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RELATIVE GOTTLIEB GROUPS OF THE PLÜCKER EMBEDDING OF SOME COMPLEX GRASSMANIANS

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Abstract. Let $\mathrm{Gr}(k,n)$ be the Grassmann manifold of k-linear subspaces in $\mathbb{C}^n$. We compute rational relative Gottlieb groups of the Plücker embedding $i : \mathrm{Gr}(2,n) \to \mathbb{C}P^{N-1}$, where $N = n(n-1)/2$.

1. Introduction

All spaces are assumed to be simply connected CW complexes of finite type. We denote by $h : X \to X_{\mathbb{Q}}$ the rationalization of X [3, §9]. Let $f : X \to Y$ be a pointed continuous mapping and $\mathrm{map}(X, Y; f)$ the component of f in the space of continuous mappings from X to Y. Consider the evaluation map $\mathrm{ev} : \mathrm{map}(X, Y; f) \to Y$. The nth evaluation subgroup of f, $G_n(Y, X; f)$, is the image of $\pi_n(\mathrm{ev})$ in $\pi_n(Y)$ [17]. In the special case where $X = Y$ and $f = 1_X$, one obtains the Gottlieb group $G_n(X)$ of X [4]. It is also obtained as the image of the connecting map of the long homotopy exact sequence of the universal fibration of fibre X. Gottlieb groups play an important role in topology. For instance, if $G_n(X) = 0$, then any fibration $X \to E \longrightarrow S^{n+1}$ admits a section [4, Corollary 2-7]. If X is a finite CW complex and Y a simply connected CW complex of finite type, then the rationalization $h : Y \to Y_{\mathbb{Q}}$ induces a rationalization $h_* : \mathrm{map}(X, Y; f) \to \mathrm{map}(X, Y; h \circ f)$ [5]. Therefore

$$\mathrm{ev}_*(\pi_*(\mathrm{map}(X, Y; f)) \otimes \mathbb{Q}) \cong \mathrm{ev}_*(\pi_*(\mathrm{map}(X, Y_{\mathbb{Q}}; h \circ f))).$$

In [18] Lee and Woo introduce relative evaluation groups $G_n^{rel}(Y, X; f)$ and obtain a long sequence

$$\cdots \to G_{n+1}^{rel}(Y, X; f) \to G_n(X) \to G_n(Y, X; f) \to G_n^{rel}(Y, X; f) \to \cdots,$$

called G-sequence [8]. This sequence is exact in some cases, for instance if f is a homotopy monomorphism [13].

In this paper we describe the rational relative Gottlieb subgroup of the Plücker embedding $i : \mathrm{Gr}(2,n) \hookrightarrow \mathbb{C}P^{N-1}$ of the Grassmanian of 2-subspaces of $\mathbb{C}^n$ into the the complex projective space of dimension $N-1$ where $N = n(n-$

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1)/2. More specifically, we show that: $G_*^{rel}(\mathbb{C}P^{N-1}, Gr(2,n)_{\mathbb{Q}}, h\circ i)$ splits as the direct sum of the suspension of $G_*(Gr(2,n)_{\mathbb{Q}})$ and $G_{2N-1}(\mathbb{C}P^{N-1}, Gr(2,n)_{\mathbb{Q}}, h\circ i)$, where $h : \mathbb{C}P^{N-1} \to \mathbb{C}P^{N-1}_{\mathbb{Q}}$ is the rationalization.

2. Relative Gottlieb groups

We work with rational homotopy models of simply connected topological spaces which were introduced by Quillen and Sullivan [14, 16]. In this section we give relevant definitions and fix notation. Details can be found in [3]. All vector spaces and algebras are over the field of rational numbers $\mathbb{Q}$.

Definition 1. A differential graded algebra (A, d) is a graded algebra $A = \oplus_{n\geq 0} A^n$ together with a differential $d : A^n \to A^{n+1}$ which is a derivation. The degree of an element a is denoted by $|a|$. We assume that (A, d) is 1-connected, that is, $H^0(A, d) = \mathbb{Q}$ and $H^1(A, d) = 0$. A graded algebra A is called commutative if $ab = (-1)^{|a||b|}ba$, $a, b \in A$. A commutative differential graded algebra (cdga for short) (A, d) is called a Sullivan algebra if $A = S(V^{even}) \otimes E(V^{odd})$, where $V = \oplus_{k\geq 2} V^k$. It will be denoted by $(\wedge V, d)$.

