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# HARDY-LITTLEWOOD PROPERTY AND $\alpha$-QUASIHYPERBOLIC METRIC

Ki Won Kim and Jeong Seog Ryu

Abstract. Hardy and Littlewood found a relation between the smoothness of the radial limit of an analytic function on the unit disk $D \subset \mathbb{C}$ and the growth of its derivative. It is reasonable to expect an analytic function to be smooth on the boundary if its derivative grows slowly, and conversely. Gehring and Martio showed this principle for uniform domains in $\mathbb{R}^2$. Astala and Gehring proved quasiconformal analogue of this principle for uniform domains in $\mathbb{R}^n$. We consider *$\alpha$-quasihyperbolic metric*, $k_D^\alpha$ and we extend it to proper domains in $\mathbb{R}^n$.

## 1. Introduction

A proper subdomain $D \subset \mathbb{R}^2$ is said to have Hardy-Littlewood property if there is a constant $c$ such that for $0 < \alpha \leq 1$, $f$ is in $\mathrm{Lip}_\alpha(D)$ with $||f||_\alpha \leq cm/\alpha$ whenever $f$ is analytic with

$$|f'(z)| \leq md(z, \partial D)^{\alpha-1} \tag{1.1}$$

in $D$ [3].

Hardy and Littlewood first considered analytic functions on the unit disk in $\mathbb{C}$ and showed that the radial limit $F(\theta) = f(e^{i\theta})$ is in $\mathrm{Lip}_\alpha$ if and only if (1.1) hold [2, Theorem 5.1], [5, Theorem 40].

**Theorem 1.1** (Hardy-Littlewood)**.** *Let $f(z)$ be a function analytic in $|z| < 1$. Then $f(z)$ is continuous in $|z| \leq 1$ and $f(e^{i\theta}) \in Lip_\alpha$ $(0 < \alpha \leq 1)$ if and only if*

$$|f'(z)| \leq m(1 - |z|)^{\alpha-1}.$$

It is reasonable to expect an analytic function to be smooth on the boundary if its derivative grows slowly, and conversely. Gehring and Martio showed this principle for uniform domains in $\mathbb{R}^2$ which include the quasidisk, the image of disk under a quasiconformal map of the extended complex plane [3, Corollary 2.2].

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**Theorem 1.2.** *If $D$ is uniform and $f$ is analytic and satisfies*

$$|f'(z)| \leq md(z, \partial D)^{\alpha-1}$$

*in $D$, then $f$ is $Lip_\alpha(D)$ with*

$$||f||_\alpha \leq \frac{cm}{\alpha},$$

*where $c$ is a constant which depends only on $\alpha$ and the constants for $D$.*

Astala and Ghering introduced the notion of average derivative $a_f(x)$ of quasiconformal map $f$ and proved quasiconformal analogue of Theorem 1.2 for uniform domains in $\mathbb{R}^n$ [1, Theorem 1.9]. They also proved its converse [1, Theorem 3.17].

**Theorem 1.3.** *Suppose that $D$ is a uniform domain in $\mathbb{R}^n$ and that $\alpha$ and $m$ are constants with $0 < \alpha \leq 1$ and $m \geq 0$. If $f$ is $K$-quasiconformal in $D$ with $f(D) \subset \mathbb{R}^n$ and if*

$$a_f(x) \leq md(x, \partial D)^{\alpha-1} \tag{1.2}$$

*for $x \in D$, then $f$ has a continuous extension to $\overline{D} \setminus \{\infty\}$ and*

$$|f(x_1) - f(x_2)| \leq cm\,(|x_1 - x_2| + d(x_1, \partial D))^\alpha \tag{1.3}$$

*for $x_1, x_2 \in \overline{D} \setminus \{\infty\}$, where $c$ is a constant which depends only on $K$, $n$, $\alpha$ and the constants for $D$.*

We consider *$\alpha$-quasihyperbolic metric*, $k_D^\alpha$ and we extend Theorem 1.3 to proper domains in $\mathbb{R}^n$ for $\alpha \leq 1$. If $D$ is not uniform, then there is an $f$ that can not have a continuous extension to $\overline{D} \setminus \{\infty\}$ even though $f$ satisfies the hypothesis of Theorem 1.3.

