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ON THE CLOSED RANGE COMPOSITION AND WEIGHTED COMPOSITION OPERATORS

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Abstract. Let ψ be an analytic function on $\mathbb{D}$, the unit disc in the complex plane, and φ be an analytic self-map of $\mathbb{D}$. Let $\mathcal{B}$ be a Banach space of functions analytic on $\mathbb{D}$. The weighted composition operator $W_{\varphi,\psi}$ on $\mathcal{B}$ is defined as $W_{\varphi,\psi}f = \psi f \circ \varphi$, and the composition operator C_φ defined by $C_\varphi f = f \circ \varphi$ for $f \in \mathcal{B}$. Consider $\alpha > -1$ and $1 \leq p < \infty$. In this paper, we prove that if $\varphi \in H^\infty(\mathbb{D})$, then C_φ has closed range on any weighted Dirichlet space $\mathcal{D}_\alpha$ if and only if $\varphi(\mathbb{D})$ satisfies the reverse Carleson condition. Also, we investigate the closed rangeness of weighted composition operators on the weighted Bergman space A^p_α.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane, G be an open subset of $\mathbb{D}$ and $H(G)$ be the class of all functions analytic on G. Let ψ be an analytic function on the unit disc and let φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ on $H(\mathbb{D})$ is defined as $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$ and $W_{\varphi,\psi}f = \psi f \circ \varphi$ defines a linear operator from $H(\mathbb{D})$ to itself; which we call the weighted composition operator generated by φ and ψ. The pseudohyperbolic distance between two points z and a in $\mathbb{D}$ is defined as $\rho(z,a) = |\varphi_a(z)|$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. The pseudohyperbolic disk with center a and radius r $(0 < r < 1)$ is

$$\Delta(a,r) = \{z : \ \rho(z,a) < r\} = \varphi_a(\Delta(0,r)) = \varphi_a(\{z : \ |z| < r\}).$$

Let $\alpha > -1$ and $1 \leq p < \infty$, the weighted Bergman space $A^p_\alpha(G)$ is the space of all functions $f \in H(G)$ for which

$$||f||^p_\alpha = \int_G |f(z)|^p (1-|z|^2)^\alpha dA(z) < \infty,$$

where A is the normalized area measure on G. This space with the above norm is a Banach space, we denote by A^p_α the space $A^p_\alpha(\mathbb{D})$. Also, the Besov type space $B_{\alpha,p}$ is the space of all functions analytic on the unit disk $\mathbb{D}$ such that their

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derivatives are in A^p_α. The space $B_{\alpha,p}$ with the norm $||f||^p_{\alpha,p} = |f(0)|^p + ||f'||^p_\alpha$ is a Banach space of analytic functions. A useful and famous relationship between the Bergman and Besov type spaces is the equivalency of their norms. Indeed, $B_{\alpha+p,p} = A^p_\alpha$ and $||\cdot||^p_{\alpha,p}$, $||\cdot||^p_\alpha$ are equivalent norms, we denote this equivalence relation by $\approx$. Many authors have benefited from this equivalency to characterize the closed range composition operators on different functional Banach spaces (for example see [3, 4, 8–12]). Because of the existence of the multiplier ψ in weighted composition operators, we could not use this relationship. So we worked with the original norm of A^p_α. However, we sometimes used this equivalent norm in this paper. The weighted Dirichlet space $\mathcal{D}_\alpha$ is $\mathcal{D}_\alpha = B_{\alpha,2}$ for $\alpha > -1$.

If φ and ψ are not constant, then $W_{\varphi,\psi}$ and C_φ are one to one on Banach spaces of analytic functions. So the closed rangeness is equivalent to their being bounded below. Hence, we study the boundedness below of these operators in our proofs. Many authors used the reverse Carleson condition to characterize closed range composition operators and we also use it in our studies.

Let $\alpha > -1$ and $1 \leq p < \infty$. We say that G, a Borel subset of $\mathbb{D}$, satisfies the reverse Carleson condition on A^p_α if there exists positive constant η such that for each $f \in A^p_\alpha$,

$$\eta \int_G |f(z)|^p (1-|z|^2)^\alpha dA(z) \geq \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z).$$

Luecking in [9] have shown that G satisfies the reverse Carleson condition if and only if the following condition holds.

($\star$) There are $\delta > 0$ and $0 < r < 1$ such that $A(G \cap \Delta(a,r)) > \delta A(\Delta(a,r))$ for each $a \in \mathbb{D}$.

That is, the condition ($\star$) holds for G if and only if G satisfies the reverse Carleson condition on all A^p_α for $\alpha > -1$ and $1 \leq p < \infty$.

We use frequently from the following theorem (Area formula), for the proof see [5, Theorem 2.35].

