

## ON ZEROS AND GROWTH OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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**ABSTRACT.** For a second order linear differential equation  $f'' + A(z)f' + B(z)f = 0$ , with  $A(z)$  and  $B(z)$  being transcendental entire functions under some restrictions, we have established that all non-trivial solutions are of infinite order. In addition, we have proved that these solutions, with a condition, have exponent of convergence of zeros equal to infinity. Also, we have extended these results to higher order linear differential equations.

### 1. Introduction

Consider a second order linear differential equation of the form

$$(1) \quad f'' + A(z)f' + B(z)f = 0, \quad B(z) \not\equiv 0,$$

where  $A(z)$  and  $B(z)$  are entire functions. The fundamental results of complex differential equations can be found in [13] and [15]. We have used the notion of value distribution Theory of meromorphic function, also known as Nevanlinna theory [23]. For an entire function  $f$ , the order of  $f$  and exponent of convergence of  $f$  are defined, respectively, in the following manner,

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ T(r, f)}{\log r}, \quad \lambda(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ N(r, \frac{1}{f})}{\log r},$$

where  $T(r, f)$  is the characteristic function of  $f(z)$  and  $N(r, \frac{1}{f})$  is the number of zeros of  $f(z)$  enclosed in the disk  $|z| < r$ . For an entire function we can replace  $T(r, f)$  with  $\log^+ M(r, f)$ , where  $M(r, f)$  is the maximum modulus of the function  $f(z)$ .

It is well known that all solutions of the equation (1) are entire functions. Using Wiman-Valiron theory, it is proved that equation (1) has all solutions of finite order if and only if both  $A(z)$  and  $B(z)$  are polynomials [15]. Therefore, if

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either  $A(z)$  or  $B(z)$  is a transcendental entire function, then almost all solutions of the equation (1) are of infinite order. So, it is natural to find conditions on coefficients of the equation (1) such that all non-trivial solutions of the equation (1) are of infinite order. Our aim in this paper is to find such  $A(z)$  and  $B(z)$ . It was Gundersen [8], who gave a necessary condition for equation (1) to have a solution of finite order:

**Theorem 1.** *A necessary condition for equation (1) to have a non-trivial solution  $f$  of finite order is*

$$(2) \quad \rho(B) \leq \rho(A).$$

We illustrate this condition with following examples:

**Example 1.**  $f(z) = e^{-z}$  satisfies  $f'' + e^{-z}f' - (e^{-z} + 1)f = 0$ , where  $\rho(A) = \rho(B) = 1$ .

**Example 2.** With  $A(z) = e^z + 2$  and  $B(z) = 1$  equation (1) has finite order solution  $f(z) = e^{-z} + 1$ , where  $\rho(B) < \rho(A)$ .

Thus if  $\rho(A) < \rho(B)$ , then all solutions of the equation (1) are of infinite order. However, given necessary condition is not sufficient, for example:

**Example 3** ([11]). If  $A(z) = P(z)e^z + Q(z)e^{-z} + R(z)$ , where  $P, Q$  and  $R$  are polynomials and  $B(z)$  is an entire function with  $\rho(B) < 1$ , then  $\rho(f)$  is infinite for all non-trivial solutions  $f$  of the equation (1).

In the same paper [8], Gundersen proved the following result:

**Theorem 2.** *Let  $f$  be a non-trivial solution of the equation (1) where either*

- (i)  $\rho(B) < \rho(A) < \frac{1}{2}$  or
- (ii)  $A(z)$  is a transcendental entire function with  $\rho(A) = 0$  and  $B(z)$  is a polynomial.

*Then  $\rho(f)$  is infinite.*

Hellerstein, Miles and Rossi [12] proved Theorem 2 for  $\rho(B) < \rho(A) = \frac{1}{2}$ . In [5], Frei showed that the second order differential equation,

$$(3) \quad f'' + e^{-z}f' + B(z)f = 0$$

possesses a solution of finite order if and only if  $B(z) = -n^2$ ,  $n \in \mathbb{N}$ . Ozawa [20] proved that the equation (3) possesses no solution of finite order when  $B(z) = az + b$ ,  $a \neq 0$ . Amemiya and Ozawa [1] and Gundersen [6] studied the equation (3) for  $B(z)$  being a particular polynomial. After this, Langley [16] showed that the differential equation

$$(4) \quad f'' + Ce^{-z}f' + B(z)f = 0$$

has all non-trivial solutions of infinite order for any nonzero constant  $C$  and for any nonconstant polynomial  $B(z)$ .