*Remark* 1.4. Let $D = \{z : 1 < |z| < 2\} \setminus \{z | \arg z = \pi\}$, then $D$ is not a uniform domain but an inner uniform domain in $\mathbb{C}$. The function $f(z) = \log z$ is analytic on $D$ and $a_f(z) = |f'(z)|$ is bounded on $D$. But $f$ can not have continuous extension to $\overline{D} \setminus \{\infty\}$.

**Theorem 1.5.** *Suppose that $D$, $D'$ are proper subdomains of $\mathbb{R}^n$ and that $f : D \to D'$ is $K$-quasiconformal in $D$. Suppose that $\alpha$ and $m$ are constants with $\alpha \leq 1$ and $m \geq 0$. Then*

$$a_f(x) \leq md(x, \partial D)^{\alpha-1} \tag{1.4}$$

*for $x \in D$ if and only if*

$$|f(x_1) - f(x_2)| \leq cm(k_D^\alpha(x_1, x_2) + d(x_1, \partial D)^\alpha) \tag{1.5}$$

*for $x_1, x_2 \in D$, where $c$ is a constant which depends only on $K, n, \alpha$.*

If $D$ is an inner uniform domain, Kim showed that $k_D$ is comparable to $j'_D$ [7, Theorem 2.1] and Langmeyer showed that $k_D^\alpha$ is comparable to $d(x_1, \partial D)^\alpha$ for $\alpha < 0$ [10, Theorem 6.5]. Theorem 1.5 can be rephrased in the following form.

**Theorem 1.6.** *Suppose that $D$, $D'$ are proper subdomains of $\mathbb{R}^n$ and that $f : D \to D'$ is $K$-quasiconformal in $D$. Suppose that $D$ is an inner uniform domain in $\mathbb{R}^n$ and that $\alpha$ and $m$ are constants with $\alpha \le 1$ and $m \ge 0$. Then*

$$a_f(x) \le md(x, \partial D)^{\alpha-1}$$

*for $x \in D$ if and only if the following inequality holds. For $x_1, x_2 \in D$,*

$$|f(x_1) - f(x_2)| \le c\, m\, (\lambda_D(x_1, x_2) + d(x_1, \partial D))^\alpha \qquad (0 < \alpha \le 1), \tag{1.6}$$

$$|f(x_1) - f(x_2)| \le c\, m\, (j'_D(x_1 x_2) + 1) \qquad (\alpha = 0), \tag{1.7}$$

$$|f(x_1) - f(x_2)| \le c\, m\, d(x_1, \partial D)^\alpha \qquad (\alpha < 0). \tag{1.8}$$

*Here $c$ is a constant which depends only on $K$, $n$, $\alpha$ and the constants for $D$.*

## 2. Preliminaries

We shall assume throughout this paper that $D$, $D'$ are proper subdomains of $\mathbb{R}^n$, $n \ge 2$ and that $d(A, B)$ denotes the euclidean distance from $A$ to $B$.

We say that $D$ is uniform if there exist constants $a$ and $b$ such that each pair of points $x_1$, $x_2$ in $D$ can be joined by a rectifiable arc $\gamma \subset D$ for which

$$\begin{cases} \ell(\gamma) \le a|x_1 - x_2|, \\ \min \ell(\gamma(x_j, x)) \le bd(x, \partial D) \end{cases} \tag{2.1}$$

for all $x \in \gamma$. Here $\ell(\gamma)$ denotes the euclidean length of $\gamma$, $\gamma(x_j, x)$ the part of $\gamma$ between $x_j$ and $x$.

