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PSEUDO-HERMITIAN 2-TYPE LEGENDRE SURFACES IN THE UNIT SPHERE S^5

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Abstract. In this paper, we show that it is Chen surfaces that non-minimal pseudo-Hermitian mass-symmetric 2-type Legendre surfaces in S^5. Moreover, we show that pseudo-Hermitian mass-symmetric 2-type Legendre surfaces in S^5 are the locally product of two pseudo-Hermitian circles.

1. Introduction

Let M^n be an n-dimensional submanifold of Euclidean space E^{m+1}. Denote by Δ the Laplacian of M^n acting on smooth functions on M^n. This Laplacian can be extended in a natural way to E^{m+1}-valued smooth functions on M^n. A submanifold M^n of E^{m+1} is said to be of *k-type* if the position vector x of M^n in E^{m+1} admits the following spectral decomposition

$$x = x_0 + x_1 + \cdots + x_k,$$

where $x_0 \in E^{m+1}$ is a fixed vector and x_i $(i = 1, \ldots, k)$ are non-constant E^{m+1}-valued smooth maps on M^n such that

$$\Delta x_i = \lambda_i x_i \quad i = 1, \ldots, k \quad \text{and} \quad \lambda_1 < \cdots < \lambda_k, \quad \lambda_i \in \mathbb{R}.$$

The study of submanifolds of finite type was introduced by B. Y. Chen in [3].

A compact submanifold M^n of a hypersphere S^m of E^{m+1} is said to be mass-symmetric in S^m if the center of mass x_0 of M^n in E^{m+1} is exactly the center of S^m in E^{m+1}. Mass-symmetric 2-type submanifolds of a hypersphere can be regarded as the "simplest" submanifolds of E^{m+1} next to minimal submanifolds. B. Y. Chen ([3]) found that mass-symmetric spherical 2-type submanifolds have some special properties. For instances, every mass-symmetric spherical 2-type submanifold has constant mean curvature. Thus, he classified 2-type surfaces

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of a hypersphere that a compact surface of a hypersphere S^3 in E^4 is the product of two plane circles with different radii if and only if it is mass-symmetric and of 2-type.

C. Baikoussis and D. E. Blair ([1]) classified integral surfaces of the unit sphere $S^5(1)$ which are mass-symmetric and of 2-type. They proved that a mass-symmetric 2-type integral surface of $S^5(1)$ is the product of a plane circle and a helix of order 4 or the product of two circles.

In this paper, we study Legendre submanifolds M^n of the unit sphere S^{2n+1} in E^{2n+2}. In Section 3, we consider the Takahashi's Theorem (Lemma 3.1) for pseudo-Hermitian geometry. Thus, we define the finite type for pseudo-Hermitian geometry and prove that Legendre submanifold M^n is of pseudo-Hermitian 1-type if and only if it is a minimal submanifold of S^{2n+1}.

In Section 4, we find that it is Chen surface that non-minimal pseudo-Hermitian mass-symmetric 2-type Legendre surfaces in S^5. Moreover, we classify that pseudo-Hermitian mass-symmetric 2-type Legendre surfaces in S^5 is the locally product of two pseudo-Hermitian circles.

2. Preliminaries

Let $\mathbb{C}^{n+1}$ be the complex Euclidean $(n+1)$-space with the standard almost complex structure J. Denoted by S^{2n+1} the unit sphere with the standard induced metric g in $\mathbb{C}^{n+1}$.

We give S^{2n+1} the usual contact structure. Define a tangent vector field ξ, a 1-form η and a $(1,1)$-type tensor field φ on S^{2n+1} satisfying

$$\xi = Jx, \quad \eta(X) = g(X,\xi), \quad \text{and} \quad \varphi = s \circ J,$$

where s denotes the orthogonal projection from T_pC^{n+1} on T_pS^{2n+1}, $p \in S^{2n+1}$, and the position vector field x of S^{2n+1} is a unit normal vector field of S^{2n+1} in $\mathbb{C}^{n+1}$.

Then we obtain for tangent vector fields X and Y on S^{2n+1}

$$\eta(X) = g(X,\xi), \quad d\eta(X,Y) = g(X,\varphi Y), \quad \varphi^2 X = -X + \eta(X)\xi. \tag{1}$$

Thus it satisfies

$$[\varphi,\varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi,\varphi]$ is the Nijenhuis torsion of φ. These formulas imply that S^{2n+1} is a Sasakian manifold.

