
$$

And

$$
-\mu_1(t)\int_{\mathbb{R}^n}\rho(x)u_t\int_0^t g(t-s)(u(t)-u(s))dsdx \qquad (4.13)
$$

$$
\begin{aligned}
&- \mu_2(t)\int_{\mathbb{R}^n}\rho(x)z(x,1,t)\int_0^t g(t-s)(u(t)-u(s))dsdx\\
&\leq \frac{1}{2}\mu_1(t)\|u_t\|^2_{L^2_\rho}+\frac{1}{2}\mu_2(t)\|z(x,1,t)\|^2_{L^2_\rho}+\|\rho\|^2_{L^2_\rho}(\alpha-l)(g\circ\Delta u),
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^n}\rho(x)u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx\\
&\leq \delta\|u_t\|^2_{L^2_\rho}-c_\delta\|\int_0^t -g'(t-s)(u(t)-u(s))ds\|^2_{L^2_\rho}\\
&\leq \delta\|u_t\|^2_{L^2_\rho}-c_\delta\|\rho\|^2_{L^2_\rho}(g'\circ\Delta u).
\end{aligned}
\tag{4.14}
$$

Using Poincare-Sobolev inequality and Lemma 2.3 and conditions in Lemma 3.3, we have

$$
\begin{aligned}
&-\int_{\mathbb{R}^n}\rho(x)u\ln|u|^k\int_0^t g(t-s)(u(t)-u(s))dsdx\\
&\leq \int_{\mathbb{R}^n}\rho(x)u\Big(\ln\frac{|u|^k}{\|u\|^2_{L^2_\rho}}+\ln\|u\|^2_{L^2_\rho}\Big)\int_0^t g(t-s)(u(t)-u(s))dsdx\\
&\leq k\Big(\ln\|u\|^2_{L^2_\rho}-\frac{n(1+c_1)}{2}\Big)\|u\|^2_{L^2_\rho}+\frac{kc_2}{2\pi}\Big\|u\int_0^t g(t-s)(u(t)-u(s))ds\Big\|^2_{L^2_\rho}\\
&\leq k\Big(\ln\|u\|^2_{L^2_\rho}-\frac{n(1+c_1)}{2}\Big)\|\rho\|^2_{L^2_\rho}\|\Delta u\|^2_2\\
&\quad+\frac{k\omega c_2}{2\pi}\|\rho\|^2_{L^2_\rho}\Big\|\Delta u\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))ds\Big\|^2_2\\
&\leq k\Big(\frac{\delta c_2}{2\pi}+\ln\|u\|^2_{L^2_\rho}-\frac{n(1+c_1)}{2}\Big)\|\rho\|^2_{L^2_\rho}\|\Delta u\|^2_2\\
&\quad+\frac{kc_\delta\omega c_2}{2\pi}\|\rho\|^2_{L^2_\rho}(\alpha-l)(g\circ\Delta u).
\end{aligned}
\tag{4.15}
$$

Combining these estimates we arrive at

$$
\begin{aligned}
\varphi'(t)\leq{}&\Big[\delta l+k\Big(\frac{\delta c_2}{2\pi}+\ln\|u\|^2_{L^2_\rho}-\frac{n(1+c_1)}{2}\Big)\|\rho\|^2_{L^2_\rho}\Big]\|\Delta u\|^2_2\\
&+\Big(1+c_\delta l+(1+\frac{kc_\delta\omega c_2}{2\pi})\|\rho\|^2_{L^2_\rho}\Big)(\alpha-l)(g\circ\Delta u)\\
&-c_\delta\|\rho\|^2_{L^2_\rho}(g'\circ\Delta u)+\Big(\delta+\frac{1}{2}\mu_1(t)-\int_0^t g(s)\Big)\|u_t\|^2_{L^2_\rho}\\
&+\frac{1}{2}\mu_2(t)\|z(x,1,t)\|^2_{L^2_\rho}.
\end{aligned}
\tag{4.16}
$$

