Commun. Korean Math. Soc. **35** (2020), No. 1, pp. 261–268
https://doi.org/10.4134/CKMS.c180534
pISSN: 1225-1763 / eISSN: 2234-3024

ON A CLASSIFICATION OF WARPED PRODUCT MANIFOLDS WITH GRADIENT YAMABE SOLITONS

Jin Hyuk Choi, Byung Hak Kim, and Sang Deok Lee

Abstract. In this paper, we study gradient Yamabe solitons in the warped product manifolds and classify the warped product manifolds with gradient Yamabe solitons. Moreover we investigate the admitness of gradient Yamabe solitons and geometric structures for some model spaces.

1. Introduction

Let M be a Riemannian manifold with a Riemannian metric g. The Yamabe soliton is a special solution to the Yamabe flow

$$\frac{\partial}{\partial t} g_{ij} = -r g_{ij} \tag{1}$$

and naturally arises as the limit of dilations of singularities in the Yamabe flow, where r is a scalar curvature of M. Such a flow was introduced by R. Hamilton [2,3]. A Riemannian metric g or (M, g) is called a Yamabe soliton if there exist a smooth vector field X and a constant ρ such that

$$(r - \rho)g = \frac{1}{2}\mathfrak{L}_X g. \tag{2}$$

In particular, if $X = \nabla h$ for some smooth function h, we call it the gradient Yamabe soliton [1]. The function h above is called the potential function. In this case, the equation (2) can be rewritten as

$$(r - \rho)g = \nabla^2 h, \tag{3}$$

where ρ is a constant. In this case, we will call M a gradient Yamabe soliton with (h, ρ). For $\rho = 0$ the Yamabe soliton is steady, for $\rho > 0$, it is shrinking and $\rho < 0$ expanding. When h is constant, we call the corresponding Yamabe soliton a trivial Yamabe soliton. In this case the scalar curvature r is a constant ρ.

Received December 24, 2018; Accepted March 26, 2019.
2010 *Mathematics Subject Classification.* Primary 53C25, 53B21.
Key words and phrases. Warped product space, gradient Yamabe soliton.
The second author was supported by the National Research Foundation of Korea (NRF-2017 R1E1A1A 03071005).

It has been known [5] that every compact Yamabe soliton has a constant scalar curvature, hence trivial. But the non-compact case is unsettled until now. For the complete non-compact case, P. Daskalopoulos and N. Sesum [2] showed that under the certain conditions on the initial data, which in particular imply that the initial metric admits the asymptotic behavior of the cylindrical metric at infinity complete non-compact solutions to the Yamabe flow develop a finite time singularity and after re-scaling the metric converges to the Barenblatt solution. Moreover, the known classification theorems of Yamabe solitons are very few and the underlying space is somewhat special such as conformal flatness [2]. Motivated by these results, it is natural to investigate the classification of the Riemannian manifolds with gradient Yamabe solitons, in particular, including the non-compact case. From this point of view, we study the warped product spaces with gradient Yamabe solitons and construct model spaces satisfying some geometric conditions using our results.

2. Gradient Yamabe solitons of the Riemannian product manifolds

Let (B, g) be an n-dimensional Riemannian manifold and $(F, \bar{g})$ be a p-dimensional Riemannian manifold. For a local coordinate system (x^a) $(a = 1, 2, \ldots, n)$ of B, the metric tensor g has components (g_{ab}) and $\bar{g}$ on F has the components $(\bar{g}_{\alpha\beta})$ for a local coordinate system (y^α) $(\alpha = 1, 2, \ldots, p)$. Then the product manifold M of B and F is an $m(= n+p)$-dimensional Riemannian manifold with a Riemannian metric G with the components $(G_{ij}) = \begin{pmatrix} g_{ab} & 0 \\ 0 & \bar{g}_{\alpha\beta} \end{pmatrix}$ with respect to the local coordinate system $x^i = (x^a, y^\alpha)$ on M and $i, j, k, \ldots = 1, \ldots, n, n+1, \ldots, n+p$.