J. R. Long introduced the notion of the deficient value and Borel direction into the studies of the equation (1). For the definition of deficient value, Borel direction and function extremal for Yang's inequality one may refer to [23].

In [19], J. R. Long proved that if  $A(z)$  is an entire function extremal for Yang's inequality and  $B(z)$  a transcendental entire function with  $\rho(B) \neq \rho(A)$ , then all solutions of the equation (1) are of infinite order. In [17], J. R. Long replaced the condition  $\rho(B) \neq \rho(A)$  with the condition that  $B(z)$  is an entire function with *Fabry gap*. We say that an entire function  $B(z) = \sum_{n=0}^{\infty} a_{\lambda_n} z^{\lambda_n}$  has Fabry gap if the sequence  $(\lambda_n)$  satisfies

$$\frac{\lambda_n}{n} \rightarrow \infty$$

as  $n \rightarrow \infty$ . An entire function  $B(z)$  with Fabry gap satisfies  $\rho(B) > 0$  [10].

X. B. Wu [21] proved that if  $A(z)$  is a non-trivial solution of  $w'' + Q(z)w = 0$ , where  $Q(z) = b_m z^m + \cdots + b_0$ ,  $b_m \neq 0$  and  $B(z)$  be an entire function with  $\mu(B) < \frac{1}{2} + \frac{1}{2(m+1)}$ , then all solutions of equation (1) are of infinite order. J. R. Long [17] replaced the condition  $\mu(B) < \frac{1}{2} + \frac{1}{2(m+1)}$  with  $B(z)$  being an entire function with *Fabry gap* such that  $\rho(B) \neq \rho(A)$ .

The main source of the problems in complex differential equations is Gundersen's "*Research questions on meromorphic functions and complex differential equations*" [9]. J. R. Long [18] gave a partial solution for a question (Question no. 5.1) asked by Gundersen in [9]. He proved that:

**Theorem 3.** *Let  $A(z) = v(z)e^{P(z)}$ , where  $v(z) (\neq 0)$  is an entire function and  $P(z) = a_n z^n + \cdots + a_0$  is a polynomial of degree  $n$  such that  $\rho(v) < n$ . Let  $B(z) = b_m z^m + \cdots + b_0$  be a non-constant polynomial of degree  $m$ . Then all non-trivial solutions of the equation (1) have infinite order if one of the following condition holds:*

- (i)  $m + 2 < 2n$ ;
- (ii)  $m + 2 > 2n$  and  $m + 2 \neq 2kn$  for all integers  $k$ ;
- (iii)  $m + 2 = 2n$  and  $\frac{a_n^2}{b_m}$  is not a negative real.

In our work, we have assumed  $B(z)$  to be a transcendental entire function in Theorem 3. We will prove the following theorem:

**Theorem 4.** *Suppose  $A(z)$  is an entire function with  $\lambda(A) < \rho(A)$  and*

- (1)  *$B(z)$  is a transcendental entire function satisfying  $\rho(B) \neq \rho(A)$  or*
- (2)  *$B(z)$  is a transcendental entire function with Fabry gap.*

*Then all non-trivial solutions of the equation (1) are of infinite order.*

For the exponent of convergence of zeros of  $f$ , where  $f$  is a non-trivial solution of the equation (1) with coefficients satisfying the condition of Theorem 4, we have next result.

**Corollary 1.** *Suppose that  $f(z) = h(z)e^{Q(z)}$  a non-trivial solution of the equation (1), where  $h(z)$  is a canonical product of zeros of  $f(z)$  and  $Q(z)$  is an entire function. Then  $\lambda(f) = \infty$ , if  $\rho(B) > \max\{\rho(A), \rho(Q)\}$ .*

In condition (2) of Theorem 4,  $B(z)$  may be a transcendental entire function with order equal to the order of an entire function  $A(z)$ . J. R. Long proved Theorem 4 for  $A(z)$  being an entire function extremal for Yang's inequality in [17] and [19]. We illustrate our result with some examples.

**Example 4.** Consider the equation

$$f'' + Q(z)e^{P(z)}f' + B(z)f = 0,$$

where  $Q(z)$ ,  $P(z)$  are polynomials and  $B(z)$  is any transcendental entire function with  $\rho(B) \neq \text{degree of } P(z)$ . Then  $\rho(f) = \infty$  for all non-trivial solutions.