On the other hand, for a given contact form we have two compatible structures: One is a Riemannian structure (or metric) and the other is a pseudo-Hermitian structure (or almost CR-structure). In pseudo-Hermitian geometry (CR-geometry) we use the *Tanaka-Webster connection* as a canonical connection instead of the Levi-Civita connection ([2]).

Now, we review the *Tanaka-Webster connection* ([6], [8]) on a contact strongly pseudo-convex CR-manifold $N = (N; \eta, L)$ with the associated contact

Riemannian structure (η, ξ, φ, g). The Tanaka-Webster connection $\hat{\nabla}$ for a Sasakian manifold is

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi, \tag{2}$$

where ∇ is the Levi-Civita connection. The Tanaka-Webster connection $\hat{\nabla}$ has the torsion

$$\hat{T}(X, Y) = 2g(X, \varphi Y)\xi$$

for all vector fields X, Y on N. Furthermore, it was proved in ([7]) that:

Proposition 2.1. *The Tanaka-Webster connection $\hat{\nabla}$ on a contact Riemannian manifold $N = (N^{2n+1}; \eta, \varphi, \xi, g)$ with the associated (integrable) CR-structure is the unique linear connection satisfying the following conditions*:
(i) $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$,
(ii) $\hat{\nabla}g = 0$, $\hat{\nabla}\varphi = 0$,
(iii-1) $\hat{T}(X, Y) = -\eta([X, Y])\xi$, X, $Y \in ker\eta$,
(iii-2) $\hat{T}(\xi, \varphi Y) = -\varphi\hat{T}(\xi, Y)$, $Y \in ker\eta$.

We define the *pseudo-Hermitian curvature tensor* (or *Tanaka-Webster curvature tensor*) $\hat{R}$ on a contact Riemannian manifold equipped with the associated CR-structure and Tanaka-Webster connection $\hat{\nabla}$ by

$$\hat{R}(X, Y)Z = \hat{\nabla}_X(\hat{\nabla}_Y Z) - \hat{\nabla}_Y(\hat{\nabla}_X Z) - \hat{\nabla}_{[X,Y]}Z \tag{3}$$

for all vector fields X, Y, Z in N. In [4] he studied the relation between pseudo-Hermitian geometry and Riemannian geometry. Indeed, for Sasakian space forms $N^{2n+1}(\epsilon)$ the holomorphic sectional curvature for $\hat{\nabla}$ is $\hat{\epsilon} = \epsilon + 3$. Thus, we see that the unit sphere $S^{2n+1}(1)$ has constant pseudo-holomorphic sectional curvature $\hat{\epsilon} = 4$. We will denote by S^{2n+1} the unit sphere.

Now, we recall the properties on Legendre submanifolds in Sasakian space forms for the Tanaka-Webster connection $\hat{\nabla}$.

Let N^{2n+1} be a contact Riemannian manifold and $f : M^m \longrightarrow N^{2n+1}$ be an isometric immersion of a Riemannian manifold M^n. Then we have the basic formulas for $\hat{\nabla}$:

$$\hat{\nabla}^f_X Y = \hat{\nabla}^o_X Y + \hat{\sigma}(X, Y) \quad \text{and} \quad \hat{\nabla}^f_X V = -\hat{S}_V X + \hat{D}_X V, \tag{4}$$

where $X, Y \in TM^m$, $V \in T^{\perp}M^m$, $\hat{\sigma}$, $\hat{S}$ and $\hat{D}$ are the *second fundamental form*, the *shape operator* and the *normal connection* with respect to $\hat{\nabla}$. The connection $\hat{\nabla}^o$ is the connection on M induced from $\hat{\nabla}$. The first formula is called the *Gauss formula* and the second formula is called the *Weingarten formula* with respect to Tanaka-Webster connection.

If η restricted to M^m vanishes, then a Riemannian manifold M^m, isometrically immersed in contact Riemannian manifold N^{2n+1}, is called an *integral submanifold*. In particular if $m = n$, it is called a *Legendre submanifold*.