If the product manifold $M = B \times F$ is a gradient Yamabe soliton with $(\tilde{h}, \tilde{\rho})$, then

$$(4)\qquad \begin{aligned} (\tilde{r} - \tilde{\rho})g_{ab} &= \nabla_a \nabla_b \tilde{h}, \\ \partial_a \tilde{h_\alpha} &= 0, \\ (\tilde{r} - \tilde{\rho})\bar{g}_{\alpha\beta} &= \bar{\nabla}_\alpha \bar{\nabla}_\beta \tilde{h}, \end{aligned}$$

where $\tilde{\rho}$ is a constant and $\tilde{h} = \tilde{h}(x_1, \ldots, x_n, y_1, \ldots, y_p)$ is a smooth function on M. Since $\partial_a \tilde{h_\alpha} = 0$, the function $\tilde{h}$ can be represented by

$$\tilde{h}(x_1, \ldots, x_n, y_1, \ldots, y_p) = k(x_1, \ldots, x_n) + l(y_1, \ldots, y_p)$$

for some functions k on B and l on F respectively. Then, using this fact and the first equation of (4), we get $(r + \bar{r} - \tilde{\rho})g = \nabla\nabla k$, from which we obtain $\bar{r} = \tilde{\rho} - r + \frac{\triangle k}{n}$. Since the left hand side is depending only on F and the right hand side is depending only on B, we see that $\bar{r}$ is a constant and that F is a trivial gradient Yamabe soliton. By the similar argument, r becomes constant. Since $\tilde{r} = r + \bar{r}$, M becomes a trivial gradient Yamabe soliton. Thus we have:

Theorem 2.1. *If the product manifold $M = B \times F$ is a gradient Yamabe soliton, then B, F and M become trivial gradient Yamabe solitons. This means* *that there is no non-trivial gradient Yamabe soliton in the Riemannian product manifold.*

If the two Riemannian manifolds B and F are trivial gradient Yamabe solitons, then the product manifold $M = B \times F$ becomes a trivial gradient Yamabe soliton because $\tilde{r} = r + \bar{r}$.

3. Gradient Yamabe solitons of warped product manifolds

Let the warped product manifold $M = R \times_f F$ be a gradient Yamabe soliton with $(\tilde{h}, \tilde{\rho})$. Then we get

$$
\begin{aligned}
\tilde{r} - \tilde{\rho} &= \tilde{h}_{11}, \\
\partial_1 \tilde{h_\alpha} &= \frac{f_1}{f}\tilde{h}_\alpha, \\
(\tilde{r} - \tilde{\rho}) f^2 \bar{g}_{\alpha\beta} &= \bar{\nabla}_\alpha \tilde{h}_\beta + f f^1 \tilde{h}_1 \bar{g}_{\alpha\beta}, \\
\tilde{r} &= \frac{\bar{r}}{f^2} - 2m\frac{\triangle f}{f} - \frac{m(m-1)}{f^2}||f_1||^2,
\end{aligned} \tag{5}
$$

where $\tilde{h} = \tilde{h}(t, y_1, \ldots, y_p)$ is the potential function, $\tilde{\rho}$ is a constant and $f_1 = \frac{df}{dt}$, $\tilde{h}_1 = \frac{\partial \tilde{h}}{\partial t}, \tilde{h}_{11} = \frac{\partial^2 \tilde{h}}{\partial t^2}$. Then we can state:

Theorem 3.1. *If the warped product $M = R \times_f F$ is a gradient Yamabe soliton with $(\tilde{h}, \tilde{\rho})$, then we have the followings*:

(a) *If $\tilde{h}_\alpha \neq 0$ for some $\alpha = 1, \ldots, p$, then $f^2 f_{11} =$ constant. So the warping function f becomes $f = -(\frac{9}{2})^{\frac{1}{3}} c^{\frac{1}{3}} t^{\frac{2}{3}}$. In this case, F can not admit a trivial gradient Yamabe soliton, where $f_{11} = \frac{\partial^2 f}{\partial t^2}$.*

(b) *If $\tilde{h}_1 = 0$, then M and F become trivial gradient Yamabe solitons. In this case, M is either a Riemannian product of R and F or the potential function $\tilde{h}$ is constant.*

(c) *If $\tilde{h}_\alpha = 0$ for all $\alpha = 1, \ldots, p$ and M is a non-trivial gradient Yamabe soliton, then F is a trivial gradient Yamabe soliton. Moreover f and $\tilde{h}$ are related by $f = \lambda \tilde{h}_1$ for some non-zero λ.*

