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TOPOLOGICALLY STABLE MEASURES IN NON-AUTONOMOUS SYSTEMS

Pramod Das and Tarun Das

Abstract. We introduce and study notions of expansivity, topological stability and persistence for Borel measures with respect to time varying bi-measurable maps on metric spaces. We prove that on Mandelkern locally compact metric spaces expansive persistent measures are topologically stable in the class of all time varying homeomorphisms.

1. Introduction

For several decades, a discrete dynamical system induced by a continuous map or a homeomorphism on a compact metric space has been the most popular and attractive formulation for a dynamical system to a large number of mathematicians all over the world. Since the tools to investigate the dynamics is mainly topological, the study of such system is a part of the mathematical field of topological dynamics. One of the broadly studied dynamical notions in topological dynamics is expansivity which was introduced [10] by Utz in the middle of the twentieth century. On the other hand, the most fundamental topological dynamical notion of shadowing was originated from Anosov closing lemma [1]. In [11], Walters proved that on compact metric spaces expansive homeomorphisms with shadowing are topologically stable. Historically, this celebrated result is popularly known as "Walters stability theorem". In [3], authors improved this result by showing that persistent expansive homeomorphisms on compact metric spaces are topologically stable. Recently, the notion of topological stability was studied [9] in the perspective of non-autonomous systems which is the original nature of many real life problems. The rich mathematical literature around "Walters stability theorem" interested authors of [5] to introduce a notion of topological stability for Borel measures so that it is possessed by any expansive measure [6] with shadowing. We naturally call

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this measurable version of "Walters stability theorem" as "Measurable stability theorem".

Our purpose in this paper is to look at these notions of expansivity, persistence and topological stability for Borel measures in the perspective of non-autonomous systems and prove "Measurable stability theorem" in this settings. In particular, we determine the size of the set of points with converging semiorbits under time varying bi-measurable maps with respect to any expansive outer regular measure on separable metric spaces. Consequently, we show that every equicontinuous time varying uniform equivalence is aperiodic with respect to expansive outer regular measure. We prove that every stable class of a time varying measurable map has measure zero with respect any positively expansive outer regular measure. Finally, we show that on Mandelkern locally compact metric spaces an expansive persistent measure is topologically stable in the class of all time varying homeomorphisms.

2. Preliminaries

Throughout the paper, $\mathbb{Z}$ (resp. $\mathbb{N}$) denotes the set of all (resp. non-negative) integers. We consider (X, d) to be any metric space unless otherwise stated and for $n \geq 1$, $f_n : X \to X$ to be a sequence of bi-measurable maps (bijective maps for which f and f^{-1} are measurable) and $f_0 : X \to X$ to be the identity map. The family $F = \{f_n\}_{n\in\mathbb{N}}$ is called a time varying bi-measurable map on X. The inverse of F is given by $F^{-1} = \{f_n^{-1}\}_{n\in\mathbb{N}}$. Let us denote

$$F_n = \begin{cases} f_n \circ f_{n-1} \circ \cdots \circ f_1 \circ f_0 & \text{for all } n \geq 0, \\ f_{-n}^{-1} \circ f_{-(n-1)}^{-1} \circ \cdots \circ f_1^{-1} \circ f_0^{-1} & \text{for all } n < 0. \end{cases}$$

It is clear that F_0 is the identity on X. We call (X, F) an invertible non-autonomous discrete dynamical system induced by a time varying bi-measurable map.

The dynamics of a self-homeomorphism f of a metric space X is a special case if $f_n = f$ for all $n \in \mathbb{N}$. We denote

$$F_{[i,j]} = \begin{cases} f_j \circ f_{j-1} \circ \cdots \circ f_{i+1} \circ f_i & \text{for any } i \leq j, \\ \text{the identity map on X} & \text{for any } i > j, \end{cases}$$

$$F_{[i,j]}^{-1} = \begin{cases} f_j^{-1} \circ f_{j-1}^{-1} \circ \cdots \circ f_{i+1}^{-1} \circ f_i^{-1} & \text{for any } i \leq j, \\ \text{the identity map on X} & \text{for any } i > j. \end{cases}$$

For $k \geq 1$, we define $F^k = \{g_n\}_{n\in\mathbb{N}}$, where $g_n = F_{[(n-1)k+1,nk]}$. The sequence $O(x_0) = \{F_n(x_0)\}_{n\in\mathbb{Z}}$ is called the orbit of x_0 under the time varying bi-measurable map F. A subset $Y \subset X$ is said to be F-invariant if $f_n(Y) \subset Y$ for all $n \in \mathbb{N}$, equivalently, $F_n(Y) \subset Y$ for all $n \in \mathbb{Z}$.

A homeomorphism $h : X \to X$ is called a uniform equivalence if both h and h^{-1} are uniformly continuous. If each f_n $(n \geq 1)$ is a homeomorphism (resp.

uniform equivalence), then $F = \{f_n\}_{n\in\mathbb{N}}$ is called a time varying homeomorphism (resp. uniform equivalence). A time varying homeomorphism F is said to be equicontinuous if $\{F_{[m,n]}, F_{[m,n]}^{-1} \mid 0 \leq m \leq n\}$ is an equicontinuous family of functions.

Let (X, d_1) and (Y, d_2) be two metric spaces. Let $F = \{f_n\}_{n\in\mathbb{N}}$ and $G = \{g_n\}_{n\in\mathbb{N}}$ be time varying bi-measurable maps on X and Y respectively. Then, F and G are said to be topologically conjugate if there exists a homeomorphism $h : X \to Y$ such that $h \circ f_n = g_n \circ h$ for all $n \in \mathbb{N}$. In particular, if h is a uniform equivalence, we say that F and G are uniformly conjugate.

Let (X, d) be a metric space and $d_1(x, y) =\min\{d(x, y), 1\}$ be the standard bounded metric on X. Let $\mathcal{H}(X)$ be the metric space of all bi-measurable maps with the metric $\eta(f, g) =\sup_{x\in X} d_1(f(x), g(x))$. If $\mathcal{G}(X)$ is the collection of all time varying bi-measurable maps, then we define a metric p on $\mathcal{G}(X)$ as $p(F, G) =\max\{ \sup_{n\in\mathbb{N}} \eta(f_n, g_n), \sup_{n\in\mathbb{N}} \eta(f_n^{-1}, g_n^{-1})\}$, where $F = \{f_n\}_{n\in\mathbb{N}}$ and $G = \{g_n\}_{n\in\mathbb{N}}$.