Let M^n be a Legendre submanifold of a Sasakian manifold N^{2n+1} and let e_i $(i = 1, \ldots, n)$ be an orthonormal frame along M^n such that $\{e_i\}$ are tangent to M^n, $\varphi e_1 = e_{n+1}, \ldots, \varphi e_n = e_{2n}, \xi = e_{2n+1}$. From (2), assuming

$$A(X,Y) = \eta(X)\varphi Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi,$$

then we see that

$$A(X,Y) = 0 \tag{5}$$

for $X, Y \in TM$, and then we find that $\hat{\sigma} = \sigma$. This implies that $\hat{\nabla}^o$ coincides with the Levi-Civita connection ∇^o of M^n. Moreover, we have

$$S_{\varphi Y}X = -\varphi\sigma(X,Y) = S_{\varphi X}Y, \quad S_\xi = 0. \tag{6}$$

Now we suppose that the ambient space $N = N^{2n+1}(\hat{\epsilon})$ is a Sasakian space form. Since φ is parallel for Tanaka-Webster connection $\hat{\nabla}$, we get

$$\hat{D}_X \varphi Y = \varphi \hat{\nabla}^o_X Y, \qquad S_{\varphi Y}X = -\varphi\sigma(X,Y).$$

Then by using a straightforward computation the equations of Gauss and Codazzi of Legendre submanifolds for Tanaka-Webster connection are given respectively by:

$$h(R^o(X,Y)Z,W) = g(\hat{R}(X,Y)Z,W) + h([S_{\varphi Z}, S_{\varphi W}]X,Y), \tag{7}$$

$$(\hat{\nabla}_X\sigma)(Y,Z) = (\hat{\nabla}_Y\sigma)(X,Z). \tag{8}$$

3. Laplace operator in pseudo-Hermitian geometry

In this section, we find the Takahashi's Theorem (Lemma 3.1) for pseudo-Hermitian geometry. Thus, we define the finite type for pseudo-Hermitian geometry and compute $\hat{\Delta}H$.

Now instead of the Levi-Civita connection ∇, using the canonical affine connection of contact Riemannian manifolds, the Tanaka-Webster connection $\hat{\nabla}$, we define the Laplace operator $\hat{\Delta}$ on M^n in E^{2n+2},

$$\hat{\Delta} = -\sum_{i=1}^{n}\left(\hat{\nabla}_{e_i}\hat{\nabla}_{e_i} - \hat{\nabla}_{\hat{\nabla}^o_{e_i}e_i}\right), \tag{9}$$

where $e_1, \ldots, e_n$ is a local orthonormal frame field and $\hat{\nabla}^o$ the induced connections on M^n.

Let $x : M^n \to S^{2n+1} \subset E^{2n+2}$ be an isometric immersion of an n-dimensional Legendre submanifold M^n into S^{2n+1} in E^{2n+2}. Let $e_1, \ldots, e_n$ be an orthonormal local frame fields such that $\hat{\nabla}^o_{e_i}e_j = \nabla^o_{e_i}e_j = 0$ in M^n. Then from (5) we find that $\hat{\sigma} = \sigma$ and $\hat{H} = H$. Thus we get

$$\begin{aligned}\hat{\Delta}x &= -\sum_{i=1}^{n}\hat{\nabla}_{e_i}\hat{\nabla}_{e_i}x = -\sum_{i=1}^{n}\hat{\nabla}_{e_i}e_i \\ &= -\sum_{i=1}^{n}\sigma(e_i,e_i) = -nH.\end{aligned}$$

Hence we have:

Lemma 3.1. *Let $x : M^n \to S^{2n+1} \subset E^{2n+2}$ be an n-dimensional Legendre submanifold into the unit sphere S^{2n+1} of Euclidean space E^{2n+2} with respect to the Tanaka-Webster connection $\hat{\nabla}$. If M^n has an orthonormal local frame fields $e_1, \dots, e_n$ such that $\hat{\nabla}^o_{e_i} e_j = 0$, then*

$$\hat{\Delta} x = -nH, \tag{10}$$

where x is the position vector and H mean curvature vector field (for the Levi-Civita connection ∇).

The allied mean curvature vector field $a(H)$ is defined by

$$a(H) = \sum_{r=n+2}^{2n+1} (trS_H S_{e_r}) e_r.$$

If $a(H)$ vanishes identically on M^n, it is called (according to [3]) a-submanifold and when $n = 2$ it is called a *Chen surface.*

By direct computation, we have ([5])

$$\hat{\Delta} H = tr(\nabla S_H) + \hat{\Delta}^{\hat{D}} H + (trS^2_{\varphi e_1}) H + a(H),$$

where $tr(\nabla S_H) = \sum_{i=1}^n (S_{D_{e_i} H} e_i + (\nabla_{e_i} S_H) e_i)$.