□

Lemma 4.3. *Suppose that* (H_1), (H_2) *hold and let* $(u_0,u_1)\in\mathcal{D}^{2,2}(\mathbb{R}^n)\times L^2_\rho(\mathbb{R}^n)$ *be given. If* (u,z) *is the solution of* (2.8), *then the derivative of the* *functional $\phi(t)$ satisfies the following inequality*

$$\begin{aligned}\phi'(t) \leq\ & \tau(\xi'(t) - 2\xi(t)) \int_0^1 \|z(x,\gamma,t)\|^2_{L^2_\rho} d\gamma \\ & + \xi(t) e^{-2\tau} \|z(x,1,t)\|^2_{L^2_\rho} - \xi(t)\|u_t\|^2_{L^2_\rho}.\end{aligned} \tag{4.17}$$

Proof. Differentiating $\phi(t)$, we get

$$\begin{aligned}\phi'(t) =\ & -\xi'(t)\tau \int_{\mathbb{R}^n} \rho(x) \int_0^1 e^{-2\tau\gamma} z^2(x,\gamma,t) dx d\gamma \\ & - 2\xi(t)\tau \int_{\mathbb{R}^n} \rho(x) \int_0^1 e^{-2\tau\gamma} z(x,\gamma,t) z_t(x,\gamma,t) dx d\gamma.\end{aligned} \tag{4.18}$$

Using the second equality in (2.8) we obtain

$$\begin{aligned}\phi'(t) &= \Big(\frac{\xi'(t)}{\xi(t)} - 2\Big)\phi(t) + \xi(t) e^{-2\tau} \|z(x,1,t)\|^2_{L^2_\rho} - \xi(t)\|u_t\|^2_{L^2_\rho} \\ &\leq \Big(\xi'(t) + 2\xi(t)\Big)\tau \int_0^1 \|z(x,\gamma,t)\|^2_{L^2_\rho} d\gamma \\ &\quad + \xi(t) e^{-2\tau} \|z(x,1,t)\|^2_{L^2_\rho} - \xi(t)\|u_t\|^2_{L^2_\rho}.\end{aligned} \tag{4.19}$$

□

Lemma 4.4. *If the functional L satisfies* (4.1)*, then there exists two constants α_1 and α_2 such that*

$$\alpha_1 E(t) \leq L(t) \leq \alpha_2 E(t). \tag{4.20}$$

Proof. Using the Cauchy Schwartz and Young's inequalities, Poincare-Sobolev inequality and Lemma 2.4, we obtain

$$\begin{aligned}|L(t) - \xi_1 E(t)| \leq\ & \Big(\frac{1}{2} + \frac{\xi_2}{2} - \varepsilon_1 \xi(t)\|u_t\|^2_{L^2_\rho} + \frac{1}{2}\|\rho\|^2_{L^2_\rho}\|\Delta u\|_2^2 \\ & + \frac{\xi_2}{2}\|\rho\|^2_{L^2_\rho}(\alpha - l)(g \circ \Delta u)\varepsilon_1 \tau(\xi'(t) + 2\xi(t))\Big) \\ & \times \int_0^1 \|z(x,\gamma,t)\|^2_{L^2_\rho} d\gamma \\ & + \varepsilon_1 \xi(t) e^{-2\tau}\|z(x,1,t)\|^2_{L^2_\rho} \leq cE(t).\end{aligned} \tag{4.21}$$

Choosing ε_1 small enough such that

$$|L(t) - \xi_1 E(t)| \leq cE(t). \tag{4.22}$$

Then we can choose ξ_1 such that

$$\alpha_1 E(t) \leq L(t) \leq \alpha_2 E(t). \tag{4.23}$$

□

Lemma 4.5. *For all $t \geq t_1 > 0$, we have*

$$\int_{t_1}^t (g \circ \Delta u)(s) ds \leq H_0^{-1}\Big(\int_{t_1}^t H_0(-g'(s)) g(s) \int_{\mathbb{R}^n} g(s)|\Delta u(t) - \Delta u(t-s)|^2 dx ds\Big),$$

where H_0 is introduced in (H_2).