Let $\mathcal{P}(X)$ be the power set of X and $H : X \to \mathcal{P}(X)$ be a set valued map on X. We define the domain of H by $Dom(H) = \{x \in X \mid H(x) \neq \phi\}$. H is said to be compact valued if $H(x)$ is compact for each $x \in X$. Recall from [5] that $d(H, Id) < \epsilon$ for some $\epsilon > 0$ if $H(x) \subset B(x, \epsilon)$ for each $x \in X$, where $B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$ is the open ϵ-ball with center x. On the other hand, $B[x, \epsilon] = \{y \in X \mid d(x, y) \leq \epsilon\}$ denotes the closed ϵ-ball with center x. If K is a subset of X, then $B[K, \epsilon] = \{x \in X \mid d(x, K) \leq \epsilon\}$, where $d(x, K) = \inf\{d(x, y) \mid y \in K\}$. H is called upper semi-continuous if for every $x \in Dom(H)$ and every open neighbourhood O of $H(x)$ there exists $\delta > 0$ such that $H(y) \subset O$ for all $y \in X$ with $d(x, y) < \delta$.

A metric space is said to be Mandelkern locally compact [7] if every bounded set is contained in a compact set.

Borel measures on X play a major role in this paper. The following are some of the well-known and useful properties of such measures.

Let μ be a Borel measure on X. We call a subset X_0 of X has measure zero if $\mu(A) = 0$ for any measurable subset A of X_0. A point $x \in X$ is called an atom for μ if $\mu(\{x\}) > 0$. μ is said to be non-atomic if it has no atom. μ is said to be outer regular if for every measurable set A and any $\epsilon > 0$, there is an open set O containing A such that $\mu(O \setminus A) < \epsilon$. The pullback measure of μ with respect to a measurable map $f : X \to Y$ is defined by $f_*(\mu)(A) = \mu(f^{-1}(A))$ for all Borel measurable set $A \subset Y$.

The following result [8, Corollary 6.1] is useful because it guarantees existence of non-atomic measure, which is a standard assumption for a Borel measure to be expansive.

Theorem 2.1. *Let X be a complete separable metric space with uncountably many points. Then, there exists a non-atomic measure on X.*

According to the remark in [4], the following version of Lusin's theorem holds true.

Theorem 2.2. *Let X be a second countable topological space endowed with an outer regular measure μ and $f : X \to X$ be a measurable map. Then, for every $\epsilon > 0$ there exists a measurable subset C_ϵ with $\mu(X \setminus C_\epsilon) < \epsilon$ such that $f |_{C_\epsilon}$ is continuous.*

Theorem 2.3. *Let μ be a Borel measure on a topological space. Then, for every measurable Lindelöf subset K with $\mu(K) > 0$ there are $z \in K$ and open neighborhood U of z such that $\mu(K \cap W) > 0$ for every open neighborhood $W \subset U$ of z.*

Proof. Otherwise, for every $z \in K$ there is open neighborhood $U_z \subset U$ satisfying $\mu(K \cap U_z) = 0$. Since K is Lindelöf, the open cover $\{K \cap U_z : z \in K\}$ of K admits a countable sub-cover, i.e., there is a sequence $\{z_l\}_{l\in\mathbb{N}}$ in K satisfying $K = \bigcup_{l\in\mathbb{N}}(K \cap U_{z_l})$. So, $\mu(K) \le \sum_{l\in\mathbb{N}} \mu(K \cap U_{z_l}) = 0$, a contradiction. □

3. Expansive measures

Here, we introduce and investigate the notion of expansivity for Borel measures with respect to time varying bi-measurable maps. This notion extends the autonomous notion of expansivity for Borel measures introduced in [6].

Definition 3.1. Let $F = \{f_n\}_{n\in\mathbb{N}}$ be a time varying bi-measurable (resp. measurable) map on X. Then, μ is said to be expansive (resp. positively expansive) with respect to F if there exists $\delta > 0$ such that $\mu(\Gamma_\delta(x)) = 0$ (resp. $\mu(\Phi_\delta(x)) = 0$) for all $x \in X$, where $\Gamma_\delta(x) = \{y \in X \mid d(F_n(x), F_n(y)) \le \delta$ for all $n \in \mathbb{Z}\}$ and $\Phi_\delta(x) = \{y \in X \mid d(F_n(x), F_n(y)) \le \delta$ for all $n \in \mathbb{N}\}$. Such δ is called an expansive (resp. positively expansive) constant for μ.

Remark 3.2. (i) An expansive measure must be non-atomic because for any $x \in X$, $\Gamma_\delta(x)$ contains x.

(ii) A non-atomic Borel measure is expansive with respect to any expansive time varying bi-measurable map introduced in [9].

Example 3.3. Let μ be an expansive measure for a self-homeomorphism f of a metric space X and Is is an isometry on X. Then, μ is expansive with respect to $F = \{f_n\}_{n\in\mathbb{N}}$, where $f_n = Is$ for $n = 1, 3, 6, 10, 15, \ldots$ and $f_n = f$ otherwise.

Proposition 3.4. *Let $F = \{f_n\}_{n\in\mathbb{N}}$ and $G = \{g_n\}_{n\in\mathbb{N}}$ be time varying bi-measurable maps on (X, d_1) and (Y, d_2), respectively. If F is uniformly conjugate to G, then μ is expansive with respect to F if and only if it is expansive with respect to G.*

Proof. Let μ be expansive with respect to F with an expansive constant $\delta > 0$. Let h be a uniform conjugacy between F and G. Then, observe that $F_n \circ h^{-1} = h^{-1} \circ G_n$ for all $n \in \mathbb{N}$. Since h^{-1} is uniformly continuous, there exists $\delta > 0$ such that $d_2(x, y) \le \delta$ implies $d_1(h^{-1}(x), h^{-1}(y)) \le \epsilon$. Let us fix $x \in Y$. Then,

$$\mu(\{y \in Y \mid d_2(G_n(x), G_n(y)) \le \delta \text{ for all } n \in \mathbb{Z}\})$$

$$\leq \mu(\{y \in Y \mid d_1(h^{-1}(G_n(x)), h^{-1}(G_n(y))) \leq \epsilon \text{ for all } n \in \mathbb{Z}\})$$
$$= \mu(\{y \in Y \mid d_1(F_n(h^{-1}(x)), F_n(h^{-1}(y))) \leq \epsilon \text{ for all } n \in \mathbb{Z}\}) = 0.$$