For each $x_1, x_2 \in D$ we set $\lambda_D(x_1, x_2) = \inf \ell(\gamma)$ where infimum is taken over all rectifiable arcs $\gamma$ joining $x_1$ and $x_2$ in $D$. We call $\lambda_D$ the *inner length metric* in $D$. We say that $D$ is an inner uniform if (2.1) holds when we replace euclidean distance $|x_1 - x_2|$ by $\lambda_D(x_1, x_2)$, i.e., $\ell(\gamma) \le a\lambda_D(x_1, x_2)$. We also use a metric $j'_D(x_1, x_2) = \frac{1}{2}\log(1 + \frac{\lambda_D(x_1, x_2)}{\min_{j=1,2} d(x_j, \partial D)})$.

For each $x_1, x_2 \in D$ we set

$$k^\alpha_D(x_1, x_2) = \inf_\gamma \int_\gamma d(x, \partial D)^{\alpha-1} ds,$$

where the infimum is taken over all rectifiable arcs $\gamma$ joining $x_1$ and $x_2$ in $D$. We call $k^\alpha_D$ the *$\alpha$-quasihyperbolic metric* in $D$. When $\alpha = 0$, we have $k^0_D = k_D$, the usual quasihyperbolic metric. If $k^\alpha_D(x_1, x_2) = \int_\gamma d(x, \partial D)^{\alpha-1} ds$, then we call $\gamma$ the *$\alpha$-quasihyperbolic geodesic*. The existence of $\alpha$-quasihyperbolic geodesic in $D$ is proved in [4, Lemma 1] for $\alpha = 0$, in [9, Lemma 2.2] for $\alpha < 0$. The same argument in [9, Lemma 2.2] shows the existence for $0 < \alpha < 1$.

Suppose that $f : D \to D'$ is $K$-quasiconformal in $D$ with Jacobian $J_f$. Then $\log J_f$ is integrable over each ball $B \subset D$. Astala and Gehring introduced average derivative $a_f$ in [1]

$$a_f(x) = \exp\left(\frac{1}{n|B_x|}\int_{B_x} \log J_f dm\right),$$

where $B_x = B(x, d(x, \partial D)/2)$ and $|B_x|$ is the $n$-dimensional measure of $B_x$. Using $a_f$, they proved following quasiconformal analogue of Koebe distortion theorem [1, Theorem 1.8].

**Lemma 2.1.** *Let $f : D \to D'$ be $K$-quasiconformal. There exists a positive constant $c$ depending only on $n$, $K$, so that for each $x \in D$*

$$\frac{1}{c} d(f(x), \partial D') \le a_f(x) d(x, \partial D) \le c d(f(x), \partial D').$$

The following is the basic property of quasiconformal mapping [8, Lemma 2.2]. (See also [12, 18.1].)

**Lemma 2.2.** *Let $f : D \to D'$ be $K$-quasiconformal. Then for any $0 < \lambda < 1$ there exist positive constants $c_1$, $c_2$ depending only on $n$, $K$, $\lambda$ such that*

$$B^n(f(x), c_1 d') \subset f(B^n(x, c_2 d) \subset B^n(f(x), \lambda d'),$$

*where $d = d(x, \partial D)$ and $d' = d(f(x), \partial D')$. Moreover, there is a constants $c_3$ depending only on $n$, $K$ such that*

$$B^n(f(x), d'/c_3) \subset f(B_x) \subset B^n(f(x), c_3 d'),$$

*and*

$$d(f(B_x), \partial D') \ge d'/c_3.$$

Using Lemmas 2.1 and 2.2, Koskela showed the following [8, Lemma 2.6].

**Lemma 2.3.** *Let $f : D \to D'$ be $K$-quasiconformal. If $\gamma \subset D$ is a rectifiable curve with $\ell(\gamma) \ge d(\gamma, \partial D)$, then*

$$diam(f\gamma) \le c \int_\gamma a_f(x) ds,$$

*where $c$ is a constant which depends only on $n, K$.*

A homeomorphism $f : D \to D'$ is called *$\eta$-quasisymmetric* if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ so that

$$|x - a| \le t|x - b| \quad \text{implies} \quad |f(x) - f(a)| \le \eta(t)|f(x) - f(b)|$$

for each $t > 0$ and for each triple $x$, $a$, $b$ of points in $D$. The connection between quasiconformality and quasisymmetry has been studied by many authors in various geometric settings. We will use the following result [6, Theorem 11.14].