Proof. (a) Suppose that $\tilde{h}_\alpha \neq 0$ for some α. Then, from the second relation of (5), we obtain $\frac{\partial_1 \tilde{h}_\alpha}{\tilde{h}_\alpha} = \frac{f_1}{f}$. So we get $\ln \tilde{h}_\alpha - \ln f = A(y_1, \ldots, y_p)$, that is, $\tilde{h}_\alpha = f e^{A(y_1,\ldots,y_p)} = f B(y_1, \ldots, y_p)$, where we have put $e^{A(y_1,\ldots,y_p)} = B(y_1, \ldots, y_p)$. Then we can see that $\tilde{h}$ is of the form $\tilde{h} = f(C(y_1, \ldots, y_p) + D(t, y_1, \ldots, \hat{y_\alpha}, \ldots, y_p))$, where $f(\hat{y}_\alpha)$ means that f is not a function of y_α. Hence $\tilde{h}_1 = f_1(C + D) + f D_1$ and $\tilde{h}_{11} = f_{11}(C + D) + 2 f_1 D_1 + f D_{11}$. From the first and fourth equations of (5), we have

$$
f_{11}(C + D) + 2f_1 D_1 + f D_{11} = \frac{\bar{r}}{f^2} - \frac{2n}{f} f_{11} - \frac{n(n-1)}{f^2}||f_1||^2 - \tilde{\rho}. \tag{6}
$$

If we differentiate both sides of (6) by y_α, then we get

$$f_{11}\frac{\partial C}{\partial y_\alpha} = \frac{1}{f^2}\frac{\partial \bar{r}}{\partial y_\alpha} \tag{7}$$

that is, $f^2 f_{11}\frac{\partial C}{\partial y_\alpha} = \frac{\partial \bar{r}}{\partial y_\alpha}$. Hence we get $\frac{d}{dt}(f^2 f_{11})\frac{\partial C}{\partial y_\alpha} = 0$. Therefore $f^2 f_{11} =$ constant or $\frac{\partial C}{\partial y_\alpha} = 0$. From the facts that $\tilde{h}_\alpha \neq 0$ for some α and $\tilde{h} = f(C+D)$, the case $\frac{\partial C}{\partial y_\alpha} = 0$ does not occur. Thus we have $f^2 f_{11} = d =$ constant and we see that the general solution of the ordinary differential equation $f^2 f_{11} = d$ is given by $f = -(\frac{9}{2})^{\frac{1}{3}} d^{\frac{1}{3}} t^{\frac{2}{3}}$. Since f is a warping function, the constant d is not equal to zero. From the equation (7) and $\tilde{h} = f(C + D)$, we see that $\frac{\partial \bar{r}}{\partial y_\alpha} = d\frac{\partial C}{\partial y_\alpha}$ does not vanish, that is F is not a trivial gradient Yamabe soliton.

(b) If $\tilde{h}_1 = 0$, then we see that M becomes a trivial gradient Yamabe soliton from the first equation of (5). From this fact and the fourth equation of (5), we can also see that $\bar{r}$ is constant, that is, F is a trivial gradient Yamabe soliton. Moreover, we see that $f_1\tilde{h}_\alpha = 0$ from the second equation of (5). This means that M is either a Riemannian products or the potential function $\tilde{h}$ is a constant.

(c) Since $\tilde{h}_\alpha = 0$ for all α, $\tilde{h}$ is only a function of t. Then, from the third and fourth equations of (5), we get

$$\bar{r} = ff^1\tilde{h}_1 + 2mf(\triangle f) + m(m-1)||f_1||^2 + \tilde{\rho}f^2. \tag{8}$$

If we differentiate the both sides of (8) with respect to y_α, then we have $\partial_\alpha \bar{r} = 0$ for all α, which means that $\bar{r}$ is constant. From the first and third equations of (5), $f^1\tilde{h}_1 = \tilde{h}_{11}f$. Since $\tilde{M}$ is a non-trivial gradient Yamabe soliton by (b) $\tilde{h}_1 \neq 0$. Hence we obtain $\frac{f_1}{f} = \frac{\tilde{h}_{11}}{\tilde{h}_1}$, from which we have $f = \lambda\tilde{h}_1$ for some non-zero constant λ. □