This shows that μ is expansive with respect to G, with expansive constant δ. The converse holds in similar manner because of the fact that h is a uniform equivalence. □

Theorem 3.5. *Let $F = \{f_n\}_{n\in\mathbb{N}}$ be a time varying bi-measurable map on X. Then, μ is expansive with respect to F if and only if it is expansive with respect to F^{-1}.*

Proof. From the fact $F_{-n} = F_n^{-1}$ it follows that $\mu(\{y \in X \mid F_n(y) \in B_d[F_n(x), \delta]$ for all $n \in \mathbb{Z}\}) = 0$ if and only if $\mu(\{y \in X \mid F_n^{-1}(y) \in B_d[F_n^{-1}(x), \delta]$ for all $n \in \mathbb{Z}\}) = 0$. This clearly shows that μ is expansive with respect to F if and only if it is expansive with respect to F^{-1}. □

Theorem 3.6. *Let $F = \{f_n\}_{n\in\mathbb{N}}$ be a time varying uniform equivalence on X such that the family $\{f_n, f_n^{-1}\}_{n\in\mathbb{N}}$ is equicontinuous. Then, μ is expansive with respect to F if and only if it is expansive with respect to $G = F^k = \{F_{[(n-1)k+1,nk]}\}_{n\in\mathbb{N}}$ for all $k \in \mathbb{Z}$.*

Proof. In view of Theorem 3.5, it is enough to prove the result for $k \geq 1$. Fix $k \geq 1$ and let e be an expansive constant for μ. Since $\{f_n, f_n^{-1}\}_{n\in\mathbb{N}}$ is equicontinuous, for any $n \in \mathbb{Z}$ and any j with $nk + 1 \leq j \leq (n + 1)k$, the homeomorphisms $F_{[nk+1,j]}$ are uniformly continuous. Thus, there is $\delta_j > 0$ such that $d(x, y) < \delta_j$ implies $d(F_{[nk+1,j]}(x), F_{[nk+1,j]}(y)) < e$ for any $n \in \mathbb{Z}$ and all j with $nk + 1 \leq j \leq (n + 1)k$. Observe that δ_j does not depend on n because of the equicontinuity of $\{f_n, f_n^{-1}\}_{n\in\mathbb{N}}$. Then, $d(x, y) < \delta$ implies $d(F_{[nk+1,j]}(x), F_{[nk+1,j]}(y)) < e$ for all $n \in \mathbb{Z}$, where δ =min$\{\delta_j \mid nk + 1 \leq j \leq (n + 1)k\}$. Observe that for any $j \in \mathbb{Z}$ there exists $n \in \mathbb{Z}$ such that $nk + 1 \leq j \leq (n + 1)k$ and $G_n = F_{nk}$ for all $n \in \mathbb{N}$, $k \in \mathbb{Z}$. Now,

$$\begin{aligned}
&\mu(\{y \in X \mid d(G_n(x), G_n(y)) \leq \delta \text{ for all } n \in \mathbb{Z}\})\\
&= \mu(\{y \in X \mid d(F_{nk}(x), F_{nk}(y)) \leq \delta \text{ for all } n \in \mathbb{Z}\})\\
&= \mu(\{y \in X \mid d(F_{[nk+1,j]}(F_{nk}(x)), F_{[nk+1,j]}(F_{nk}(y))) \leq e\\
&\qquad \text{for all } n \in \mathbb{Z} \text{ and all } j \in \mathbb{Z}\})\\
&= \mu(\{y \in X \mid d(F_j(x), F_j(y)) \leq e \text{ for all } j \in \mathbb{Z}\}) = 0.
\end{aligned}$$

This shows that μ is expansive with respect to F^k with expansive constant δ.

Conversely, suppose that μ is expansive with respect to F^k with expansive constant δ. Then for $x \in X$, $\mu(\{y \in X \mid d(F_n^k(x), F_n^k(y)) \leq \delta$ for all $n \in \mathbb{Z}\}) = 0$ which implies $\mu(\{y \in X \mid d(F_{nk}(x), F_{nk}(y)) \leq \delta$ for all $n \in \mathbb{Z}\}) = 0$. This further implies that $\mu(\{y \in X \mid d(F_n(x), F_n(y)) \leq \delta$ for all $n \in \mathbb{Z}\}) = 0$. Therefore, μ is expansive with respect to F with expansive constant δ. □

The following result is interesting in view of the fact that an equicontinuous homeomorphism on a compact metric space cannot be expansive.

Theorem 3.7. *Let $F = \{f_n\}_{n\in\mathbb{N}}$ be an equicontinuous time varying homeomorphism on a Lindelöf metric space X. Then, there exists no Borel measure which is expansive with respect to F.*

Proof. Suppose that μ is an expansive measure with respect to F. Let e be an expansive constant for μ. Since F is equicontinuous time varying homeomorphism, the family $\{F_m, F_{-m}\}_{m\in\mathbb{N}}$ is equicontinuous. Then, for e there exists $\delta > 0$ such that $d(x,y) < \delta$ implies $d(F_n(x), F_n(y)) < e$ for all $n \in \mathbb{Z}$. Thus, $B(x,\delta) \subset \Gamma_e(x)$ and hence, $\mu(B(x,\delta)) \leq \mu(\Gamma_e(x)) = 0$ for all $x \in X$. Now, $\{B(x,\delta) \mid x \in X\}$ is an open cover for X and since X is Lindelöf there is $\{x_i\}_{i\in\mathbb{N}}$ such that $\{B(x_i,\delta) \mid i \in \mathbb{N}\}$ is an open covering for X. So, $\mu(X) \leq \Sigma_{i\in\mathbb{N}}\mu(B(x_i,\delta))$, which implies $\mu(X) = 0$, a contradiction. $\square$

The following corollary of the above result can be verified using Theorem 2.1.

Corollary 3.8. *An equicontinuous time varying homeomorphism on a complete separable metric space with uncountably many points cannot be expansive.*

Definition 3.9. Let $F = \{f_n\}_{n\in\mathbb{N}}$ be a time varying bi-measurable map on X. Then,

(i) the ω-limit set of a point $x \in X$ is given by $\omega(F,x) = \{y \in X \mid \lim_{k\to\infty} F_{n_k}(x) = y$ for some strictly increasing sequence of integers$\}$.
(ii) the α-limit set of a point $x \in X$ is given by $\alpha(F,x) = \{y \in X \mid \lim_{k\to\infty} F_{n_k}(x) = y$ for some strictly decreasing sequence of integers$\}$.