Definition 2. A Sullivan algebra $(\wedge V, d)$ is called minimal if $dV \subset \wedge^{\geq 2} V$. A Sullivan model of (A, d) is given by a Sullivan algebra $(\wedge V, d)$ together with a quasi-isomorphism $f : (\wedge V, d) \to (A, d)$. It is unique up to isomorphism.

If X is a simply connected space of finite type, then the (minimal) Sullivan model of X is the (minimal) Sullivan model of the cdga $A_{PL}(X)$ of polynomial differential forms on X [3, §10]. A simply connected topological space X is called formal if there exists a quasi-isomorphism $(\wedge V, d) \to H^*(X, \mathbb{Q})$, where $(\wedge V, d)$ is a Sullivan model of X.

The complex Grassmann manifold $\mathrm{Gr}(k, n)$ is the space of k dimensional subspaces of $\mathbb{C}^n$. As $\mathrm{Gr}(k, n) = U(n)/(U(k) \times U(n-k))$ is a homogeneous space of equal rank as well as a Kähler manifold, hence it is formal [16, §12], [2]. A Sullivan model of $\mathrm{Gr}(k, n)$ can be computed using Proposition 15.6 of [3] and it is explicitly given by Murillo [11, Theorem 2]. Moreover $\mathrm{Gr}(k, n)$ is a complex manifold of dimension $k(n-k)$ for which the rational cohomology is the quotient algebra

$$\wedge(y_2, y_4, \ldots, y_{2k})/(h_{n-k+1}, \ldots, h_{n-1}, h_n),$$

where h_j is the polynomial of degree $2j$ in the Taylor expansion of the expression $1/(1 + y_2 + \cdots + y_{2k})$ [6]. A Sullivan model is given by

$$(\wedge(y_2, \ldots, y_{2k}, y_{2n-2k+1}, \ldots, y_{2n-1}), d),$$

where $dy_{2i} = 0$, $dy_{2n-2k+2j-1} = h_{n-k+j}$, $j = 1, \ldots, k$. Moreover this model is minimal. In particular, the minimal Sullivan model of $\mathrm{Gr}(2, n)$ is

$$\wedge(y_2, y_4, y_{2n-3}, y_{2n-1}),\ dy_2 = dy_4 = 0,\ dy_{2n-3} = h_{n-1},\ dy_{2n-1} = h_n.$$

It can be shown by induction that

$$N = \begin{pmatrix} n \\ k \end{pmatrix} > k(n-k).$$

Therefore $H^{\geq 2N}(\mathrm{Gr}(k,n),\mathbb{Q}) = 0$. Hence y_2^N is a coboundary in

$$(\wedge(y_2,\dots,y_{2k},y_{2n-2k+1},\dots,y_{2n-1}),d).$$

We recall that the minimal Sullivan model of $\mathbb{C}P^{N-1}$ is given by

$$(\wedge(x_2,x_{2N-1}),d),$$

where $dx_2 = 0$ and $dx_{2N-1} = x_2^N$. A Sullivan model of the inclusion $i : \mathrm{Gr}(2,n) \to \mathbb{C}P^{N-1}$ is then

$$\phi : (\wedge(x_2,x_{2N-1}),d) \to (\wedge(y_2,y_4,y_{2n-3},y_{2n-1}),d), \tag{1}$$

where $\phi(x_2) = y_2$ and $\phi(x_{2N-1}) = y$, where $dy = y_2^N$.

Recall that if $\phi : (A,d_A) \to (B,d_B)$ is a map of chain complexes, the mapping cone of ϕ, denoted by $\mathrm{Rel}(\phi)$ is given by

$$\mathrm{Rel}(\phi)_* = (sA_{*-1} \oplus B_*, D),$$

where the differential is defined by $D(sa,b) = (-sd_A(a), \phi(a) + d_B b)$ [9] or [10, p. 46]. Define chain maps $J : B_n \to \mathrm{Rel}_n(\phi)$ and $P : \mathrm{Rel}_n(\phi) \to A_{n-1}$ by $J(b) = (0,b)$ and $P(sa,b) = a$. There is an exact sequence of chain complexes

$$0 \to B_* \xrightarrow{J} \mathrm{Rel}_*(\phi) \xrightarrow{P} A_{*-1} \to 0$$

which induces a long exact sequence in homology [10, Proposition 4.3].

Definition 3. Let X be a topological space. We say $\alpha \in \pi_n(X)$ is a Gottlieb element if the map: $f \vee 1_X : S^n \vee X \to X$ extends to $S^n \times X$, where f represents the homotopy class α [4].