Theorem 1.1. *If g and W are non-negative measurable functions on $\mathbb{D}$, then*

$$\int_{\mathbb{D}} g(\varphi(z))|\varphi'(z)|^2 W(z) dA(z) = \int_{\varphi(\mathbb{D})} g(w) \Big(\sum_{\varphi(z)=w} W(z) \Big) dA(w).$$

For φ a holomorphic map on $\mathbb{D}$ and $w \neq \varphi(0)$ a point of the plane, let $z_j(w)$ be the points of the disk for which $\varphi(z_j(w)) = w$, with multiplicities. Let $\alpha > -1$, the generalized Nevanlinna counting function is

$$N_{\varphi,\alpha}(w) = \sum_j (1-|z_j(w)|^2)^\alpha,$$

where we understand $N_{\varphi,\alpha}(w) = 0$ for w is not in $\varphi(\mathbb{D})$.

The closed range composition operators on $\mathcal{D}_\alpha$ have been characterized in the case $\alpha > 0$. For $\alpha \geq 1$, Zorboska in [12], showed that C_φ has closed range on $\mathcal{D}_\alpha$ if and only if there exists $\varepsilon > 0$ such that the set $\{z \in \mathbb{D} : \frac{N_{\varphi,\alpha}(z)}{(1-|z|^2)^\alpha} \geq \varepsilon\}$ satisfies the reverse Carleson condition. Pau and Perez [11], extended this result for $0 < \alpha < 1$. Akeroyd and Fulmer [2], have been used the set $\varphi(\{z \in \mathbb{D} : \frac{1-|z|^2}{1-|\varphi(z)|^2} \geq \varepsilon\})$, instead of the set that Zorboska used, to characterize closed range composition operators on A^p_α for $\alpha > -1$ and $p \geq 1$. Thus if C_φ has closed range on some A^p_α, then it is also on any A^p_α. Moreover, Akeroyd and Dutta in [1], addressed the question: Does closed rangeness of the composition operator C_φ on one space confer closed rangeness of C_φ on a smaller space?

By using Schwarz -Pick Theorem, for any $-1 < \alpha \leq 0$ there exists $\varepsilon > 0$ such that $\{z \in \mathbb{D} : \frac{N_{\varphi,\alpha}(z)}{(1-|z|^2)^\alpha} \geq \varepsilon\} = \varphi(\mathbb{D})$. Jovovic and MacCluer in [6] showed that if $\varphi(\mathbb{D})$ satisfies the reverse Carleson condition, then the composition operator C_φ has closed range on $\mathcal{D}_0$. In this paper we show that the reverse Carleson condition for $\varphi(\mathbb{D})$ implies the closed rangeness of composition operators on $\mathcal{D}_\alpha$ for all $-1 < \alpha \leq 0$. Moreover, we show that if $\varphi' \in H^\infty(\mathbb{D})$, then C_φ has closed range on $\mathcal{D}_\alpha$ for any $\alpha > -1$ if and only if $\varphi(\mathbb{D})$ satisfies the reverse Carleson condition.

In Section 2, we will study the closed range composition operators on $\mathcal{D}_\alpha$. In Section 3, we study the closed range weighted composition operators on A^p_α. Indeed, we show that if the set

$$\varphi\Big(\{z \in \mathbb{D};\ \frac{|\psi(z)|^p(1-|z|^2)^{\alpha+2}}{(1-|\varphi(z)|^2)^{\alpha+2}} \geq \varepsilon\}\Big)$$

satisfies the reverse Carleson condition, then $W_{\varphi,\psi}$ has closed range in A^p_α. Also, we prove the converse of this theorem by setting additional assumptions on φ and ψ.

2. Closed range composition operators on $\mathcal{D}_\alpha$

In this section, we will study the closed range composition operators on weighted Dirichlet space $\mathcal{D}_\alpha$. If φ is an analytic self-map of the unit disk such that $\varphi(0) = 0$, then the boundedness or closed rangeness of C_φ on $\mathcal{D}_\alpha$ is equivalent to the boundedness or closed rangeness on $\mathcal{D}_{\alpha,0} = \{f \in \mathcal{D}_\alpha :\ f(0) = 0\}$. Since a composition operator induced by a disk automorphism is a bounded operator on $\mathcal{D}_\alpha$, we suppose $\varphi(0) = 0$. Thus, in this section we assume that the functions in $\mathcal{D}_\alpha$ vanish at the origin.