We say that a point $x \in X$ has converging semiorbits under F if both $\omega(F,x)$ and $\alpha(F,x)$ consists of single point. The set of such points is denoted by $A(F)$.

For given $x, y \in X$ and $m, n \in \mathbb{N}^+$, we define
$A(x,y,n,m) = \{z \in X \mid \max\{d(F_{-i}(z),x), d(F_i(z),y)\} \leq \frac{1}{n}$ for all $i \geq m\}$.

Lemma 3.10. *Let $F = \{f_n\}_{n\in\mathbb{N}}$ be a time varying bi-measurable map on a separable metric space X. Then, there exists a sequence $x_k \in X$ such that*
$$A(F) \subset \bigcap_{n\in\mathbb{N}^+} \bigcup_{k,k',m\in\mathbb{N}^+} A(x_k, x_{k'}, n, m).$$

Proof. If $z \in A(F)$, then $\alpha(F,z)$ and $\omega(F,z)$ reduce to single points x and y respectively. Then, for each $n \in \mathbb{N}^+$ there exists $m \in \mathbb{N}^+$ such that $d(F_{-i}(z),x) \leq \frac{1}{2n}$ and $d(F_i(z),y) \leq \frac{1}{2n}$ for all $i \geq m$. If x_k is dense in X, there are $k, k' \in \mathbb{N}^+$ such that $d(x_k,x) \leq \frac{1}{2n}$ and $d(x_{k'},y) \leq \frac{1}{2n}$. Therefore,
$$\max\{d(F_{-i}(z),x_k), d(F_i(z),x_{k'})\} \leq \frac{1}{n}$$
for all $i \geq m$. This completes a proof. $\square$

The following result is an extension of [2, Lemma 2.6] to non-autonomous systems.

Theorem 3.11. *Let $F = \{f_n\}_{n\in\mathbb{N}}$ be a time varying bi-measurable map on a separable metric space X. If μ is an expansive outer regular measure with respect to F, then the set of points with converging semiorbits under F has measure zero.*

Proof. Suppose there exists $A \subset A(F)$ such that $\mu(A) > 0$. By Lemma 3.10, there is a sequence $x_k \in X$ such that $A(F) \subset \bigcap_{n\in\mathbb{N}^+}\bigcup_{k,k',m\in\mathbb{N}^+} A(x_k, x_{k'}, n, m)$. It follows that $A \subset \bigcup_{k,k',m\in\mathbb{N}^+} A(x_k, x_{k'}, n, m)$ for all $n \in \mathbb{N}^+$. Thus, we can choose $k, k', n, m \in \mathbb{N}^+$ with $\frac{1}{n} \leq \frac{e}{2}$ such that $\mu(A(x_k, x_{k'}, n, m)) > 0$. Hereafter, we fix such $k, k', n, m \in \mathbb{N}^+$ and for simplicity we put $B = A(x_k, x_{k'}, n, m)$.

Since X is separable, it is a second countable metric space. Since μ is outer regular, therefore Theorem 2.2 implies that for every $\epsilon > 0$ there exists a measurable set $C_\epsilon \subset X$ with $\mu(X \setminus C_\epsilon) < \epsilon$ such that $F_i \mid_{C_\epsilon}$ is continuous for all $\mid i \mid \leq m$. Taking $\epsilon = \frac{\mu(B)}{2}$ we get a measurable set $C = C_{\frac{\mu(B)}{2}}$ such that $F_i \mid_C$ is continuous for all $\mid i \mid \leq m$ and $\mu(B \cap C) > 0$.

Further, since $K = B \cap C$ is a Lindelöf subspace of X, by Theorem 2.3 there are $z \in B \cap C$ and $\delta_0 > 0$ such that $\mu(B \cap C \cap B[z, \delta]) > 0$ for all $0 < \delta < \delta_0$. Since $z \in C$ and $F_i \mid_C$ is continuous for all $\mid i \mid \leq m$, we can fix $0 < \delta < \delta_0$ such that $d(F_i(z), F_i(w)) \leq e$ for all $\mid i \mid \leq m$, whenever $d(z, w) \leq \delta$ with $w \in C$.

We now prove that $B \cap C \cap B[z, \delta] \subset \Gamma_e(z)$. Let $w \in B \cap C \cap B[z, \delta]$ which implies $w \in C \cap B[z, \delta]$ and hence, $d(F_i(z), F_i(w)) \leq e$ for all $\mid i \mid \leq m$. Again $z, w \in B = A(x_k, x_{k'}, n, m)$, so observe that $d(F_i(z), F_i(w)) \leq e$ for all $\mid i \mid \geq m$. Combining we get $d(F_i(z), F_i(w)) \leq e$ for all $i \in \mathbb{Z}$ which implies $w \in \Gamma_e(z)$ and hence, $B \cap C \cap B[z, \delta] \subset \Gamma_e(z)$. Thus $\mu(B \cap C \cap B[z, \delta]) = 0$, which is a contradiction. □

Example 3.12. For $n \geq 0$, let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = x$ if $x \in \mathbb{Q}$ and $f_n(x) = (n+1)x$ if $x \in \mathbb{R} \setminus \mathbb{Q}$. Then, the Lebesgue measure on $\mathbb{R}$ is expansive with respect to the time varying bi-measurable map $F = \{f_n\}_{n\in\mathbb{N}}$. Thus, the set of points with converging semiorbits under F has measure zero with respect to μ.

Definition 3.13. Let $F = \{f_n\}_{n\in\mathbb{N}}$ be a time varying bi-measurable map on X. A point $p \in X$ is said to be periodic if there exists an integer $k > 0$ such that $F_{ik+j}(p) = F_j(p)$ for all $i \in \mathbb{Z}$ and $0 \leq j < k$. The positive integer k is said to be a period of p. A set $A \subset X$ is said to be periodic if there exists $k > 0$ such that each point in A is periodic with period k. A time varying bi-measurable map F is said to be aperiodic with respect to μ if every measurable periodic subset of X has measure zero.