Theorem 3.2. *Let $M = R \times_{e^t} F$ be a gradient Yamabe soliton. Then M is a trivial shrinking gradient Yamabe soliton with a constant potential function and the scalar curvature of F is equal to zero.*

Proof. Since e^t is not a solution of the equation $f^2 f_{11}=$ constant, we see that $\tilde{h}_\alpha = 0$ for all α from Theorem 3.1(b) and that $\bar{r}$ becomes constant by use of Theorem 3.1(a). From these facts and equation (5), we obtain $\tilde{h}_{11} = \tilde{r} - \tilde{\rho} = \tilde{h}_1$ and that $\tilde{h}$ is given by $\tilde{h} = \lambda e^t + \mu$ for some constants λ and μ. Hence we get $\tilde{r} = \tilde{\rho} + \lambda e^t$ and $\bar{r} = e^{2t}(\tilde{\rho} + \lambda e^t + p^2 + p)$. Consequently we see that $\lambda = 0$, $\tilde{\rho} + p^2 + p = 0$ and finally $\bar{r} = 0$ and $\tilde{r} = \tilde{\rho} = -p^2 - p$. Hence we see that $\tilde{h}$ is constant, $\tilde{\rho}$ is negative constant and $\bar{r} = 0$.

It is known [5] that in a compact Riemannian manifold with a gradient Yamabe soliton, the scalar curvature becomes constant. From Theorem 3.2, we can see that the warped product $R \times_{e^t} F$ with a gradient Yamabe soliton has a constant scalar curvature without topological conditions such as compactness.

Since the scalar curvature $\tilde{r}$ of $R \times_{e^t} S^n$ is $re^{-2t} - p^2 - p$ for the nonzero scalar curvature r of S^n, by Theorem 3.2, we can state:

Corollary 3.3. *The warped product manifold $R\times_{e^t}S^n$ can not admit a gradient Yamabe soliton.*

In [4], it is known that the warped product space $R\times_{ce^t}F(c>0)$ of the line R and Kaehler manifold F admits an almost contact metric structure (ϕ,ξ,η,g) which satisfy

$$\begin{aligned}\nabla_X\phi\cdot Y &= -\eta(Y)\phi X - g(X,\phi,Y)\xi,\\ \nabla_X\xi &= X-\eta(X)\xi.\end{aligned} \tag{9}$$

From this fact and Corollary 3.3, we have:

Theorem 3.4. *The warped product manifold $M=R\times_{e^t}S^{2n}$ can not admit a gradient Yamabe soliton but admit an almost contact structure satisfying* (9)*.*

Next consider the warped product manifold $M=S^1(k)\times_f F$ with a gradient Yamabe soliton. Then we have

$$\begin{aligned}(\tilde r-\tilde\rho)\frac{k^2}{k^2-t^2} &= \partial_1\tilde h_1 - \frac{t}{k^2-t^2}\tilde h_1,\\ \partial_1\tilde h_\alpha &= \frac{f_1}{f}\tilde h_\alpha,\\ (\tilde r-\tilde\rho)f^2\bar g_{\alpha\beta} &= \bar\nabla_\alpha\tilde h_\beta + ff^1\tilde h_1\bar g_{\alpha\beta},\\ \tilde r &= \frac{\bar r}{f^2} - 2m\frac{\triangle f}{f} - \frac{p(p-1)}{f^2}||f_1||^2,\end{aligned} \tag{10}$$

where $\tilde h=\tilde h(t,y_1,\dots,y_p)$ is a potential function and $\tilde\rho$ is a constant.