Now, assume that M^n is a Legendre submanifold of the unit sphere S^{2n+1} of E^{2n+2}. Let H, σ, S, and $\hat{D}$ denote the mean curvature vector, the second fundamental form, the Weingarten maps and the normal connection of M^n in E^{2n+2} for the Tanaka-Webster connection $\hat{\nabla}$, respectively. Denote by H', σ', S' and $\hat{D}'$ the corresponding for M^n in S^{2n+1}. Then we have

$$H = H' - x, \quad \hat{D} x = 0. \tag{11}$$

We put $H' = \frac{trS_{\varphi e_1}}{n} \varphi e_1$. Then we get

$$\begin{aligned} \hat{\Delta}^{\hat{D}} H &= \hat{\Delta}^{\hat{D}'} H', \\ a(H) &= a'(H') - n \mid H \mid^2 x, \\ trS^2_{\varphi e_1} H &= (trS^2_{\varphi e_1} + n) H'. \end{aligned}$$

Hence we obtain (cf. [3], p. 273):

Lemma 3.2. *Let M^n be a Legendre submanifold of the unit sphere S^{2n+1} in E^{2n+2} for the Tanaka-Webster connection $\hat{\nabla}$. Then we have*

$$\hat{\Delta} H = tr(\nabla S_H) + \hat{\Delta}^{\hat{D}'} H' + (trS^2_{\varphi e_1} + n) H' + a'(H') - n \mid H \mid^2 x, \tag{12}$$

where $a'(H')$ is the allied mean curvature vector field of M^n in S^{2n+1}.

A submanifold M^m of contact Riemannian manifold N^{2n+1} in E^{2n+2} is said to be of *pseudo-Hermitian k-type* if the position vector x of M^m in contact

Riemannian manifold N^{2n+1} in E^{2n+2} for the Tanaka-Webster connection $\hat{\nabla}$ admits the following spectral decomposition

$$x = x_0 + x_1 + \cdots + x_k,$$

where $x_0 \in E^{2n+2}$ is fixed vector and x_i $(i = 1, \ldots, k)$ are non-constant E^{2m+2}-valued smooth maps on M^m such that

$$\hat{\Delta} x_i = \lambda_i x_i, \quad i = 1, \ldots, k \quad \text{and} \quad \lambda_1 < \cdots < \lambda_k, \quad \lambda_i \in R.$$

Since $H = H' - x$ for a Legendre submanifold M^n of the unit sphere S^{2n+1} in E^{2n+2}, using (10) we have

$$\hat{\Delta} x = -nH = -n(H' - x) = -nH' + nx.$$

Hence we obtain:

Proposition 3.3. *Let M^n be a Legendre submanifold of the unit sphere S^{2n+1} in E^{2n+2}. Then M^n is of pseudo-Hermitian* 1*-type if and only if it is a minimal submanifold of S^{2n+1}.*

4. Pseudo-Hermitian 2-type Legendre surface

We consider the hypersphere $S^5 \subset C^3 \cong E^6$ centered at the origin. Assume that

$$x : M \to S^5 \tag{13}$$

is a mass-symmetric 2-type immersion of a Legendre surface M into S^5 for the Tanaka-Webster connection $\hat{\nabla}$.

Denote by $\hat{\nabla}$ the Tanaka-Webster connection of E^6 and by $\hat{\nabla}^o$, $\hat{\nabla}'$ the induced connections on M and S^5, respectively. Let H, σ, S, and $\hat{D}$ denote the mean curvature vector, the second fundamental form, the Weingarten maps and the normal connection of M in E^6, respectively. Denote by H', σ', S' and $\hat{D}'$ the corresponding for M in S^5. Then we have $H = H' - x$.

Since M is pseudo-Hermitian 2-type and mass-symmetric, the position vector x of M with respect to the origin of E^6 can be written as follows:

$$x = x_p + x_q, \quad \hat{\Delta} x_p = \lambda_p x_p, \quad \hat{\Delta} x_q = \lambda_q x_q, \tag{14}$$

where x_p, x_q are non-constant E^6-valued maps on M.