Corollary 3.14. *If $F = \{f_n\}_{n\in\mathbb{N}}$ is a time varying uniform equivalence on a separable metric space X such that $\{f_n, f_n^{-1}\}_{n\in\mathbb{N}}$ is equicontinuous, then it is aperiodic with respect to any expansive outer regular measure.*

Proof. Let μ be an expansive outer regular measure with respect to F. Let m be a positive integer and A be a measurable subset such that for each $x \in A$, we have $F_{im+j}(x) = F_j(x)$ for all $i \in \mathbb{Z}$ and $0 \leq j < m$. Thus, the orbit of x is given by $\{F_0(x), F_1(x), \dots, F_{m-1}(x)\}$. Since $F_{nm}(x) = \{x\}$ for all $n \geq 1$, we have $A \subset A(F^m)$, where $F^m = \{g_k = F_{[(n-1)m+1,nm]}\}_{n\in\mathbb{N}}$. By Theorem 3.6, μ is expansive with respect to F^m. So by Theorem 3.11, $\mu(A) \leq \mu(A(F^m)) = 0$. This completes our proof. □

Definition 3.15. Let $F = \{f_n\}_{n\in\mathbb{N}}$ be a time varying measurable map on X. The stable class of $p \in X$ is defined as $W^s(p) = \{x \in X \mid$ for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(F_n(p), F_n(x)) < \epsilon$ for all $n \geq N\}$.

Theorem 3.16. *Let $F = \{f_n\}_{n\in\mathbb{N}}$ be a time varying measurable map on a separable metric space X and μ is a positively expansive outer regular measure with respect to F. Then, any stable class of F has measure zero.*

Proof. Let $p \in X$ and e be a positively expansive constant for μ. If $A_N = \{x \in X \mid d(F_n(p), F_n(x)) < \frac{e}{2}$ for all $n \geq N\}$, then it is easy to verify that $W^s(p) \subset \bigcup_{N\geq 0} A_N$ and each A_N is measurable. It is enough to show that $\mu(\bigcup_{N\geq 1} A_N) = 0$.

If possible, suppose $\mu(\bigcup_{N\geq 1} A_N) > 0$. Then, there is $M \geq 1$ such that $\mu(A_M) > 0$.

Since X is separable, it is a second countable metric space. Since μ is outer regular, therefore Theorem 2.2 implies that for every $\epsilon > 0$ there exists a measurable set $C_\epsilon \subset X$ with $\mu(X \setminus C_\epsilon) < \epsilon$ such that $F_i \mid_{C_\epsilon}$ is continuous for all $0 \leq i \leq M$. Taking $\epsilon = \frac{\mu(A_M)}{2}$ we get a measurable set $C = C_{\frac{\mu(A_M)}{2}}$ such that $F_i \mid_C$ is continuous for all $0 \leq i \leq M$ and $\mu(A_M \cap C) > 0$.

Further, since $K = A_M \cap C$ is a Lindelöf subspace of X, by Theorem 2.3 there are $z \in A_M \cap C$ and $\delta_0 > 0$ such that $\mu(A_M \cap C \cap B[z,\delta]) > 0$ for all $0 < \delta < \delta_0$. Since $z \in C$ and $F_i \mid_C$ is continuous for all $0 \leq i \leq M$, we can fix $0 < \delta < \delta_0$ such that $d(F_i(z), F_i(w)) \leq e$ for all $0 \leq i \leq M$, whenever $d(z,w) \leq \delta$ with $w \in C$.

We now prove that $A_M \cap C \cap B[z,\delta] \subset \Phi_e(z)$. Let $w \in A_M \cap C \cap B[z,\delta]$ which implies $w \in C \cap B[z,\delta]$ and hence, $d(F_i(z), F_i(w)) \leq e$ for all $0 \leq i \leq M$. Again $z, w \in A_M$, so observe that $d(F_i(z), F_i(w)) \leq e$ for all $i \geq M$. Combining we get $d(F_i(z), F_i(w)) \leq e$ for all $i \in \mathbb{N}$ which implies $w \in \Phi_e(z)$ and hence, $A_M \cap C \cap B[z,\delta] \subset \Phi_e(z)$. Thus $\mu(B \cap C \cap B[z,\delta]) = 0$, which is a contradiction. □

4. Measurable stability theorem

The main purpose of this section is to provide a sufficient condition for an expansive measure to possess topological stability. In particular, we prove the following result.

Theorem 4.1. *Let $F = \{f_n\}_{n\in\mathbb{N}}$ be a time varying homeomorphism on a Mandelkern locally compact metric space X. If μ is expansive and persistent with respect to F, it is topologically stable with respect to F in the class of all time varying homeomorphisms.*

The following notions of persistence are generalizations of the autonomous notion of persistence introduced in [3].

Definition 4.2. Let $F = \{f_n\}_{n\in\mathbb{N}}$ be a time varying bi-measurable map on X. Then,

(i) F is said to be persistent if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $G = \{g_n\}_{n\in\mathbb{N}}$ is another time varying bi-measurable map with $p(F, G) < \delta$, then for each $x \in X$ there exists $y \in X$ such that $d(F_n(y), G_n(x)) < \epsilon$ for all $n \in \mathbb{Z}$.

(ii) μ is said to be persistent with respect to F if for every $\epsilon > 0$ there exists $\delta > 0$ and a measurable set $B \subset X$ with $\mu(X \setminus B) = 0$ such that if $G = \{g_n\}_{n\in\mathbb{N}}$ is another time varying bi-measurable map with $p(F, G) < \delta$, then for each $x \in B$ there exists $y \in X$ such that $d(F_n(y), G_n(x)) < \epsilon$ for all $n \in \mathbb{Z}$.

Remark 4.3. Any Borel measure is persistent with respect to a persistent time varying bi-measurable map.

Theorem 4.4. *Let $F = \{f_n\}_{n\in\mathbb{N}}$ be a time varying bi-measurable map on X and $h : X \to X$ be a uniform equivalence. If μ is persistent with respect to F, then $h_*^{-1}(\mu)$ is persistent with respect to $F' = \{f'_n\}_{n\in\mathbb{N}}$, where $f'_n = h^{-1} \circ f_n \circ h$ for all $n \in \mathbb{N}$.*

Proof. Fix $0 < \epsilon < 1$. Then, there exists $0 < \epsilon' < \epsilon$ such that $d_1(a, b) < \epsilon'$ implies $d_1(h^{-1}(a), h^{-1}(b)) < \epsilon$. Let $0 < \delta' < 1$ and a measurable set $B \subset X$ with $\mu(X \setminus B) = 0$ be given for ϵ' by the persistence of μ with respect to F. Again, there exists $0 < \delta < 1$ such that $d_1(a, b) < \delta$ implies $d_1(h(a), h(b)) < \delta'$.