**Lemma 2.4.** *A homeomorphism $f : D \to D'$ between domains in $\mathbb{R}^n$, $n \ge 2$, is $K$-quasiconformal if and only if there is $\eta$ such that $f$ is $\eta$-quasisymmetric in each ball $B(x, \frac{1}{2} d(x, \partial D))$ for $x \in D$. The statement is quantitative involving $K$, $\eta$, and the dimension $n$.*

We will use the following modified version of [11, Cor. 5.3].

**Lemma 2.5.** *If a homeomorphism $f : D \to D'$ is $\eta'$-quasisymmetric in each ball $B_{x,\lambda} = B(x, \lambda d(x, \partial D))$ for $x \in D$ with a constant $0 < \lambda < 1$, then there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that*

$$\frac{d(f(x), f(y))}{d(f(x), \partial D')} \leq \eta\left(\frac{d(x, y)}{d(x, \partial D)}\right) \tag{2.2}$$

*for all $x \in D$ and $y \in B_{x,\lambda}$.*

## 3. Proof of Theorems

**Proof of Theorem 1.5**

(1.4) $\Longrightarrow$ (1.5). Choose *$\alpha$-quasihyperbolic geodesic* $\gamma \subset D$ joining $x_1$ and $x_2$.

**Case 1 :** $\ell(\gamma) \geq d(\gamma, \partial D)$

By Lemma 2.3

$$\begin{aligned} |f(x_1) - f(x_2)| &\leq diam(f\gamma) \leq c \int_\gamma a_f(x) ds \\ &\leq c \int_\gamma m d(x, \partial D)^{\alpha - 1} ds = cmk_D^\alpha(x_1, x_2). \end{aligned}$$

**Case 2 :** $\ell(\gamma) < d(\gamma, \partial D)$

We follow the similar arguments in the proof of Lemma 2.3. Since $d(\gamma, \partial D) \leq \frac{1}{2} d(x_1, \partial D) + \frac{1}{2} d(x_2, \partial D)$, we get $\gamma \subset B_{x_1} \cup B_{x_2}$. By Lemma 2.2, we get

$$|f(x_1) - f(x_2)| \leq \sum_{i=1,2} diam(f(B_{x_i})) \leq c_1 \sum_{i=1,2} d(f(x_i), \partial D').$$

By Lemma 2.1 and (1.4)

$$\sum_{i=1,2} d(f(x_i), \partial D') \leq c_2 \sum_{i=1,2} d(x_i, \partial D) a_f(x_i) \leq c_2 m \sum_{i=1,2} d(x_i, \partial D)^\alpha.$$

Since $|x_1 - x_2| \leq \ell(\gamma) < d(\gamma, \partial D) \leq \min(d(x_1, \partial D), d(x_2, \partial D))$,

$$d(x_i, \partial D) \leq d(x_j, \partial D) + |x_1 - x_2| \leq d(x_j, \partial D) + \ell(\gamma) \leq 2d(x_j, \partial D)$$

for $\{i, j\} = \{1, 2\}$. Thus

$$\frac{1}{2} d(x_1, \partial D) \leq d(x_2, \partial D) \leq 2d(x_1, \partial D)$$

and

$$|f(x_1) - f(x_2)| \leq c_3 d(x_1, \partial D)^\alpha.$$

(1.5) $\Longrightarrow$ (1.4). If we consider the inverse of the homeomorphsim $f$ in Lemma 2.4 and Lemma 2.5, then the following holds.

**Lemma 3.1.** *If a homeomorphism $f : D \to D'$ is $\eta'$-quasisymmetric in each ball $B_x = B(x, \frac{1}{2}d(x, \partial D))$ for $x \in D$. Then there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that*

$$\frac{d(x, y)}{d(x, \partial D)} \leq \eta\left(\frac{d(f(x), f(y))}{d(f(x), \partial D')}\right) \tag{3.1}$$

*for all $x \in D$ and $y \in B_x$.*

We will use following technical Lemma.