**Example 5.** If equation is given by

$$f'' + \sin(z)e^{P(z)}f' + \cos(z^{\frac{n}{2}})f = 0,$$

where  $P(z)$  is a polynomial of degree  $m > 1$ ,  $m \neq \frac{n}{2}$  and  $n \in \mathbb{N}$ , then all non-trivial solutions are of infinite order.

We have organised the paper in the following manner. In Section 2, we give results which will be useful in proving our main result. In Section 3, we will prove our main theorem. In Section 4, we will extend our result to higher order linear differential equations.

## 2. Auxiliary result

In this section, we present some known results, which will be useful in proving Theorem 4. These results involves linear measure, logarithmic measure and logarithmic density of sets, therefore we recall these concepts:

The linear measure of a set  $E \subset [0, \infty)$  is defined as  $m(E) = \int_E dt$ . The logarithmic measure of a set  $F \subset [1, \infty)$  is given by  $m_1(F) = \int_F \frac{dt}{t}$ . The upper and lower logarithmic densities of a set  $F \subset [0, \infty)$  are given, respectively, by

$$\overline{\log dens}(F) = \limsup_{r \rightarrow \infty} \frac{m_1(F \cap [1, r])}{\log r}, \quad \underline{\log dens}(F) = \liminf_{r \rightarrow \infty} \frac{m_1(F \cap [1, r])}{\log r}.$$

Also, logarithmic density of a set  $F \subset [1, \infty)$  is defined as

$$\log dens(F) = \overline{\log dens}(F) = \underline{\log dens}(F).$$

The following lemma is due to Gundersen [7] which has been used extensively in the literature.

**Lemma 1.** *Let  $f$  be a transcendental entire function of finite order  $\rho$  and let  $\Gamma = \{(k_1, j_1), (k_2, j_2), \dots, (k_m, j_m)\}$  denote finite set of distinct pairs of integers that satisfy  $k_i > j_i \geq 0$  for  $i = 1, 2, \dots, m$  and let  $\epsilon > 0$  be a given constant. Then the following three statements hold:*

- (i) *there exists a set  $E_1 \subset [0, 2\pi)$  that has linear measure zero, such that if  $\psi_0 \in [0, 2\pi) \setminus E_1$ , then there is a constant  $R_0 = R_0(\psi_0) > 0$  so that for all  $z$  satisfying  $\arg z = \psi_0$  and  $|z| \geq R_0$  and for all  $(k, j) \in \Gamma$ , we have*

$$(5) \quad \left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho-1+\epsilon)}.$$

- (ii) *there exists a set  $E_2 \subset (1, \infty)$  that has finite logarithmic measure, such that for all  $z$  satisfying  $|z| \notin E_2 \cup [0, 1]$  and for all  $(k, j) \in \Gamma$ , the inequality (5) holds.*
- (iii) *there exists a set  $E_3 \subset [0, \infty)$  that has finite linear measure, such that for all  $z$  satisfying  $|z| \notin E_3$  and for all  $(k, j) \in \Gamma$ , we have*

$$(6) \quad \left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho+\epsilon)}.$$

For the statement of next lemma we need the notion of *critical rays* which is defined as follows:

**Definition 1** ([18]). Let  $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ ,  $a_n \neq 0$  and  $\delta(P, \theta) = \operatorname{Re}(a_n e^{i n \theta})$ . A ray  $\gamma = r e^{i \theta}$  is called a *critical ray* of  $e^{P(z)}$  if  $\delta(P, \theta) = 0$ .

It can be easily seen that there are  $2n$  different critical rays of  $e^{P(z)}$  which divides the whole complex plane into  $2n$  distinct sectors of equal length  $\frac{\pi}{n}$ . Also  $\delta(P, \theta) > 0$  in  $n$  sectors and  $\delta(P, \theta) < 0$  in remaining  $n$  sectors. We note that  $\delta(P, \theta)$  is alternatively positive and negative in the  $2n$  sectors. We now fix some notations:

$$E^+ = \{\theta \in [0, 2\pi] : \delta(P, \theta) \geq 0\},$$

$$E^- = \{\theta \in [0, 2\pi] : \delta(P, \theta) \leq 0\}.$$

Let  $\alpha > 0$  and  $\beta > 0$  be such that  $\alpha < \beta$  then

$$\Omega(\alpha, \beta) = \{z \in \mathbb{C} : \alpha < \arg z < \beta\}.$$

The following result gives estimates for absolute value of  $A(z)$  outside a negligible set (i.e., the set of measure zero).