Theorem 3.5. *Let $M=S^1\times_f F$ be a gradient Yamabe soliton with $(\tilde h,\tilde\rho)$. Then we have the followings*:

(a) *If $\tilde h_\alpha\neq 0$ for some α, then the warping function f satisfies $f^2(f_{11}(k^2-t^2)-tf_1)=\nu=$ constant. Moreover, if the scalar curvature $\bar r$ of F is constant, then the warping function f becomes $f=\lambda\sin^{-1}\frac{t}{k}+\mu$ for some constants λ and μ.*

(b) *If $\tilde h_1=0$, then M and F become trivial gradient Yamabe solitons.*

(c) *If $\tilde h_\alpha=0$ for all α, then F is a trivial gradient Yamabe soliton.*

Proof. (a) Assume that $\tilde h_\alpha\neq0$ for some α. Then from the second equation of (10), we have $\frac{\partial_1\tilde h_\alpha}{h_\alpha}=\frac{f_1}{f}$. So we get $\tilde h_\alpha=fe^{A(y_1,\dots,y_p)}$. If we put $B(y_1,\dots,y_p)=e^{A(y_1,\dots,y_p)}$, then $\tilde h_\alpha=B(y_1,\dots,y_p)$ and so $\tilde h$ is of the form $\tilde h=f(C(y_1,\dots,y_p)+D(t,y_1,\dots,\hat y_\alpha,\dots,y_p))$. Hence $\tilde h_1=f_1(C+D)+fD_1$ and $\tilde h_{11}=f_{11}(C+D)+2f_1D_1+fD_{11}$. From the first and fourth equation of (10), we obtain

$$\begin{aligned}&f_{11}(C+D)+2f_1D_1+fD_{11}-\frac{t}{k^2-t^2}(f_1(C+D)+fD_1)\\ &=(\frac{\bar r}{f^2}-2p\frac{\triangle f}{f}-\frac{p(p-1)}{f^2}||f_1||^2-\tilde\rho)\frac{k^2}{k^2-t^2}.\end{aligned} \tag{11}$$

If we differentiate both sides of (11) by y_α, then we get $f_{11}\frac{\partial C}{\partial y_\alpha}-\frac{t}{k^2-t^2}f_1\frac{\partial C}{\partial y_\alpha}=\frac{1}{f^2}\frac{\partial \bar{r}}{\partial y_\alpha}\frac{k^2}{k^2-t^2}$, that is

$$\frac{f^2f_{11}(k^2-t^2)-tf^2f_1}{k^2}\frac{\partial C}{\partial y_\alpha}=\frac{\partial \bar{r}}{\partial y_\alpha}. \tag{12}$$

Hence we get $\frac{d}{dt}(f^2f_{11}(k^2-t^2)-tf^2f_1)\frac{\partial C}{\partial y_\alpha}=0$. Therefore $f^2f_{11}(k^2-t^2)-tf^2f_1=constant$ or $\frac{\partial C}{\partial y_\alpha}=0$. From the fact that $\tilde{h}_\alpha\neq 0$ for some α and $\tilde{h}=f(C+D)$, $\frac{\partial C}{\partial y_\alpha}=0$ does not occurs. Thus we have $f^2(f_{11}(k^2-t^2)-tf_1)=constant$. Moreover, if $\bar{r}$ =constant, then we get

$$f_{11}(k^2-t^2)-tf_1=0 \tag{13}$$

from the equation (12), and the differential equation (13) gives us the general solutions $f=\lambda\sin^{-1}\frac{t}{k}+\mu$ for some constants λ and μ. Since f is positive, we see that λ and μ satisfy $\mu>\frac{|\lambda|}{2}\pi$. The function $f=\nu\cos^{-1}\frac{t}{k}+\omega$ for some constants ν and ω is also the general solution of the equation (13) by the same method. So we can restrict the range of f due to positive function.

(b) From the assumption $\tilde{h}_1=0$, $\tilde{h}$ depends only on F and the first equation of (10) gives $\tilde{r}=\tilde{\rho}$, which means that M is a trivial gradient Yamabe soliton. Then from this fact and the fourth equation of (10), we see that $\bar{r}$ is constant.

(c) From the assumptions and the third equation of (10), we get $\tilde{r}-\tilde{\rho}=\frac{f_1}{f}\tilde{h}_1$, from which $\tilde{r}$ depends only on S^1. Then, this fact and the fourth equation of (10) give $\frac{\bar{r}}{f^2}-2p\frac{\triangle f}{f}-\frac{p(p-1)}{f^2}||f_1||^2=\tilde{\rho}+\frac{f_1}{f}\tilde{h}_1$. Hence $\bar{r}$ becomes constant.