Let $e_i (i = 1, \ldots, 5)$ be an orthonormal frame field along M^2 such that e_1, e_2 are tangent to M^2, $\varphi e_1 = e_3, \varphi e_2 = e_4, \xi = e_5$. We denote by $\{\hat{\omega}_i\}$, $i = 1, \ldots, 5$ the dual frame field of the frame $\{e_i\}$ for the Tanaka-Webster connection $\hat{\nabla}$. Then we have

$$\hat{\nabla} e_i = \Sigma_{i=1}^{6} \hat{\omega}_i^j e_j.$$

From (2) we get

$$\hat{D}_X V = D_X V + \eta(V) \varphi X - g(\varphi X, V) \xi,$$

where $X \in TM$ and $V \in TM^{\perp}$. Using the above equation, we have

$$\hat{\omega}_3^4 = \omega_3^4, \quad \hat{\omega}_5^j = \omega_5^j = 0, \quad \hat{\omega}_5^3(e_i) = \omega_5^3(e_i) + g(e_i, e_1) = 0, \tag{15}$$

$$\hat{\omega}_5^4(e_i) = \omega_5^4(e_i) + g(e_i, e_2) = 0, \quad \hat{\omega}_6^k = \omega_6^k = 0,$$

where $i = 1, 2$, $j = 1, 2, 5, 6$, and $k = 3, 4, 5, 6$.

On the other hand, from (10) we have $\hat{\Delta}x = -2H$. By using (14) we have

$$\hat{\Delta}H = \frac{trS_{\varphi e_1}}{2}(\lambda_p + \lambda_q)\varphi e_1 - (\lambda_p + \lambda_q - \frac{\lambda_p\lambda_q}{2})x. \tag{16}$$

From (12) and (16) we have that $trS_{\varphi e_1}$ is a constant. When $trS_{\varphi e_1} = 0$, M is a minimal surface of S^5 and pseudo-Hermitian 1-type. We may assume that $trS_{\varphi e_1} = constant \neq 0$.

By direct computation, we get

$$\begin{aligned} \hat{\Delta}^{\hat{D}'}H' &= \sum_{i=1}^{2}(\hat{D}'_{\nabla_{e_i}e_i}H' - \hat{D}'_{e_i}\hat{D}'_{e_i}H') = \frac{trS_{\varphi e_1}}{2}\hat{\Delta}^{\hat{D}}\varphi e_1 \\ &= \frac{trS_{\varphi e_1}}{2}[\mid \hat{D}\varphi e_1 \mid^2 \varphi e_1 - (tr\nabla\omega_3^4)\varphi e_2], \end{aligned} \tag{17}$$

where

$$\mid \hat{D}\varphi e_1 \mid = \Sigma_{i=1}^2 \mid \hat{D}_{e_i}\varphi e_1 \mid = \Sigma_{i=1}^2 \omega_3^4(e_i) \tag{18}$$

and

$$tr\nabla\omega_3^4 = \Sigma_{i=1}^2(\nabla_{e_i}\omega_3^4)(e_i) = \Sigma_{i=1}^2(e_i\omega_3^4(e_i) - \omega_3^4(\nabla_{e_i}e_i)). \tag{19}$$

Since

$$\begin{aligned} a'(H') &= \frac{trS_{\varphi e_1}}{2}(trS_{\varphi e_1}S_{\varphi e_2})\varphi e_2 \quad \text{and} \\ tr(\nabla S_H) &= \frac{trS_{\varphi e_1}}{2}\sum_{i=1}^{2}((\nabla_{e_i}S_{\varphi e_1})e_i + \omega_3^4(e_i)S_{\varphi e_2}e_i), \end{aligned}$$

from (12), (16) and (17), therefore we have the following equations:

$$\sum_{i=1}^{2}((\nabla_{e_i}S_{\varphi e_1}(e_i)) + \omega_3^4(e_i)S_{\varphi e_2}e_i) = 0, \tag{20}$$

$$\mid \hat{D}\varphi e_1 \mid^2 + trS_{\varphi e_1}^2 = \lambda_p + \lambda_q - 2, \tag{21}$$

$$tr\nabla\omega_3^4 - trS_{\varphi e_1}S_{\varphi e_2} = 0. \tag{22}$$

Since $trS_{\varphi e_1}$ is a constant, we get

$$\begin{aligned} 0 = grad\ trS_{\varphi e_1} &= \sum_{i=1}^{2}((\hat{\nabla}_{e_i}S_{\varphi e_1})(e_i) - S_{\hat{D}_{e_i}\varphi e_1}e_i) \\ &= \sum_{i=1}^{2}((\nabla_{e_i}S_{\varphi e_1})(e_i) - \omega_3^4(e_i)S_{\varphi e_2}e_i). \end{aligned}$$