**Lemma 3.2.** *For any homeomorphism $\eta : [0, \infty) \to [0, \infty)$, there exist constants $a$, $t_0 < 1/2$ so that $\eta(t_0) \leq 1/2$ and*

$$\eta(t_0) = at_0. \tag{3.2}$$

*Proof.* We set $b_1 = \min(\eta^{-1}(1/2), 1/2)$, then $\eta(t) \leq \frac{1}{2}$ for $0 \leq t \leq b_1$. Let $m_1 = \frac{\eta(b_1)}{b_1}$ and let $b_0 = \frac{1}{2}b_1$, $m_0 = \frac{\eta(b_0)}{b_0}$. If $m_1 = m_0$, then choose $a = m_0$ and $t_0 = b_0$. If $m_1 \neq m_0$, then choose a constant $a = \frac{1}{2}(m_1 + m_0)$. Then we can find $t_0 < 1/2$ such that the line $l(t) = (t, at)$ and curve $\beta(t) = (t, \eta(t))$ intersect at $t = t_0$. $\square$

Choose $y$ such that $d(f(x), f(y)) = t_0 d(f(x), \partial D')$. By (3.1) and (3.2)

$$\frac{d(x, y)}{d(x, \partial D)} \leq \eta(t_0) = k. \tag{3.3}$$

Choose a line $\gamma$ joining $x$ and $y$. Since $\ell(\gamma) = d(x, y) \leq kd(x, \partial D)$ and $(1 - k)d(x, \partial D) \leq d(\gamma(s), \partial D)$,

$$k_D^\alpha(x, y) \leq \int_\gamma \frac{ds}{d(\gamma(s), \partial D)^{1-\alpha}} \leq \frac{k}{(1-k)^{1-\alpha}} d(x, \partial D)^\alpha. \tag{3.4}$$

By (1.5) and (3.4)

$$\begin{aligned} t_0 d(f(x), \partial D') &= d(f(x), f(y)) \\ &\leq m(k_D^\alpha(x, y) + d(x, \partial D)^\alpha) \\ &\leq m_2 d(x, \partial D)^\alpha. \end{aligned}$$

By Lemma 2.1

$$a_f(x) \leq c\frac{d(f(x), \partial D')}{d(x, \partial D)} \leq c\frac{m_2}{t_0} d(x, \partial D)^{\alpha - 1}.$$

This completes the proof. $\square$

**Proof of Theorem 1.6**

(1.4) $\Longleftrightarrow$ (1.6). If $x_2 \in B(x_1, d(x_1, \partial D))$, then $d(x_1, x_2) = \lambda_D(x_1, x_2)$. The proof follows if we use $\lambda_D(x_1, x_2)$ instead of $d(x_1, x_2) = |x_1 - x_2|$ in the proof of Theorem 1.3 and Theorem 1.5.

*(1.4) $\Longleftrightarrow$ (1.7).* It is easy to see that

$$j'_D(x_1x_2) \le k_D(x_1, x_2). \tag{3.5}$$

Kim showed

$$k_D(x_1, x_2) \le b j'_D(x_1x_2) \tag{3.6}$$

for an inner uniform domain $D$ [7, Theorem 2.1]. Therefore (1.5) and (1.7) are equivalent if $\alpha = 0$.

*(1.4) $\Longleftrightarrow$ (1.8).* In the proof of [10, Theorem 6.5], Langmeyer showed

$$k_D^\alpha(x_1, x_2) \le c \left( \min_{j=1,2} d(x_j, \partial D) \right)^\alpha \tag{3.7}$$

for $\alpha < 0$. Therefore (1.5) and (1.8) are equivalent if $\alpha < 0$. □

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Ki Won Kim
Department of Mathematics Education
College of Education
Silla University
Busan 46958, Korea
*Email address*: kwkim@silla.ac.kr

Jeong Seog Ryu
Department of Mathematics Education
College of Education
Hongik University
Seoul 04066, Korea
*Email address*: jsryu@hongik.ac.kr