Suppose $G = \{g_n\}_{n\in\mathbb{N}}$ is another time varying bi-measurable map such that $p(F', G) < \delta$, i.e., $\max\{\sup_{n\geq 0} \eta(f'_n, g_n), \sup_{n\geq 0} \eta(f'^{-1}_n, g_n^{-1})\} < \delta$. This implies that $d_1((h^{-1} \circ f_n \circ h)(x), g_n(x)) < \delta$ and $d_1((h^{-1} \circ f_n^{-1} \circ h)(x), g_n^{-1}(x)) < \delta$ for all $x \in X$, $n \in \mathbb{N}$. Thus, $d_1(f_n(h(x)), (h \circ g_n \circ h^{-1})(h(x))) < \delta'$ and $d_1(f_n^{-1}(h(x)), (h \circ g_n \circ h^{-1})^{-1}(h(x))) < \delta'$ for all $x \in X$, $n \in \mathbb{N}$, which implies $d_1(f_n(h(x)), g'_n(h(x))) < \delta'$ and $d_1(f_n^{-1}(h(x)), g'^{-1}_n(h(x))) < \delta'$ for all $x \in X$, $n \in \mathbb{N}$, where $g'_n = h \circ g_n \circ h^{-1}$ for all $n \in \mathbb{N}$. If $G' = \{g'_n\}_{n\in\mathbb{N}}$, then by the persistence of μ with respect to F, for each $x \in B$ there exists $y \in X$ such that $d(F_n(y), G'_n(x)) < \epsilon'$ for all $n \in \mathbb{Z}$ which implies that $d(h^{-1}(F_n(y)), h^{-1}(G'_n(x))) < \epsilon$ for all $n \in \mathbb{Z}$. From this we have that

$$d(F'_n(h^{-1}(y)), G_n(h^{-1}(x))) < \epsilon$$

for all $n \in \mathbb{Z}$. Since $h_*^{-1}(\mu)(X \setminus h^{-1}(B)) = \mu(h(X) \setminus B) = 0$, therefore we conclude that $h_*^{-1}(\mu)$ is persistent with respect to F'. $\square$

The following definition extends the autonomous notion of topological stability for Borel measures to non-autonomous systems.

Definition 4.5. Let $F = \{f_n\}_{n\in\mathbb{N}}$ be a time varying bi-measurable map on X. Then, a Borel measure μ is said to be topologically stable with respect to F if for every $\epsilon > 0$ there exists $0 < \delta < 1$ such that if $G = \{g_n\}_{n\in\mathbb{N}}$ is another time varying bi-measurable map on X with $p(F, G) < \delta$, then there exists an upper semi-continuous compact valued map $H : X \to \mathcal{P}(X)$ with measurable domain satisfying the following conditions.

(i) $\mu(X \setminus Dom(H)) = 0$,
(ii) $\mu \circ H = 0$,
(iii) $d(H, Id) < \epsilon$,
(iv) $F_n(H(x)) \subset B[G_n(x), \epsilon]$ for all $n \in \mathbb{Z}$.

Theorem 4.6. *Let $F = \{f_n\}_{n\in\mathbb{N}}$ be a time varying bi-measurable map on X. If F is topologically stable, then every non-atomic Borel measure μ is topologically stable with respect to F.*

Proof. Fix $0 < \epsilon < 1$ and let $0 < \delta < 1$ be given for ϵ by the topological stability of F. Let $G = \{g_n\}_{n\in\mathbb{N}}$ be another time varying bi-measurable map such that $p(F, G) < \delta$. By topological stability of F, there exists a continuous map $h : X \to X$ such that $d(h(x), x) < \epsilon$ and $d(F_n(h(x)), G_n(x)) \leq \epsilon$ for all $n \in \mathbb{Z}$. Define the compact valued map $H : X \to \mathcal{P}(X)$ by $H(x) = \{h(x)\}$ for all $x \in X$. H is upper semi-continuous because of continuity of h. Since $Dom(H) = X$, $\mu(X \setminus Dom(H)) = 0$ and since μ is non-atomic, $\mu \circ H = 0$. Further, $d(H(x), x) = d(h(x), x) < \epsilon$ which gives $d(H, Id) < \epsilon$. Finally, $F_n(H(x)) = F_n(h(x)) \subset B[G_n(x), \epsilon]$ for all $n \in \mathbb{Z}$. This completes our proof. $\square$

The following corollary of the above result can be verified using Theorem 2.1.

Corollary 4.7. *Every complete separable metric space supporting topologically stable bi-measurable map without supporting topologically stable measure is countable.*

Theorem 4.8. *Let $F = \{f_n\}_{n\in\mathbb{N}}$ be a time varying bi-measurable map on X and $h : X \to X$ be a uniform equivalence. If μ is topologically stable with respect to F, then $h_*^{-1}(\mu)$ is topologically stable with respect to $F' = \{f'_n\}_{n\in\mathbb{N}}$, where $f'_n = h^{-1} \circ f_n \circ h$ for all $n \in \mathbb{N}$.*

Proof. Fix $0 < \epsilon < 1$. Then, there exists $0 < \epsilon' < \epsilon$ such that $d_1(a, b) < \epsilon'$ implies $d_1(h^{-1}(a), h^{-1}(b)) < \epsilon$. Let $0 < \delta' < 1$ and a measurable set $B \subset X$ with $\mu(X \setminus B) = 0$ be given for ϵ' by the persistence of μ with respect to F. Again, there exists $0 < \delta < 1$ such that $d_1(a, b) < \delta$ implies $d_1(h(a), h(b)) < \delta'$.

Suppose $G = \{g_n\}_{n\in\mathbb{N}}$ is another time varying bi-measurable map such that $p(F', G) < \delta$, i.e., $\max\{\sup_{n\geq 0} \eta(f'_n, g_n), \sup_{n\geq 0} \eta(f'^{-1}_n, g_n^{-1})\} < \delta$. This implies that $d_1((h^{-1} \circ f_n \circ h)(x), g_n(x)) < \delta$ and $d_1((h^{-1} \circ f_n^{-1} \circ h)(x), g_n^{-1}(x)) < \delta$

for all $x \in X$, $n \in \mathbb{N}$. Thus, $d_1(f_n(h(x)), (h \circ g_n \circ h^{-1})(h(x))) < \delta'$ and $d_1(f_n^{-1}(h(x)), (h \circ g_n \circ h^{-1})^{-1}(h(x))) < \delta'$ for all $x \in X$, $n \in \mathbb{N}$, which implies $d_1(f_n(h(x)), g'_n(h(x))) < \delta'$ and $d_1(f_n^{-1}(h(x)), g'^{-1}_n(h(x))) < \delta'$ for all $x \in X$, $n \in \mathbb{N}$, where $g'_n = h \circ g_n \circ h^{-1}$ for all $n \in \mathbb{N}$. If $G' = \{g'_n\}_{n\in\mathbb{N}}$, then by topological stability of μ with respect to F, there exists an upper semi-continuous compact valued map $H : X \to \mathcal{P}(X)$ with measurable domain satisfying $\mu(X \setminus Dom(H)) = 0$, $\mu \circ H = 0$, $d(H, Id) < \epsilon'$, $F_n(H(x)) \subset B[G'_n(x), \epsilon']$ for all $n \in \mathbb{Z}$.