From this and (20) we have

$$\sum_{i=1}^{2}(\nabla_{e_i}S_{\varphi e_1})(e_i)=0, \tag{23}$$

and

$$\sum_{i=1}^{2}\omega_3^4(e_i)S_{\varphi e_2}e_i=0. \tag{24}$$

Let M be a Legendre surface in Sasakian space forms S^5. Then since $S_{\varphi e_1}e_2=S_{\varphi e_2}e_1$ (by [2]) and $trS_{\varphi e_2}=0$, if

$$S_{\varphi e_1}=\begin{bmatrix}a & b\\ b & c\end{bmatrix},\quad \text{then}\quad S_{\varphi e_2}=\begin{bmatrix}b & c\\ c & -b\end{bmatrix}, \tag{25}$$

where a,b,c are functions on M. By the similar way with ([1]), we have $\det S_{\varphi e_2}\neq 0$. Since $\det S_{\varphi e_2}\neq 0$, using (24) we obtain $\omega_3^4=0$.

From (18) and (21) we get $trS_{\varphi e_1}$ is a constant. From (19) and (22) we get $trS_{\varphi e_1}S_{\varphi e_2}=0$. Applying this to (25) we find that $b=0$, $a=constant$, $c=constant$. Therefore we obtain:

Proposition 4.1. *Let M be a non-minimal pseudo-Hermitian mass-symmetric 2-type Legendre surface in the unit sphere S^5 in E^6. Then M is a Chen surface.*

Lemma 4.2. *Let M be a pseudo-Hermitian mass-symmetric 2-type Legendre surface in the unit sphere S^5 in E^6. Then M is flat.*

Proof. Let M be a Legendre surface in a Sasakian manifold S^5 for pseudo-Hermitian geometry. Then using (4) we get

$$\begin{aligned}\hat\nabla_{e_j}\varphi e_i&=\hat\nabla'_{e_j}\varphi e_i=(\hat\nabla'_{e_j}\varphi)e_i+\varphi(\hat\nabla'_{e_j}e_i)\\&=\varphi(\nabla^o_{e_j}e_i+\sigma'(e_i,e_j)).\end{aligned} \tag{26}$$

On the other hand,

$$\hat\nabla_{e_j}\varphi e_i=-S_{\varphi e_i}e_j+\hat D_{e_j}\varphi e_i. \tag{27}$$

Using $S_{\varphi e_1}e_2=S_{\varphi e_2}e_1$ we have

$$\begin{aligned}\varphi(\sigma'(e_i,e_j))&=\varphi(g(S_{\varphi e_1}e_i,e_j)\varphi e_1+g(S_{\varphi e_2}e_i,e_j)\varphi e_2)\\&=-g(S_{\varphi e_i}e_1,e_j)e_1-g(S_{\varphi e_i}e_2,e_j)e_2\\&=-S_{\varphi e_i}e_j,\quad i,j=1,2.\end{aligned} \tag{28}$$

From (26), (27) and (28), we get $\hat D_{e_j}\varphi e_i=\varphi(\nabla^o_{e_j}e_i)$. Thus $\omega_3^4=0$ implies that $\hat D_{e_j}\varphi e_i=0$. Hence we have $\varphi(\nabla^o_{e_j}e_i)=0$. Using (1) obtain $\nabla^o_{e_j}e_i=0$. □