Proof. By properties of E' and by (H_4) we have for $t \geq t_1$,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_0^{t_1} g(t-s)|\Delta u(t)-\Delta u(s)|^2 dsdx \\
\leq & -\frac{1}{k}\int_{\mathbb{R}^n}\int_0^{t_1} g(t-s)|\Delta u(t)-\Delta u(s)|^2 dsdx \\
\leq & -cE'(t).
\end{aligned}$$

We define now

$$\chi(t)=\int_{t_1}^{t} H_0(-g'(s))(g\circ\Delta u)(t)ds.$$

Since $\int_0^\infty H_0(-g'(s))g(s)ds < +\infty$, we have

$$\begin{aligned}
\chi(t) &= \int_{t_1}^{t} H_0(-g'(s))\int_{\mathbb{R}^n} g(s)|\Delta u(s)-\Delta u(t)|^2 dxds \\
&\leq 2\int_{t_1}^{t} H_0(-g'(s))g(s)(\|\Delta u(s)\|_2^2-\|\Delta u(t)\|_2^2)dxds \qquad (4.24)\\
&\leq cE(0)\int_{t_1}^{t} H_0(-g'(s))g(s) < 1.
\end{aligned}$$

We define again a new functional $\lambda(t)$ related with $\chi(t)$ as

$$\lambda(t)=-\int_{t_1}^{t} H_0(-g'(s))g'(s)\int_{\mathbb{R}^n} g(s)|\Delta u(t)-\Delta u(t-s)|^2 dxds. \qquad (4.25)$$

From (H_1)-(H_2) and for some positive constant k_0, we conclude for all $t \geq t_1$

$$\begin{aligned}
\lambda(t) &\leq -k_0\int (g'(s))\int_{\mathbb{R}^n}|\Delta u(t)-\Delta u(t-s)|^2 dxds \\
&\leq -k_0\int_{t_1}^{t}(g'(s))\int_{\mathbb{R}^n}|\Delta u(t)|^2+|\Delta u(t-s)|^2 dxds \\
&\leq -cE(0)\int_{t_1}^{t} g'(s)ds \leq -cE(0)g(t_1) < \min\{r, H(r), H_0(r)\}.
\end{aligned}$$

Using the properties of $H_0(\theta x)\leq \theta H_0(x)$ and hypothesis in (H_2), (4.25), (4.24) and Jensen's inequality we get

$$\begin{aligned}
\lambda(t) &= \frac{1}{\chi(t)}\int_{t_1}^{t} H_0\Big(H_0^{-1}(-g'(s))\Big)\chi(t)H_0(-g'(s))g'(s) \\
&\qquad \times\int_{\mathbb{R}^n} g(s)|\Delta u(t)-\Delta u(t-s)|^2 dxds \\
&\geq H_0\int_{t_1}^{t}\int_{\mathbb{R}^n} g(s)|\Delta u(t)-\Delta u(t-s)|^2 dxds.
\end{aligned}$$

Which implies

$$\int_{t_1}^{t}\int_{\mathbb{R}^n} g(s)|\Delta u(t)-\Delta u(t-s)|^2dxds \le H_0^{-1}(\lambda(t)).$$

□

At this point, we can prove the second main result the asymptotic behavior

Theorem 4.6. *Let $(u_0,u_1)\in \mathcal{D}^{2,2}(\mathbb{R}^n)\times L^2_\rho(\mathbb{R}^n)$ be given. Assume that g satisfies (H_1). Then, for each $t_0>0$, there exist positive constants n_1,n_2,n_3,n_4 and k such that, for any solution of the problem (2.8), the energy satisfies*