**Lemma 2** ([2]). *Let  $A(z) = v(z)e^{P(z)}$  be an entire function with  $\lambda(A) < \rho(A) = n$ , where  $P(z)$  is a polynomial of degree  $n$ . Then for every  $\epsilon > 0$  there exists  $E \subset [0, 2\pi)$  of linear measure zero such that*

- (i) *for  $\theta \in E^+ \setminus E$  there exists  $R > 1$  such that*

$$(7) \quad |A(re^{i\theta})| \geq \exp((1 - \epsilon)\delta(P, \theta)r^n)$$

*for  $r > R$ .*

- (ii) *for  $\theta \in E^- \setminus E$  there exists  $R > 1$  such that*

$$(8) \quad |A(re^{i\theta})| \leq \exp((1 - \epsilon)\delta(P, \theta)r^n)$$

*for  $r > R$ .*

Next lemma is from [4] and give estimates for an entire function of order less than one.

**Lemma 3.** *Let  $w(z)$  be an entire function of order  $\rho$ , where  $0 < \rho < \frac{1}{2}$  and let  $\epsilon > 0$  be a given constant. Then there exists a set  $S \subset [0, \infty)$  that has upper logarithmic density at least  $1 - 2\rho$  such that  $|w(z)| > \exp(|z|^{\rho-\epsilon})$  for all  $z$  satisfying  $|z| \in S$ .*

The following lemma is from [15].

**Lemma 4.** *Let  $g : (0, \infty) \rightarrow \mathbb{R}$ ,  $h : (0, \infty) \rightarrow \mathbb{R}$  be monotone increasing functions such that  $g(r) < h(r)$  outside of an exceptional set  $E$  of finite logarithmic measure. Then, for any  $\alpha > 1$ , there exists  $r_0 > 0$  such that  $g(r) < h(\alpha r)$  holds for all  $r > r_0$ .*

Next lemma give property of an entire function with Fabry gap and can be found in [17], [22].

**Lemma 5.** *Let  $g(z) = \sum_{n=0}^{\infty} a_{\lambda_n} z^{\lambda_n}$  be an entire function of finite order with Fabry gap, and  $h(z)$  be an entire function with  $\rho(h) = \sigma \in (0, \infty)$ . Then for any given  $\epsilon \in (0, \sigma)$ , there exists a set  $H \subset (1, +\infty)$  satisfying  $\overline{\log dens}(H) \geq \xi$ , where  $\xi \in (0, 1)$  is a constant such that for all  $|z| = r \in H$ , one has*

$$\log M(r, h) > r^{\sigma-\epsilon}, \quad \log m(r, g) > (1 - \xi) \log M(r, g),$$

where  $M(r, h) = \max\{|h(z)| : |z| = r\}$ ,  $m(r, g) = \min\{|g(z)| : |z| = r\}$  and  $M(r, g) = \max\{|g(z)| : |z| = r\}$ .

The following remark follows from the above lemma.

*Remark 1.* Suppose that  $g(z) = \sum_{n=0}^{\infty} a_{\lambda_n} z^{\lambda_n}$  be an entire function of order  $\sigma \in (0, \infty)$  with Fabry gaps then for any given  $\epsilon > 0$ , ( $0 < 2\epsilon < \sigma$ ), there exists a set  $H \subset (1, +\infty)$  satisfying  $\overline{\log dens}(H) \geq \xi$ , where  $\xi \in (0, 1)$  is a constant such that for all  $|z| = r \in H$ , one has

$$|g(z)| > M(r, g)^{(1-\xi)} > \exp((1 - \xi)r^{\sigma-\epsilon}) > \exp(r^{\sigma-2\epsilon}).$$

Next result gives the lower bound for the order of a solution of the equation (1) and its proof can be found in [11].

**Lemma 6.** *Let  $f$  be a finite order solution of the equation (1) with  $\rho(A) > \rho(B)$ . Then  $\rho(f) \geq \rho(A)$ .*

This lemma is not true if  $\rho(A) = \rho(B)$ , for example,  $f(z) = z$  is a solution of differential equation  $f'' + (ze^z)f' - e^zf = 0$ . We are now prepared to give the proof of our main result.