Then, $K = h^{-1} \circ H \circ h$ is an upper semi-continuous compact valued map of X with measurable domain such that

(i) $h_*^{-1}(\mu)(X \setminus Dom(K)) = \mu(h(X \setminus Dom(K))) = \mu(X \setminus h(Dom(K))) = \mu(X \setminus Dom(H)) = 0$.

(ii) $h_*^{-1}(\mu)(K(x)) = \mu(h(K(x))) = \mu(H(h(x))) = 0$ (since $\mu \circ H = 0$). Thus, $h_*^{-1}(\mu) \circ K = 0$.

(iii) $d(H, Id) < \epsilon'$ implies that $d((h \circ K \circ h^{-1})(x), x) < \epsilon'$ for all $x \in X$. Then, we get that $d(K(h^{-1}(x)), h^{-1}(x)) < \epsilon$ for all $x \in X$ and thus, $d(K, Id) < \epsilon$.

(iv) For each $n \in \mathbb{Z}$, we have $F'_n(K(x)) = (h^{-1} \circ F_n \circ h)(K(x)) = (h^{-1} \circ F_n)(H(h(x))) \subset h^{-1}(B[(h \circ G_n \circ h^{-1})(h(x)), \epsilon']) = h^{-1}(B[h(G_n(x)), \epsilon']) \subset B[G_n(x), \epsilon]$.

Thus, we conclude that $h_*^{-1}(\mu)$ is topologically stable with respect to F'. □

Theorem 4.9. *Every topologically stable measure of an expansive time varying bi-measurable map is non-atomic* (*hence, expansive*)*.*

Proof. Let μ be a topologically stable measure with respect to an expansive time varying bi-measurable map $F = \{f_n\}_{n\in\mathbb{N}}$ on X. Let $0 < \epsilon < 1$ be an expansive constant for F and $0 < \delta < 1$, a measurable set $B \subset X$ with $\mu(X \setminus B) = 0$ be given for ϵ by the topological stability of μ. Taking $G = F$ in the definition of topological stability of μ, we get an upper semi-continuous compact valued map $H : X \to \mathcal{P}(X)$ with measurable domain such that $\mu(X \setminus Dom(H)) = 0$, $\mu \circ H = 0$, $d(H, Id) < \epsilon$ and $F_n(H(x)) \subset B[F_n(x), \epsilon]$ for all $n \in \mathbb{Z}$. If $x \in Dom(H)$, then there exists $y \in H(x)$. Thus, $F_n(y) \in B[F_n(x), \epsilon]$ for all $n \in \mathbb{Z}$ and hence $d(F_n(x), F_n(y)) \leq \epsilon$ for all $n \in \mathbb{Z}$. Since ϵ is an expansive constant, we must have $x = y$. Therefore, $H(x) = \{x\}$ for all $x \in Dom(H)$. If possible, suppose that z is an atom for μ. Since $\mu(X \setminus Dom(H)) = 0$, $z \in Dom(H)$. Thus, $H(z) = \{z\}$ and hence, $\mu(H(z)) = 0$. This is a contradiction. □

Theorem 4.10. *Every topologically stable measure with respect to a time varying bi-measurable map is persistent.*

Proof. Fix $0 < \epsilon < 1$ and let $0 < \delta < 1$ be given for ϵ by the topological stability of μ with respect to a time varying bi-measurable map $F = \{f_n\}_{n\in\mathbb{N}}$. Let $G = \{g_n\}_{n\in\mathbb{N}}$ be another time varying bi-measurable map such that $p(F, G) < \delta$.

By topological stability of μ there exists upper semi-continuous compact valued map $H : X \to \mathcal{P}(X)$ with measurable domain such that $\mu(X \setminus Dom(H)) = 0 = \mu \circ H$, $d(H, Id) < \epsilon$ and $F_n(H(x)) \subset B[G_n(x), \epsilon]$ for all $n \in \mathbb{Z}$. It then follows that there exists $y \in H(x)$ for all $x \in Dom(H)$. Therefore, $F_n(y) \in B[G_n(x), \epsilon]$ for all $n \in \mathbb{Z}$ which implies $d(F_n(y), G_n(x)) \leq \epsilon$ for all $n \in \mathbb{Z}$. This completes our proof. $\square$

We now prove Theorem 4.1 which provides sufficient condition for persistent measure to be topologically stable.

Proof of Theorem 4.1. Let μ be expansive and persistent with respect to F. Let e be an expansive constant for μ. Take $0 < \epsilon < 1$ and $0 < \epsilon' <\min(\frac{e}{2}, \epsilon)$. Let $0 < \delta < 1$ and $B \subset X$ with $\mu(X \setminus B) = 0$ be given for ϵ' by the persistence of μ. Let $G = \{g_n\}_{n \in \mathbb{N}}$ be another time varying homeomorphism such that $p(F, G) < \delta$. Define the compact valued map $H : X \to \mathcal{P}(X)$ given by $H(x) = \bigcap_{n \in \mathbb{Z}} F_n^{-1}(B[G_n(x), \epsilon'])$

First we prove that $Dom(H)$ is measurable. Take a sequence $x_k \in Dom(H)$ converging to some $x \in X$. Since $x_k \in Dom(H)$, we can choose a sequence $y_k \in X$ such that $d(F_n(y_k), G_n(x_k)) \leq \epsilon'$ for all $k \in \mathbb{N}$, $n \in \mathbb{Z}$. Thus, $y_k \in B[x_k, \epsilon']$ for all $k \in \mathbb{N}$. Let K be a compact neighborhood of x. Since $x_k \to x$ as $k \to \infty$, there exists $N \in \mathbb{N}$ such that $x_k \in K$ for all $k \geq N$. So, $y_k \in B[K, \epsilon']$ for all $k \geq N$. Since X is Mandelkern locally compact, $B[K, \epsilon']$ is contained in a compact set. So, there exists a subsequence of y_k converging to some point y in X. Therefore, $d(F_n(y), G_n(x)) \leq \epsilon'$ for all $n \in \mathbb{Z}$ which implies $y \in H(x)$. This shows that $x \in Dom(H)$ which means $Dom(H)$ is closed and hence, measurable.