Finally, consider the warped product manifold $M=B\times_f F$ of an n-dimensional Riemannian manifold (B,g) and a p-dimensional Riemannian manifold $(F,\bar{g})$. If M is a gradient Yamabe soliton, then we have

$$\begin{aligned}(\tilde{r}-\tilde{\rho})g_{ab}&=\nabla_a\tilde{h}_b,\\ \partial_a\tilde{h}_\beta&=\frac{f_a}{f}\tilde{h}_\beta,\\ (\tilde{r}-\tilde{\rho})f^2\bar{g}_{\alpha\beta}&=\bar{\nabla}_\alpha\tilde{h}_\beta+ff^a\tilde{h}_a\bar{g}_{\alpha\beta},\\ \tilde{r}&=r+\frac{\bar{r}}{f^2}-2p\frac{\triangle f}{f}-\frac{p(p-1)}{f^2}||f_e||^2,\end{aligned} \tag{14}$$

where $\tilde{h}=\tilde{u}(x_1,\ldots,x_n,y_1,\ldots,y_p)$ is a potential function and $\tilde{\rho}$ is a constant. □

Theorem 3.6. *If the warped product manifold $M=B\times_f F$ is a gradient Yamabe soliton with $(\tilde{h},\tilde{\rho})$, then we have the followings*:

(a) *If $\tilde{h}_\alpha\neq 0$ for some α, then $f^2\triangle f$ is constant. In particular, if B is compact, then M is the Riemannian product of B and F.*

(b) *If $\tilde{h}_a=0$ for all a, then M is either a Riemannian product of the Riemannian manifolds B and F or the potential function $\tilde{h}$ is constant. In any case, M and F are trivial gradient Yamabe solitons.*

(c) *If* $\tilde{h}_\alpha = 0$ *for all* α*, then* F *is trivial gradient Yamabe soliton and the potential function* $\tilde{h}$ *can be expressed by* $\tilde{h} = k + l$ *for some functions* k *and* l *on* B *and* F *respectively. Moreover if* B *is compact, then the integral of* $\tilde{r}$ *on* B *is equal to* $\tilde{\rho}\, vol(B)$*.*

Proof. (a) Assume that $\tilde{h}_\alpha \neq 0$ for some α. Then we get $\frac{\partial_a \tilde{h}_\alpha}{\tilde{h}_\alpha} = \frac{f_a}{f}$ from the second equation of (14), that is $\partial_a(\ln \tilde{h}_\alpha - \ln f) = 0$. So we get $\tilde{h}_\alpha = f e^{A(y)}$ for some $A(y) = A(y_1, \dots, y_p)$. We denote $e^{A(y_1,\dots,y_p)} = B(y)$, then $\tilde{h}_\alpha = fB(y)$. Hence we obtain $\tilde{h} = f(C + D)$ for some $C = C(y_1, \dots, y_p), D = D(x_1, \dots, x_n, y_1, \dots, \hat{y}_\alpha, \dots, y_p)$ and then $\tilde{h}_a = f_a C + f_a D + f D_a$. From the first equation of (14),

$$\tilde{r} = \tilde{\rho} + g^{ab}(\nabla_a \tilde{h}_b) \tag{15}$$

and if we differentiate (15) with respect to y_α, then we get $\triangle f \frac{\partial C}{\partial y_\alpha} = \frac{1}{f^2}\frac{\partial \bar{r}}{\partial y_\alpha}$, that is

$$f^2 \triangle f \frac{\partial C}{\partial y_\alpha} = \frac{\partial \bar{r}}{\partial y_\alpha}. \tag{16}$$

If we differentiate (16) by x_a, then we see that $\frac{\partial}{\partial x_a}(f^2\triangle f)\frac{\partial C}{\partial y_\alpha} = 0$. So we see that $f^2 \triangle f$ is constant or $\frac{\partial C}{\partial y_\alpha} = 0$. Since $\tilde{h}_\alpha = f\frac{\partial C}{\partial y_\alpha}$, $\frac{\partial C}{\partial y_\alpha} = 0$ does not occur because $\tilde{h}_\alpha \neq 0$ for some α on F. Consequently $f^2\triangle f$ is constant. Since $\triangle f^3 = 6f||df||^2 + 3f^2\triangle f$, by Green's Theorem, $\int_B f^2 \triangle f dv = -2\int_B f||df||^2 dv \leq 0$ on the compact manifold of B. Then we see that the constant $f^2\triangle f$ is non-positive, and hence $\triangle f \leq 0$ on B. Consequently f is constant.