### 3. Proof of Theorem 4

*Proof.* If  $\rho(A) = \infty$ , then the conclusion holds (from equation (1)). And if  $\rho(A) < \rho(B)$ , then by Theorem 2, all non-trivial solutions  $f$  of the equation (1) are of infinite order. Thus we consider that  $\rho(B) \leq \rho(A) < \infty$ .

Since  $\lambda(A) < \rho(A)$  therefore  $A(z) = v(z)e^{P(z)}$ , where  $v(z)$  is an entire function and  $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ ,  $a_n \neq 0$ ,  $\lambda(A) = \rho(v) < \rho(A) = n$ .

Let us suppose that there exists a non-trivial solution  $f$  of the equation (1) such that  $\rho(f) < \infty$ . Then by Lemma 1, there exists a set  $E_1 \subset [0, 2\pi)$  that has linear measure zero, such that if  $\psi_0 \in [0, 2\pi) \setminus E_1$ , then there is a constant  $R_0 = R_0(\psi_0) > 0$  so that for all  $z$  satisfying  $\arg z = \psi_0$  and  $|z| \geq R_0$ , we have

$$(9) \quad \left| \frac{f^{(k)}(z)}{f(z)} \right| \leq |z|^{2\rho(f)}, \quad k = 1, 2.$$

- (1) Let  $B(z)$  be a transcendental entire function with  $\rho(B) \neq \rho(A)$ . In this case we need to consider  $\rho(B) < \rho(A)$ . We consider the following cases on  $\rho(B)$ .

- (a) Suppose that  $0 < \rho(B) \leq \frac{1}{2}$ . Then from Lemma 3, there exists a set  $S \subset [0, \infty)$  that has upper logarithmic density at least  $1 - 2\rho(B)$  such that

$$(10) \quad |B(z)| > \exp(|z|^{\rho(B)-\epsilon})$$

for all  $z$ , satisfying  $|z| \in S$ . From equation (1), (8), (9) and (10), for all  $z$ , satisfying  $\arg z = \psi_0 \in E^- \setminus (E \cup E_1)$  and  $|z| = r \in S$ ,  $|z| = r > R_0(\psi_0)$  we have

$$\begin{aligned} \exp(r^{\rho(B)-\epsilon}) &< |B(z)| \\ &\leq \left| \frac{f''(z)}{f(z)} \right| + |A(z)| \left| \frac{f'(z)}{f(z)} \right| \\ &\leq r^{2\rho(f)}(1 + o(1)) \end{aligned}$$

which is a contradiction for arbitrary large  $r$ .

- (b) When  $\frac{1}{2} \leq \rho(B) < \infty$  then using Phragmén-Lindelöf principle, there exists a sector  $\Omega(\alpha, \beta)$ ;  $0 \leq \alpha < \beta \leq 2\pi$  with  $\beta - \alpha \geq \frac{\pi}{\rho(B)}$  such that

$$(11) \quad \limsup_{r \rightarrow \infty} \frac{\log^+ \log^+ |B(re^{i\theta})|}{\log r} = \rho(B)$$

for all  $\theta \in \Omega(\alpha, \beta)$ . Since  $\rho(B) < \rho(A)$  this implies that there exists  $\theta_0 \in \Omega(\alpha, \beta) \cap (E^- \setminus E)$ . Thus from equation (8) and (11), for  $\arg z = \theta_0$  we have,

$$(12) \quad |A(re^{i\theta_0})| \leq \exp((1 - \epsilon)\delta(P, \theta_0)r^n)$$

and

$$(13) \quad \exp(r^{\rho(B)-\epsilon}) < |B(re^{i\theta_0})|$$

for sufficiently large  $r$ . Now from equations (1), (9), (12) and (13), for all  $z = re^{i\theta_0}$ , satisfying  $\theta_0 \in \Omega(\alpha, \beta) \cap (E^- \setminus (E \cup E_1))$

and  $|z| = r > R_0(\theta_0)$  we have,

$$\begin{aligned} \exp(r^{\rho(B)-\epsilon}) &< |B(z)| \\ &\leq \left| \frac{f''(z)}{f(z)} \right| + |A(z)| \left| \frac{f'(z)}{f(z)} \right| \\ &\leq r^{2\rho(f)}(1 + o(1)) \end{aligned}$$

which gives a contradiction for arbitrary large  $r$ .