Gottlieb elements form a subgroup of $\pi_*(X)$ which will be denoted by $G_*(X)$. It comes from the definition that $G_*(X)$ is the image of $\pi_*(\mathrm{ev}) : \pi_*(\mathrm{aut}_1 X, 1_X) \to \pi_*(X,x_0)$, where ev is the evaluation map at x_0.

Definition 4. Let $\phi : (A,d) \to (B,d)$ be a morphism of cdga's. A ϕ-derivation of degree k is a linear mapping $\theta : A^n \to B^{n-k}$ such that $\theta(ab) = \theta(a)\phi(b) + (-1)^{k|a|}\phi(a)\theta(b)$. We denote by $\mathrm{Der}_n(A,B;\phi)$ the vector space of ϕ-derivations of degree n and by $\mathrm{Der}(A,B;\phi) = \oplus_n \mathrm{Der}_n(A,B;\phi)$ the $\mathbb{Z}$-graded vector space of all ϕ-derivations. The differential on $\mathrm{Der}(A,B;\phi)$ is defined by $\delta\theta = d\theta - (-1)^k\theta d$. We will restrict to derivations of positive degree, however in degree one, we only consider those derivations which are cycles.

If $\phi : A \to A$ is the identity mapping, we simply write $\mathrm{Der}\, A$ for $\mathrm{Der}(A,A;1_A)$. Moreover if $A = \wedge V$ where $\{v_1, v_2, \dots\}$ is a basis of V and $\phi : (\wedge V, d) \to (B,d)$ is a morphism of cdga's, we denote by (v_i, b) the unique ϕ-derivation θ such that $\theta(v_i) = b$ and zero on other elements of the basis.

Define the Gottlieb group of $(\wedge V, d)$

$$G_n(\wedge V) = \{[\theta] \in H_n(\mathrm{Der}\,\wedge V, \delta) : \theta(v) = 1,\ v \in V^n\}.$$

Hence $G_*(\wedge V) \cong \mathrm{im}\, H_*(\epsilon_*)$, where $\epsilon_* : \mathrm{Der}\,\wedge V \to \mathrm{Der}(\wedge V, \mathbb{Q}; \epsilon)$ is the post composition with the augmentation map $\epsilon : \wedge V \to \mathbb{Q}$. If X is simply connected and $(\wedge V, d)$ is the minimal Sullivan model of X, then $G_n(\wedge V) \cong G_n(X_{\mathbb{Q}})$, where $h : X \to X_{\mathbb{Q}}$ is the rationalization [3, Propostion 29.8].

Let $\phi : (\wedge V, d) \to (\wedge W, d)$ be a Sullivan model of $f : X \to Y$. It induces a chain map $\phi^* : \mathrm{Der}(\wedge W) \to \mathrm{Der}(\wedge V, \wedge W; \phi)$ by pre-composition by ϕ. We get the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Der}\,\wedge W \arrow[r, "\phi^*"] \arrow[d, "\epsilon_*"] & \mathrm{Der}(\wedge V, \wedge W; \phi) \arrow[r, "J"] \arrow[d, "\epsilon_*"] & \mathrm{Rel}(\phi^*) \arrow[d, "(\epsilon_*, \epsilon_*)"] \\
\mathrm{Der}(\wedge W, \mathbb{Q}, \epsilon) \arrow[r, "\widehat{\phi^*}"] & \mathrm{Der}(\wedge V, \mathbb{Q}, \epsilon) \arrow[r, "\hat{J}"] & \mathrm{Rel}(\widehat{\phi^*}).
\end{tikzcd} \tag{2}$$

Then rational evaluation subgroups are corresponding images in the lower ladder induced in homology by vertical maps. Therefore there is a long sequence

$$\cdots \to G_n(\wedge W) \overset{H(\widehat{\phi}^*)}{\to} G_n(\wedge V, \wedge W; \phi) \overset{H(\widehat{J})}{\to} G_n^{rel}(\wedge V, \wedge W; \phi) \overset{H(\widehat{P})}{\to} \cdots .$$

We will use the following result for our computations [9, Theorem 2.1], [1, Corollary 7].