We now prove that $\mu(X \setminus Dom(H)) = 0$. By the persistence of μ, for each $x \in B$ there exists $y \in X$ such that $d(F_n(y), G_n(x)) \leq \epsilon'$ for all $n \in \mathbb{Z}$ which implies $y \in H(x)$. This means that $H(x) \neq \phi$ for all $x \in B$ and thus, $B \subset Dom(H)$. Therefore, $\mu(X \setminus Dom(H)) = 0$.

Afterwards, we prove that H is upper semi-continuous. Let $x \in Dom(H)$ and O be an open neighbourhood of $H(x)$.

Define $H(y) = \bigcap_{m \geq 0} H_m(y)$, where $H_m(y) = \bigcap_{n=-m}^{m} F_n^{-1}(B[G_n(y), \epsilon'])$.

Clearly, there exists $m \in \mathbb{Z}$ such that $H_m(y) \subset O$. We assert that there exists $\eta > 0$ such that $H_m(y) \subset O$ for all $y \in X$ with $d(x, y) < \eta$. If not, there exist y_k converging to x and $z_k \in H_m(y_k) \setminus O$ for all $k \in \mathbb{N}$. Since $y_k \to x$, for any compact set K containing x there exists natural number N such that $y_k \in K$ for all $k \geq N$. Thus for all $k \geq N$, we have $z_k \in H_m(K)$. Since $H_m(K)$ is compact, we can assume that z_k converges to a point, say z and observe that $z \notin O$. But $z_k \in H_m(y_k)$ for all $k \in \mathbb{N}$ implies that $d(F_n(z_k), G_n(y_k)) \leq \epsilon'$ for all $k \in \mathbb{N}$ and $-m \leq n \leq m$. Then, $d(F_n(z), G_n(x)) \leq \epsilon'$ for $-m \leq n \leq m$. So, $z \in H_m(x) \subset O$. This leads to a contradiction. Hence, H is upper semi-continuous.

Now we prove $\mu \circ H = 0$. Take $x \in X$ and $y \in H(x)$. If $z \in H(x)$, then we have $d(F_n(z), G_n(x)) \leq \epsilon'$ for every $n \in \mathbb{Z}$. Since $y \in H(x)$, we have

$d(F_n(y), G_n(x)) \leq \epsilon'$ for every $n \in \mathbb{Z}$. Therefore, $d(F_n(y), F_n(z)) \leq 2\epsilon'$ for all $n \in \mathbb{Z}$. Since $2\epsilon' < e$, we conclude that $z \in \Gamma_e(y)$ which implies that $H(x) \subset \Gamma_e(y)$. Since e is an expansive constant of μ, $\mu(H(x)) \leq \mu(\Gamma_e(y)) = 0$. This proves $\mu \circ H = 0$.

It follows from the definition of H that $H(x) \subset B[x, \epsilon']$. Since $\epsilon' < \epsilon$, we also have $d(H, Id) \leq \epsilon$.

Finally, from the definition of $H(x)$ it is clear that $F_n(H(x)) \subset B[G_n(x), \epsilon]$ for all $n \in \mathbb{Z}$. ☐

Definition 4.11. Let $F = \{f_n\}_{n\in\mathbb{N}}$ be a time varying bi-measurable map on X. Then, a sequence $\{x_n\}_{n\in\mathbb{Z}}$ is said to be a δ-pseudo orbit for F if $d(f_{n+1}(x_n), x_{n+1}) < \delta$ for all $n \geq 0$ and $d(f_{-n}^{-1}(x_{n+1}), x_n) < \delta$ for all $n \leq -1$. A sequence $\{x_n\}_{n\in\mathbb{Z}}$ is said to be ϵ-shadowed if there exists some point y in X such that $d(F_n(y), x_n) < \epsilon$ for all $n \in \mathbb{Z}$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be through $B \subset X$ if $x_0 \in B$. A measure μ is said to have shadowing with respect to F if for every $\epsilon > 0$ there exists $\delta > 0$ and a measurable set $B \subset X$ with $\mu(X \setminus B) = 0$ such that every δ-pseudo orbit through B is ϵ-shadowed by some point in X. If $B = X$, then we say that F has shadowing [9].

Remark 4.12. If μ has shadowing with respect to F, then it is persistent with respect to F. Indeed, fix $0 < \epsilon < 1$ and let $0 < \delta < 1$ and a measurable set $B \subset X$ with $\mu(X \setminus B) = 0$ be given for $\epsilon > 0$ by the shadowing of μ with respect to F. Let $G = \{g_n\}_{n\in\mathbb{N}}$ be another time varying bi-measurable map such that $p(F, G) < \delta$. Then for any $x \in B$, we have $\eta(f_{n+1}(G_n(x)), G_{n+1}(x)) = \eta(f_{n+1}(G_n(x)), g_{n+1}(G_n(x))) < \delta$ for all $n \geq 0$ and $\eta(f_{-n}^{-1}(G_{n+1}(x)), G_n(x)) = \eta(f_{-n}^{-1}(G_{n+1}(x)), f_{-n}^{-1}(G_{n+1}(x))) < \delta$ for all $n \leq -1$.

By looking at the construction of η, one can conclude that $\{G_n(x)\}_{n\in\mathbb{N}}$ is a δ-pseudo orbit of F through B.

So by the shadowing of μ, there exists $y \in X$ such that $d(F_n(y), G_n(x)) < \epsilon$ for all $n \in \mathbb{Z}$. This means that μ is persistent with respect to F.

The following corollary of Theorem 4.1 is an immediate consequence of Remark 4.12.

Corollary 4.13. *Let $F = \{f_n\}_{n\in\mathbb{N}}$ be a time varying homeomorphism on a Mandelkern locally compact metric space X. If μ is expansive and has shadowing with respect to F, then it is topologically stable with respect to F in the class of all time varying homeomorphisms.*

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Pramod Das
Department of Mathematics
Faculty of Mathematical Sciences
University of Delhi
Delhi-110007, India
Email address: pramod.math.ju@gmail.com

Tarun Das
Department of Mathematics
Faculty of Mathematical Sciences
University of Delhi
Delhi-110007, India
Email address: tarukd@gmail.com