- (c) Now suppose that  $B(z)$  is a transcendental entire function with  $\rho(B) = 0$ , then using a result from [3], for all  $\theta \in [0, 2\pi)$  one has,

$$(14) \quad \limsup_{r \rightarrow \infty} \frac{\log |B(re^{i\theta})|}{\log r} = \infty$$

this implies that for any large  $G > 0$  there exists  $R(G) > 0$  such that

$$(15) \quad r^G < |B(re^{i\theta})|$$

for all  $\theta \in [0, 2\pi)$  and for all  $r > R(G)$ . From equations (1), (8), (9) and (15), for all  $z = re^{i\theta}$  satisfying  $\arg z = \theta \in E^- \setminus (E \cup E_1)$  and  $|z| = r > R$  we have,

$$\begin{aligned} r^G &< |B(z)| \\ &\leq \left| \frac{f''(z)}{f(z)} \right| + |A(z)| \left| \frac{f'(z)}{f(z)} \right| \\ &\leq r^{2\rho(f)}(1 + o(1)) \end{aligned}$$

which is a contradiction for arbitrary large  $r$ .

Thus all non-trivial solutions of the equation (1) are of infinite order in this case.

- (2) Let  $B(z)$  be a transcendental entire function with Fabry gap. Then from Lemma (5), for any given  $\epsilon > 0$ , ( $0 < 2\epsilon < \rho(B)$ ), there exists a set  $H \subset (1, +\infty)$  satisfying  $\overline{\log dens}(H) \geq \xi$ , where  $\xi \in (0, 1)$  is a constant such that for all  $|z| = r \in H$ , one has

$$(16) \quad |B(z)| > \exp(r^{\rho(B)-2\epsilon}).$$

From equation (1), (8), (9) and (16), for all  $z$  satisfying  $\arg z = \psi_0 \in E^- \setminus (E \cup E_1)$  and  $|z| = r \in H$ ,  $r > R_0(\psi_0)$ , we have

$$\begin{aligned} \exp(r^{\rho(B)-2\epsilon}) &< |B(z)| \leq \left| \frac{f''(z)}{f(z)} \right| + |A(z)| \left| \frac{f'(z)}{f(z)} \right| \\ &\leq r^{2\rho(f)}(1 + o(1)) \end{aligned}$$

which again gives a contradiction for arbitrary large  $r$ .

We thus conclude that all non-trivial solutions of the equation (1) are of infinite order.  $\square$



We next give proof of Corollary 1, which involves Theorem 1 and order of sum of entire functions, see [14].

*Proof.* Since  $f(z) = h(z)e^{Q(z)}$ , where  $h(z)$  is canonical product of zeros of  $f$  and  $Q(z)$  is entire function, be a solution of equation (1) therefore  $\rho(f) = \infty$ .

From equation (1), we have

$$(17) \quad h'' + (A(z) + 2Q'(z))h' + (B(z) + Q''(z) + (Q')^2(z) + A(z)Q'(z))h = 0.$$

If  $\lambda(f) = \rho(h) < \rho(f) = \infty$ , from Theorem 1 we have that  $\rho(A + 2Q') \geq \rho(B + Q'' + Q'^2 + AQ')$ . Which does not hold under given condition. Thus, equation (17) has no non-trivial solution of finite order. Hence,  $\lambda(f) = \rho(h) = \infty$ .  $\square$

#### 4. Further results

In this section we will extend our result to higher order linear differential equations. We consider the higher order linear differential equation as follows:

$$(18) \quad f^{(m)} + A_{(m-1)}(z)f^{(m-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$

where  $m \geq 2$  and  $A_0, A_1, \dots, A_{(m-1)}$  are entire functions. Then it is well known that all solutions of the equation (18) are entire functions. Moreover, if  $A_0, A_1, \dots, A_{(m-1)}$  are polynomials, then all solutions of the equation (18) are of finite order and vice-versa [15]. Therefore, if any of the coefficient is a transcendental entire function, then equation (18) will possess almost all solutions of infinite order. In the next theorem, we give conditions on coefficients of the equation (18) so that all solutions are of infinite order.

**Theorem 5.** *Suppose there exists an integer  $j \in \{1, 2, \dots, m-1\}$  such that  $\lambda(A_j) < \rho(A_j)$ . Suppose that  $A_0$  is a transcendental entire function satisfying  $\rho(A_i) < \rho(A_0)$  where  $i = 1, 2, \dots, m-1, i \neq j$  with*

- (1)  $\rho(A_0) \neq \rho(A_j)$  or
- (2)  $A_0(z)$  being a transcendental entire function with Fabry gap.