Theorem 5. *Let $f : X \to Y$ be a map between simply connected CW complexes where X is of finite type and $\phi : (\wedge V, d) \to (\wedge W, d)$ its Sullivan model. The long exact sequence induced by the map $f_* : \mathrm{map}(X, X; 1_X) \to \mathrm{map}(X, Y; f)$ on rational homotopy groups is equivalent to the long exact sequence of*

$$\phi^* : \mathrm{Der}(\wedge W, d) \to \mathrm{Der}(\wedge V, \wedge W; \phi).$$

We consider the particular case where f is the inclusion $i : \mathrm{Gr}(2, n) \to \mathbb{C}P^{N-1}$ for which the Sullivan model is given by (1). A direct computation shows $G_*(\wedge W) = \langle [y^*_{2n-3}], [y^*_{2n-1}] \rangle$. See [15] for a general result on rational Gottlieb groups of flag manifolds.

Lemma 6. *Let $N = n(n-1)/2$. Then $2(N-1) > 4n - 8$ for $n \geq 4$.*

Proof. Here $2(N-1) = n(n-1) - 2$. The inequality $n(n-1) > 4n - 8$ holds for $n = 4$. For $n \geq 5$, then $n(n-1) \geq 4n$, hence $n(n-1) - 2 > 4n - 8$. $\square$

Lemma 7. *Let $\phi : (\wedge(x_2, x_{2N-1}), d) \to (\wedge(y_2, y_4, y_{2n-3}, y_{2n-1}), d)$, where $\phi(x_2) = y_2$, $\phi(x_{2N-1}) = y$ and $dy = y_2^N$. There is a ϕ-derivation α_2 such that $\alpha_2(x_2) = 1$ and which is a cycle.*

Proof. We need to define α_2 on x_{2N-1} so that

$$d\alpha_2(x_{2N-1}) - \alpha_2(dx_{2N-1}) = 0.$$

Hence

$$\begin{aligned} d\alpha_2(x_{2N-1}) - \alpha_2(dx_{2N-1}) &= d\alpha_2(x_{2N-1}) - \alpha_2(x_2^N) \\ &= d\alpha_2(x_{2N-1}) - Ny_2^{N-1}. \end{aligned}$$

As the dimension of $\mathrm{Gr}(2,n)$ is less than $2(N-1)$ by Lemma 6, then Ny_2^{N-1} is a coboundary, that is, $Ny_2^{N-1} = dz$. Define $\alpha_2(x_{2N-1}) = z$. Moreover $\delta\alpha_2 = 0$. As $\alpha_2(x_2) = 1$, then α_2 cannot be a boundary, hence it represents a non zero homology class in $H_*(\mathrm{Der}(\wedge V, \wedge W; \phi), \delta)$. □

It is easily seen that the ϕ-derivation $\alpha_{2N-1} = (x_{2N-1}, 1)$ represents a non zero homology class as well. As $H(\epsilon_*)([\alpha_2]) = [x_2^*]$ and $H(\epsilon_*)([\alpha_{2N-1}]) = [x_{2N-1}^*]$, we get the following result.

Proposition 8. $G_*(\wedge V, \wedge W; \phi) = \mathbb{Q}\langle [x_2^*], [x_{2N-1}^*]\rangle$.

We decompose $\mathrm{Rel}_*(\phi^*) = \mathrm{Ker}(\epsilon_*, \epsilon_*) \oplus K$. As K is isomorphic to $\mathrm{im}(\epsilon_*, \epsilon_*)$, a non zero element in $G_*^{rel}(\wedge V, \wedge W; \phi)$ is represented by an element in K, modulo the kernel of (ϵ_*, ϵ_*). Moreover the vector space K is spanned by

$$\{(0, \alpha_2), (0, \alpha_{2N-1}), (s\beta_2, 0), (s\beta_4, 0), (s\beta_{2n-3}, 0), (s\beta_{2n-1}, 0)\},$$

where $\beta_j = (y_j, 1)$.

Lemma 9. $(0, \alpha_2) = J(\alpha_2)$ *represents a non zero homology class in* $H_2(\mathrm{Rel}_*(\phi^*))$.

Proof. As $(0, \alpha_2)$ is a cycle, it remains to show that it is not a boundary. Otherwise there exist $\beta_2' \in \mathrm{Der}(\wedge W)$ and $\alpha_3 \in \mathrm{Der}(\wedge V, \wedge W; \phi)$ such that $d(s\beta_2', \alpha_3) = (-s\delta\beta_2', \delta\alpha_3 + \phi^*(\beta_2')) = (0, \alpha_2)$. Hence $\delta\beta_2' = 0$ and $\phi^*(\beta_2') = \alpha_2 - \delta\alpha_3$. For degree reasons $\alpha_3(x_2) = 0$, hence $(\delta\alpha_3)(x_2) = 0$. Therefore $\phi^*(\beta_2')(x_2) = \alpha_2(x_2) = 1$. We deduce that $\beta_2'(y_2) = 1$. Moreover β_2' cannot be a cycle, otherwise $H(\epsilon_*)([\beta_2']) = [y_2^*] \neq 0$ would be a non zero Gottlieb element in $\pi_2(X_\mathbb{Q})$, which is impossible [3, Proposition 28.8]. We conclude that $[(0, \alpha_2)]$ is non zero homology class in $H_2(\mathrm{Rel}(\phi^*))$. □

We can now determine $G_*^{rel}(\wedge V; \wedge W; \phi)$.