$$E(t)\le n_3H_1^{-1}(n_1+n_2),\quad \forall t\ge 0, \tag{4.26}$$

where

$$H_1(t)=\int_t^1 (sH_0'(n_4s))^{-1}ds.$$

Proof. From the definition of $L(t)$ we obtain

$$L'(t)=\xi_1E'(t)+\psi'(t)+\xi_2\varphi'(t)+\varepsilon_1\phi(t). \tag{4.27}$$

Then

$$\begin{aligned} L'(t)\le\ & -m_0\|u_t\|^2_{L^2_\rho}-M_1\|\Delta u\|_2^2+M_2(g\circ\Delta u)\\ &+\Big(\frac{\xi_1}{2}-c_\delta\xi_2\|\rho\|^2_{L^2_\rho}\Big)(g'\circ\Delta u)\\ &-(\frac{1+\xi_2}{2}+\varepsilon_1\xi(t)e^{-2\tau})\mu_2(t)\|z(x,1,t)\|^2_{L^2_\rho}\\ &+\frac{\xi(t)}{2\tau}\|z(x,1,t)\|^2_{L^2_\rho}-\mu_2(t)\int_{\mathbb{R}^n}\rho(x)u_tz(x,1,t)dx\\ &+\frac{\xi'(t)}{2}\int_{\mathbb{R}^n}\int_0^1\rho(x)z^2(x,\gamma,t)d\gamma dx, \end{aligned} \tag{4.28}$$

hence

$$\begin{aligned} L'(t)\le\ & -M_0\|u_t\|^2_{L^2_\rho}-M_1\|\Delta u\|_2^2+M_2(g\circ\Delta u)\\ &+\Big(\frac{1}{2}-c_\delta\xi_2\|\rho\|^2_{L^2_\rho}\Big)(\alpha-l)(g'\circ\Delta u)-M_3\|z(x,1,t)\|^2_{L^2_\rho}\\ &+\Big(\varepsilon_1(\xi'(t)+2\xi(t))\tau+\frac{\xi'(t)}{2}\Big)\int_{\mathbb{R}^n}\int_0^1\rho(x)z^2(x,\gamma,t)d\gamma dx, \end{aligned} \tag{4.29}$$

where

$$\begin{aligned} M_0=\ &\Big(\xi_1\Big(\frac{\mu_2(t)}{2}+\varepsilon_1\xi(t)-\mu_1(t)-\frac{\xi(t)}{2\tau}\Big)-1+\frac{\mu_1(t)}{2}\\ &+\xi_2\Big(\delta-\int_0^{t_1}g(s)ds+\frac{\mu_1(t)}{2}\Big)\Big),\\ M_1=\ &\frac{\xi_1}{2}g(t)-\Big(\delta-l+\frac{k\omega c_2}{2\pi}+k\ln\|u\|^2_{L^2_\rho}-\frac{kn}{2}(1+c_1)+\frac{1}{2}(\mu_1(t)+\mu_2(t))\Big)\\ &-\xi_2\Big(\delta l+k\frac{\delta\omega c_2}{2\pi}+k\ln\|u\|^2_{L^2_\rho}-\frac{kn(1+c_1)}{2}\Big)\|\rho\|^2_{L^2_\rho}-\varepsilon_1\xi(t), \end{aligned}$$

$$M_2 = \xi_2\Big(1 + c_\delta l + c_\delta\|\rho\|_\infty + (1 + \frac{kc_\delta\omega c_2}{2\pi})\|\rho\|^2_{L^2_\rho}\Big)(\alpha - l) + (\alpha - l)c_\delta > 0,$$

$$M_3 = \Big(\frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2} + \frac{\mu_2(t)\xi_2}{2} - \varepsilon_1\xi(t)e^{-2\tau}\Big).$$

At this point, we choose δ so small such that

$$\xi_1 > 2c_\delta\|\rho\|^2_{L^2_\rho}\xi_2.$$

Whence δ is fixed, we can choose ξ_2 such that $M_0 > 0$

$$\begin{aligned} M_0 = &\Big(\xi_1\Big(\frac{\mu_2(t)}{2} - \mu_1(t) - \frac{\xi(t)}{2\tau}\Big) + \varepsilon_1\xi(t) - 1 + \frac{\mu_1(t)}{2} \\ &+ \xi_2\Big(\delta - \int_0^{t_1} g(s)ds + \frac{\mu_1(t)}{2}\Big)\Big) \geq \xi_2\Big(\delta - \int_0^t g(s)ds\Big), \end{aligned}$$

then for $t > t_1$ we can choose

$$\xi_2 > \Big(\int_0^{t_1} g(s)ds - \delta\Big)^{-1}.$$

Now choosing ε_1 small enough such that $M_3 > 0$. After this conditions we estimate that

$$L'(t) \leq M_2(g \circ \Delta u) - cE'(t).$$

Now we set $F(t) = L(t) + cE(t)$, which is equivalent to $E(t)$. Then

$$F'(t) \leq -cE(t) + c\int_{\mathbb{R}^n}\int_{t_1}^t g(t-s)|\Delta u(t) - \Delta u(s)|^2 dsdx, \forall t > t_1.$$