*Then every non-trivial solution of the equation (18) is of infinite order.*

*Proof.* First let us suppose that  $\rho(A_j) < \rho(A_0)$ . Then suppose that there exist a solution  $f \not\equiv 0$  of the equation (18) such that  $\rho(f) < \infty$ , then by (ii) of Lemma 1, there exists a set  $E_2 \subset (1, \infty)$  that has finite logarithmic measure, such that for all  $z$  satisfying  $|z| \notin E_2 \cup [0, 1]$  such that

$$(19) \quad \left| \frac{f^{(k)}(z)}{f(z)} \right| \leq |z|^{m\rho(f)},$$

where  $k = 1, 2, \dots, m$ . Using equation (18) and (19), we have

$$\begin{aligned} |A_0(z)| &\leq \left| \frac{f^{(m)}(z)}{f(z)} \right| + |A_{(m-1)}(z)| \left| \frac{f^{(m-1)}(z)}{f(z)} \right| + \cdots + |A_1(z)| \left| \frac{f'(z)}{f(z)} \right| \\ &\leq |z|^{m\rho(f)} \{1 + |A_{(m-1)}(z)| + \cdots + |A_1(z)|\} \end{aligned}$$

for all  $z$  satisfying  $|z| \notin E_2 \cup [0, 1]$ . From here we get that

$$(20) \quad T(r, A_0) \leq m\rho(f) \log r + (m-1)T(r, A_i) + O(1),$$

where  $T(r, A_i) = \max\{T(r, A_k) : k = 1, 2, \dots, m-1\}$  and  $|z| = r \notin E_2 \cup [0, 1]$ . Using Lemma 4, this implies that  $\rho(A_0) \leq \rho(A_i)$ , which is a contradiction. Thus all non-trivial solutions of the equation (18) are of infinite order in this case.

Now consider  $\rho(A_0) \leq \rho(A_j)$  and there exists a non-trivial solution  $f$  of finite order then by (i) of Lemma 1, there exists a set  $E_1 \subset [0, 2\pi)$  with linear measure zero such that if  $\psi_0 \in [0, 2\pi) \setminus E_1$ , then there is a constant  $R_0 = R_0(\psi_0) > 0$  so that for all  $z$  satisfying  $\arg z = \psi_0$  and  $|z| \geq R_0$  we have

$$(21) \quad \left| \frac{f^{(k)}(z)}{f(z)} \right| \leq |z|^{m\rho(f)}, \quad k = 1, 2, \dots, m-1.$$

Since  $\rho(A_i) < \rho(A_0)$  for all  $i = 1, 2, \dots, m-1, i \neq j$  then for any constant  $\eta > 0$  such that

$$\max\{\rho(A_i) : i = 1, 2, \dots, m-1, i \neq j\} < \eta < \rho(A_0)$$

there exists  $R_0 > 0$  such that

$$(22) \quad |A_i(z)| \leq \exp |z|^\eta,$$

where  $i = 1, 2, \dots, m-1, i \neq j$  and  $|z| = r > R_0$ .

Also  $\lambda(A_j) < \rho(A_j) = n$  then  $A_j(z) = v(z)e^{P(z)}$ , where  $v(z)$  is an entire function and  $P(z)$  is a polynomial of degree  $n$ .

- (1) Let  $A_0(z)$  be a transcendental entire function with  $\rho(A_0) \neq \rho(A_j)$ . In this case we need to consider that  $\rho(A_0) < \rho(A_j)$ . We will discuss following three cases:

- (a) Suppose  $0 < \rho(A_0) < \frac{1}{2}$  then by Lemma 3, for  $0 < \epsilon < (\rho(A_0) - \eta)$  there exists a set  $S \subset [0, \infty)$  that has upper logarithmic density at least  $1 - 2\rho(A_0)$  such that

$$(23) \quad |A_0(z)| > \exp(|z|^{\rho(A_0) - \epsilon})$$

for all  $z$  satisfying  $|z| \in S$ . Now using equation (8), (18), (21), (22) and (23) we have

$$\begin{aligned} \exp(|z|^{\rho(A_0) - \epsilon}) &< |A_0(z)| \\ &\leq \left| \frac{f^{(m)}(z)}{f(z)} \right| + |A_{(m-1)}(z)| \left| \frac{f^{(m-1)}(z)}{f(z)} \right| \\ &\quad + \dots + |A_1(z)| \left| \frac{f'(z)}{f(z)} \right| \\ &\leq r^{m\rho(f)} \{1 + \exp r^\eta + \dots \\ &\quad + \exp((1 - \epsilon)\delta(P, \psi_0)r^n) + \dots + \exp r^\eta\} \\ &= r^{m\rho(f)} \{1 + (m-2)\exp r^\eta + o(1)\} \end{aligned}$$

for all  $z$  satisfying  $|z| = r \in S$  and  $\arg z = \psi_0 \in E^- \setminus (E \cup E_1)$ .  
From here we will get contradiction for sufficiently large  $r$ .