(b) From the first equation of (14) and the assumptions, it is easily verified that $\tilde{r} = \tilde{\rho}$, that is M is a trivial gradient Yamabe soliton. Since $\bar{r} = f^2\tilde{\rho} - f^2 r + 2p(f\triangle f) + p(p-1)||f_e||^2$ and the right hand side of the upper equation depends only on B, $\bar{r}$ is constant. Since f_a and $\tilde{h}_\alpha$ are functions depending only on B and F respectively, the warping function f and potential function $\tilde{h}$ become constants.

(c) Assume that $\tilde{h}_\alpha = 0$ for all α. Then there exist a function k on B and a function l on F such that $\tilde{h} = k + l$ because $\partial_\alpha \partial_a \tilde{h} = 0$ from the second equation of (14). So we obtain $\tilde{r} = \tilde{\rho} + \frac{1}{n}\triangle k = \tilde{\rho} + \frac{f_a}{f}k_a$ from the first and third equation of (14). Moreover we see that $\tilde{r}$ is a quantity of B and $\bar{r}$ is constant because $\bar{r} = f^2\tilde{r} - f^2 r + 2pf\triangle f + p(p-1)||f_e||^2$ is satisfied from the fourth equation of (14). Since $\tilde{r} = \tilde{\rho} + \frac{\triangle k}{n}$, if B is compact, then we get $\int_B \tilde{r} dv = \tilde{\rho} vol(B)$. Hence the integral of $\tilde{r}$ on B is zero or positive or negative if the soliton is steady or shrinking or expending respectively. From the third equation of (14), then we obtain $\tilde{r} = \frac{f_a}{f}k_a + \tilde{\rho}$ and that

$$\bar{r} = -f^2 r + f^2(\tfrac{f^a k_a}{f} + \tilde{\rho}) + 2pf\triangle f + p(p-1)||f_e||^2 \tag{17}$$

is satisfied. Then we easily see that $\bar{r}$ is constant because the right hand side of the equation (17) depends only on B. □

Remark 3.7. If we combine the results of Theorem 3.5(a) and Theorem 3.6(a), then we see that the constant ν in Theorem 3.5(a) is equal to zero.

If we combine the results of Theorem 3.5 and Theorem 3.6(c), we get:

Corollary 3.8. *The integral of scalar curvature $\tilde{r}$ on $M = S^1 \times_f F$ is equal to $\tilde{\rho} vol(S^1)$ if the warped product manifold M is a gradient Yamabe soliton with $(\tilde{h}, \tilde{\rho})$ and $\tilde{h}_\alpha = 0$ for all α.*

References

[1] H.-D. Cao, X. Sun, and Y. Zhang, *On the structure of gradient Yamabe solitons*, Math. Res. Lett. **19** (2012), no. 4, 767–774. https://doi.org/10.4310/MRL.2012.v19.n4.a3

[2] P. Daskalopoulos and N. Sesum, *The classification of locally conformally flat Yamabe solitons*, Adv. Math. **240** (2013), 346–369. https://doi.org/10.1016/j.aim.2013.03.011

[3] R. Hamilton, *Lectures on geometric flows*, unpublished, 1989.

[4] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tôhoku Math. J. (2) **24** (1972), 93–103. https://doi.org/10.2748/tmj/1178241594

[5] S.-Y. Hsu, *A note on compact gradient Yamabe solitons*, J. Math. Anal. Appl. **388** (2012), no. 2, 725–726. https://doi.org/10.1016/j.jmaa.2011.09.062

Jin Hyuk Choi
Humanitas
Kyung Hee University
Yongin 17104, Korea
Email address: jinhchoi@khu.ac.kr

Byung Hak Kim
Department of Applied Mathematics and Institute of Natural Sciences
Kyung Hee University
Yongin 17104, Korea
Email address: bhkim@khu.ac.kr

Sang Deok Lee
Department of Mathematics
Dankook University
Cheonan 31116, Korea
Email address: sdlee@dankook.ac.kr