Using Lemma 4.5, we obtain

$$F'(t) = L'(t) + cE'(t) \leq -cE(t) + cH_0^{-1}(\lambda(t)).$$

Now, we will use the fact that $E' < 0, H' > 0, H'' > 0$ on $(0, r]$ to define the functional

$$F_1(t) = H_0'\Big(\alpha_0\frac{E(t)}{E(0)}\Big)F(t) + cE(t), \alpha_0 < r, c > 0.$$

Where $F_1(t) \sim E(t)$ and

$$\begin{aligned} F_1'(t) &= \alpha_0\frac{E'(t)}{E(0)}H_0''\Big(\alpha_0\frac{E(t)}{E(0)}\Big)F(t) + H_0'\Big(\alpha_0\frac{E(t)}{E(0)}\Big) + cE'(t) \\ &\leq -cE(t)H_0'\Big(\alpha_0\frac{E(t)}{E(0)}\Big)F(t) + cH_0'\Big(\alpha_0\frac{E(t)}{E(0)}\Big)H^{-1}(\lambda(t)) + cE'(t). \end{aligned}$$

Let H_0^* given in (H_2) and using Young's inequality in (H_2) with $A = H'\Big(\alpha_0\frac{E(t)}{E(0)}\Big)$, $B = H_0^{-1}(\lambda(t))$ to get

$$\begin{aligned} F_1'(t) &\leq -cE(t)H_0'\Big(\alpha_0\frac{E(t)}{E(0)}\Big) + cH_0^*\Big(H_0'\Big(\alpha_0\frac{E(t)}{E(0)}\Big)\Big) + c(\lambda(t)) + cE'(t) \\ &\leq -cE(t)H_0'\Big(\alpha_0\frac{E(t)}{E(0)}\Big) + c\alpha_0\frac{E(t)}{E(0)}H_0'\Big(\alpha_0\frac{E(t)}{E(0)}\Big) - c'E'(t) + cE'(t). \end{aligned}$$

Choosing α_0, c, c', such that for all $t \geq t_1$ we have

$$F_1'(t) \leq -k\frac{E(t)}{E(0)}H_0'\Big(\alpha_0\frac{E(t)}{E(0)}\Big) = -kH_2\frac{E(t)}{E(0)},$$

where $S_2(t) = H_0'(\alpha_0 t)$. By using the strict convexity of H_0 on $(0, r]$, to find that H_2', H_2 are strict positives function on $(0, 1]$, then

$$R(t) = \gamma\frac{k_1 F_1(t)}{E(0)} \sim E(t),\ \gamma \in (0,1), \tag{4.30}$$

and

$$R'(t) \leq -\gamma k_0 H_2(R(t)),\ k_0 \in (0, +\infty),\ t \geq t_1.$$

hence, a simple integration gives

$$R(t) \leq H_1^{-1}(n_1 t + n_2),\ n_1, n_2 \in (0, +\infty),\ t \geq t_1,$$

here $H_1(t) = \int_t^1 H^{-1}(s)ds$.

From (4.30), for a positive constant n_3, we have

$$E(t) \leq n_3 H_1^{-1}(n_1 t + n_2), \quad n_1, n_2 \in (0, +\infty), \quad t \geq t_1.$$

The fact that H_1 is strictly decreasing function on $(0, 1]$ and due to properties of S_2, we have

$$\lim_{t\to 0} H_1(t) = +\infty.$$

Then

$$E(t*) \leq n_3 H_1^{-1}(n_1 t + n_2), \quad n_1, n_2 \in (0, +\infty), \quad t \geq 0. \qquad \square$$

Remark 4.7. Noting that, we have obtained all results without any conditions on the exponent k in the logarithmic nonlinearities.

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