- (b) Now suppose that  $\rho(A_0) \geq \frac{1}{2}$ , then using Phragmén-Lindelöf principle, there exists a sector  $\Omega(\alpha, \beta)$ ;  $0 \leq \alpha < \beta \leq 2\pi$  with  $\beta - \alpha \geq \frac{\pi}{\rho(A_0)}$  such that

$$(24) \quad \limsup_{r \rightarrow \infty} \frac{\log^+ \log^+ |A_0(re^{i\theta})|}{\log r} = \rho(A_0)$$

for all  $\theta \in \Omega(\alpha, \beta)$ . Since  $\rho(A_0) < \rho(A_j)$  this implies that there exists  $\theta_0 \in \Omega(\alpha, \beta) \cap (E^- \setminus E)$  such that

$$(25) \quad |A_j(re^{i\theta_0})| \leq \exp((1 - \epsilon)\delta(P, \theta_0)r^n)$$

and from equation (24), we have

$$(26) \quad |A_0(re^{i\theta_0})| > \exp r^{\rho(A_0) - \epsilon}.$$

Thus we get contradiction using equation (18), (21), (22), (25) and (26) for sufficiently large  $r$  by using similar argument as in case (1a).

- (c) Suppose that  $A_0$  is a transcendental entire function with  $\rho(A_0) = 0$ , then using a result from [3], for all  $\theta \in [0, 2\pi)$  one has,

$$(27) \quad \limsup_{r \rightarrow \infty} \frac{\log |A_0(re^{i\theta})|}{\log r} = \infty$$

this implies that for any large  $G > 0$  there exists  $R(G) > 0$  such that

$$(28) \quad r^G < |A_0(re^{i\theta})|$$

for all  $\theta \in [0, 2\pi)$  and for all  $r > R(G)$ . From equations (8), (18), (21), (22) and (28) we get a contradiction for sufficiently large  $r$  using similar argument as in case (1a).

Thus, we conclude that all non-trivial solutions of the equation (18) are of infinite order in this case.

- (2) Suppose that  $A_0(z)$  is a transcendental entire function with Fabry gap then using Lemma 5, for any given  $\epsilon > 0$ , ( $0 < 2\epsilon < \rho(A_0) - \eta$ ), there exists a set  $H \subset (1, +\infty)$  satisfying  $\overline{\log dens}(H) \geq \xi$ , where  $\xi \in (0, 1)$  is a constant such that for all  $|z| = r \in H$ , one has

$$(29) \quad |B(z)| > \exp(r^{\rho(A_0) - 2\epsilon}).$$

From equation (8), (18), (21), (22) and (29), for all  $z$  satisfying  $\arg z = \psi_0 \in E^- \setminus (E \cup E_1)$  and  $|z| = r \in H$ ,  $r > R_0(\psi_0)$ , we have

$$\begin{aligned} \exp(r^{\rho(A_0) - 2\epsilon}) &< |A_0(z)| \\ &\leq \left| \frac{f^{(m)}(z)}{f(z)} \right| + |A_{(m-1)}(z)| \left| \frac{f^{(m-1)}(z)}{f(z)} \right| \end{aligned}$$

$$\begin{aligned}
& + \cdots + |A_1(z)| \left| \frac{f'(z)}{f(z)} \right| \\
& \leq r^{m\rho(f)} \{1 + \exp r^\eta \\
& \quad + \cdots + \exp((1-\epsilon)\delta(P, \psi_0)r^n) \\
& \quad + \cdots + \exp r^\eta\} \\
& = r^{m\rho(f)} \{1 + (m-2)\exp r^\eta + o(1)\}
\end{aligned}$$

which is a contradiction for arbitrary large  $r$ .

Thus all non-trivial solutions  $f$  of the equation (18) are of infinite order.  $\square$

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