Since M is flat, $\hat K(e_1,e_2)=1+ac-c^2=0$. Hence from $c\neq 0$ we have

$$a=\frac{c^2-1}{c}. \tag{29}$$

Definition. If γ is a curve in a contact Riemannian manifold N, parametrized by arc-length s, we say that γ is a Frenet curve of osculating order r when there exist orthornormal vector fields $E_1, E_2, \ldots, E_r$, along γ such that

$$\dot{\gamma} = E_1,\ \hat{\nabla}_{\dot{\gamma}} E_1 = \hat{\kappa}_1 E_2,\ \hat{\nabla}_{\dot{\gamma}} E_2 = -\hat{\kappa}_1 E_1 + \hat{\kappa}_2 E_3, \ldots,$$
$$\hat{\nabla}_{\dot{\gamma}} E_{r-1} = -\hat{\kappa}_{r-2} E_{r-2} + \hat{\kappa}_{r-1} E_r,\ \hat{\nabla}_{\dot{\gamma}} E_r = -\hat{\kappa}_{r-1} E_{r-1},$$

where $\hat{\kappa}_1, \hat{\kappa}_2, \ldots, \hat{\kappa}_{r-1}$ are positive C^∞ functions of s. $\hat{\kappa}_j$ is called the j-th pseudo-Hermitian curvature of γ. A geodesic is a Frenet curve of osculating order 1, a pseudo-Hermitian circle is a Frenet curve of osculating order 2 with $\hat{\kappa}_1$ a constant; a pseudo-Hermitian helix of order r is a Frenet curve of osculating order r, such that $\hat{\kappa}_1, \hat{\kappa}_2, \ldots, \hat{\kappa}_{r-1}$ are constants.

Theorem 4.3. *Let M be a pseudo-Hermitian mass-symmetric 2-type Legendre surface in the unit sphere S^5 in E^6. Then M is locally product of two pseudo-Hermitian circles.*

Proof. For the second fundamental form σ of M in E^6, we have

$$\sigma(e_1, e_1) = a\varphi e_1 - x,$$
$$\sigma(e_1, e_2) = c\varphi e_2,$$
$$\sigma(e_2, e_2) = c\varphi e_1 - x,$$

for a constant a, $c \neq 0$. From this and the definition (2) of the Tanaka-Webster connection $\hat{\nabla}$ we get

$$\hat{\nabla}_{e_1} e_1 = a\varphi e_1 - x,\ \hat{\nabla}_{e_1} e_2 = c\varphi e_2,\ \hat{\nabla}_{e_1} \varphi e_1 = -a e_1,$$
$$\hat{\nabla}_{e_1} \varphi e_2 = -c e_2,\ \hat{\nabla}_{e_1} \xi = 0,\ \hat{\nabla}_{e_1} x = e_1 \tag{30}$$

and

$$\hat{\nabla}_{e_2} e_1 = c\varphi e_2,\ \hat{\nabla}_{e_2} e_2 = c\varphi e_1 - x,\ \hat{\nabla}_{e_2} \varphi e_1 = -c e_2,$$
$$\hat{\nabla}_{e_2} \varphi e_2 = -c e_1,\ \hat{\nabla}_{e_2} \xi = 0,\ \hat{\nabla}_{e_2} x = e_2. \tag{31}$$

Let $e_1 = E_1$, from (30) we have

$$\hat{\nabla}_{E_1} E_1 = \bar{\nabla}_{E_1} E_1 = a\varphi e_1 - x = \kappa_1 E_2,$$

where $E_2 = \frac{a\varphi e_1 - x}{\sqrt{a^2+1}}, \kappa_1 = \sqrt{a^2+1}$.

$$\hat{\nabla}_{E_1} E_2 = \frac{a}{\sqrt{a^2+1}} \hat{\nabla}_{e_1} \varphi e_1 - \frac{1}{\sqrt{a^2+1}} \hat{\nabla}_{e_1} x = -\sqrt{a^2+1} e_1 = -k_1 E_1.$$

Thus $\hat{\kappa}_2 = 0$ and e_1-curve is a pseudo-Hermitian circle.

Now we put $e_2 = E_1$. From (31) we have

$$\hat{\nabla}_{E_1} E_1 = \bar{\nabla}_{E_1} E_1 = c\varphi e_1 - x = \kappa_1 E_2,$$

where $E_2 = \frac{c\varphi e_1 - x}{\sqrt{c^2+1}}$, $\kappa_1 = \sqrt{c^2+1}$.

$$\hat{\nabla}_{E_1} E_2 = \frac{c}{\sqrt{c^2+1}} \hat{\nabla}_{e_2} \varphi e_1 - \frac{1}{\sqrt{c^2+1}} \hat{\nabla}_{e_2} x = -\sqrt{c^2+1} e_2 = -k_1 E_1.$$

Hence $\hat{\kappa}_2 = 0$ and e_2-curve is a pseudo-Hermitian circle. $\square$

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