Theorem 10. *The relative rational Gottlieb group* $G_*^{rel}(\wedge V; \wedge W; \phi)$ *is isomorphic to the direct sum* $sG_*(\wedge W) \oplus G_{2N-1}(\wedge V, \wedge W; \phi)$. *Moreover the sequence*

$$0 = G_2(\wedge W) \longrightarrow G_2(\wedge V, \wedge W; \phi) \xrightarrow{H(\widehat{J})} G_2^{rel}(\wedge V, \wedge W; \phi) \longrightarrow 0 \tag{3}$$

is not exact.

Proof. A short calculation shows that $d\beta_2 \neq 0$ and $d\beta_4 \neq 0$ (see [12]). Moreover $(s\beta_{2n-3}, 0)$ and $(s\beta_{2n-1}, 0)$ represent non zero homology classes in $\mathrm{Rel}_*(\phi^*)$. However, in $\mathrm{Rel}_*(\widehat{\phi}^*)$, one gets $d(sy_2^*, 0) = (0, \widehat{\phi^*}(y_2^*)) = (0, x_2^*)$. Hence $H(\epsilon_*, \epsilon_*)([(0, \alpha_2)]) = [(0, x_2^*)]$ is zero in $H(\mathrm{Rel}_*(\widehat{\phi}^*))$. We conclude that

$$G_2^{rel}(\wedge V, \wedge W; \phi) = 0.$$

Therefore

$$\begin{aligned} G_*^{rel}(\wedge V, \wedge W; \phi^*) &= \operatorname{im} H(\epsilon_*, \epsilon_*) \\ &= \langle [(sy^*_{2n-3}, 0)], [(sy^*_{2n-1}, 0)], [(0, x^*_{2N-1})] \rangle \\ &= sG_*(\wedge W) \oplus G_{2N-1}(\wedge V, \wedge W; \phi). \end{aligned}$$

The sequence (3) is not exact as $H(\widehat{J})([x_2^*]) = 0$. Hence $H(\widehat{J})$ is not injective. □

Remark 11. It was incorrectly stated that $G_*^{rel}(\wedge V, \wedge W; \phi^*) = 0$ in [7].

Example 12. Consider $\mathrm{Gr}(2,4)$ for which the minimal Sullivan model is

$$(\wedge(y_2, y_4, y_5, y_7), d),$$

where $dy_2 = dy_4 = 0$, $dy_5 = y_2^3 - 2y_2y_4$, $dy_7 = y_4^2 - y_2^2y_4$. The inclusion $\mathrm{Gr}(2,4) \to \mathbb{C}P^5$ is modelled by

$$\phi : (\wedge(x_2, x_{11}), d) \to (\wedge(y_2, y_4, y_5, y_7), d),$$

where $\phi(x_2) = y_2$. As $\mathrm{Gr}(2,4)$ is a smooth manifold of dimension 8, then the cohomology class $[y_2^6]$ is zero. Therefore there exits $y \in \wedge W$ such that $dy = y_2^6$. A short computation shows that $y_2^5 = d(y_2^2y_5 - 2y_4y_5 - 4y_2y_7)$, therefore $y_2^6 = d(y_2^3y_5 - 2y_2y_4y_5 - 4y_2^2y_7)$. Hence $\phi(x_{11}) = y_2^3y_5 - 2y_2y_4y_5 - 4y_2^2y_7$.

Let $\alpha_2 \in Der(\wedge V, \wedge W; \phi)$ be the ϕ-derivation defined by $\alpha_2(x_2) = 1$ and $\alpha_2(x_{11}) = 6(y_2^2y_5 - 2y_4y_5 - 4y_2y_7)$. Then α_2 is a cycle, hence $H(\epsilon_*)([\alpha_2]) = [x_2^*] \in G_2(\wedge V, \wedge W; \phi)$ is in the kernel of $H(\widehat{J})$.

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