Lemma 2.1. *Let φ be an analytic self map of the unit disk and φ' be a bounded function on $\mathbb{D}$. Then for each $\alpha > -1$, C_φ is bounded on $\mathcal{D}_\alpha$. Moreover if $\alpha > 0$, then*

$$\sup_{z\in\mathbb{D}} \frac{N_{\varphi,\alpha}(z)}{(1-|z|^2)^\alpha} < \infty. \tag{2.1}$$

Proof. If f is in $\mathcal{D}_\alpha$, then f' is in A^2_α. Since C_φ is bounded on A^2_α we have

$$\|f \circ \varphi\|^2_{\alpha,2} = |f(\varphi(0))|^2 + \int_{\mathbb{D}} |f'(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^\alpha dA(z)$$

$$\leq |f(\varphi(0))|^2 + ||\varphi'||_\infty^2 \Big(\int_{\mathbb{D}} |f'(\varphi(z))|^2 (1-|z|^2)^\alpha dA(z) \Big) < \infty.$$

Also, by [7, Theorem 1.3 and Remark 2.6] for $\alpha > 0$, (2.1) holds. □

Theorem 2.2. *Let φ be an analytic self-map of the unit disk such that $\varphi' \in H^\infty(\mathbb{D})$. Then the following statements are equivalent.*

(i) *C_φ has closed range on $\mathcal{D}_\alpha$ for some $\alpha > -1$.*
(ii) *C_φ has closed range on $\mathcal{D}_\alpha$ for all $\alpha > -1$.*
(iii) *$\varphi(\mathbb{D})$ satisfies the reverse Carleson condition on $\mathcal{D}_\alpha$ for all $\alpha > -1$.*
(iv) *$\varphi(\mathbb{D})$ satisfies the reverse Carleson condition on $\mathcal{D}_\alpha$ for some $\alpha > -1$.*
(v) *$\varphi(\mathbb{D})$ satisfies the condition $(\star)$.*

Proof. (i) $\Rightarrow$ (ii): We show that C_φ has closed range on $\mathcal{D}_{\alpha+2}$. Let f be in $\mathcal{D}_{\alpha+2}$ where $f(0) = 0$. Consider an analytic function F on $\mathbb{D}$ such that $F' = f$ and $F(0) = 0$. There exist positive constants C_1 and C_2 such that

$$\begin{aligned}
\|f\|_{\alpha+2}^2 &= \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{\alpha+2} dA(z) \\
&\approx \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^{\alpha} dA(z) \\
&= \int_{\mathbb{D}} |F'(z)|^2 (1-|z|^2)^{\alpha} dA(z) \\
&\leq C_1 \int_{\mathbb{D}} |F'(\varphi(z))|^2 |\varphi'(z)|^2 (1-|z|^2)^{\alpha} dA(z) \\
&\leq C_1 C_2 \int_{\mathbb{D}} |F'(\varphi(z))|^2 (1-|z|^2)^{\alpha} dA(z) \\
&\approx \int_{\mathbb{D}} |F''(\varphi(z))|^2 |\varphi'(z)|^2 (1-|z|^2)^{\alpha+2} dA(z) \\
&= \int_{\mathbb{D}} |f'(\varphi(z))|^2 |\varphi'(z)|^2 (1-|z|^2)^{\alpha+2} dA(z) \\
&= \|f \circ \varphi\|_{\alpha+2}^2.
\end{aligned}$$

Akeroyd and Fulmer in [2] showed that if C_φ has closed range on $\mathcal{D}_\alpha$ for $\alpha > 1$, then there exists some $\varepsilon > 0$ such that $G_\varepsilon = \varphi(\Omega_\varepsilon) = \varphi(\{z \in \mathbb{D} : \frac{1-|z|^2}{1-|\varphi(z)|^2} > \varepsilon\})$ satisfies the condition $(\star)$. Hence G_ε and $\varphi(\mathbb{D})$ satisfy the reverse Carleson condition on all A_α^p for $\alpha > -1$ and $1 \leq p < \infty$. Consider $\beta > -1$.

- If $\beta > 0$, we consider $G_\varepsilon = \varphi(\Omega_\varepsilon)$ and
- if $-1 < \beta \leq 0$, we take $\Omega_\varepsilon = \mathbb{D}$, $G_\varepsilon = \varphi(\mathbb{D})$ and $\varepsilon = 1$.

We notice than, for $-1 < \alpha \leq 0$, by Schwarz Lemma $(1-|\varphi(z)|^2)^\alpha \leq (1-|z|^2)^\alpha$. Therefore,

$$\|C_\varphi f\|_{\beta,2}^2 = \int_{\mathbb{D}} |f'(\varphi(z))|^2 |\varphi'(z)|^2 (1-|z|^2)^{\beta} dA(z)$$

$$\geq \int_{\Omega_\varepsilon} |f'(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^\beta dA(z)$$
$$\geq \varepsilon^\beta \int_{\Omega_\varepsilon} |f'(\varphi(z))|^2|\varphi'(z)|^2(1-|\varphi(z)|^2)^\beta dA(z)$$
$$= \varepsilon^\beta \int_{G_\varepsilon} |f'(z)|^2(1-|z|^2)^\beta \eta_\varphi(z) dA(z)$$
$$\geq \varepsilon^\beta \int_{G_\varepsilon} |f'(z)|^2(1-|z|^2)^\beta dA(z)$$
$$\geq \frac{\varepsilon^\beta}{\eta} \int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^\beta dA(z).$$

Where $\eta_\varphi(z) = \sum\limits_{i,\varphi(z_i)=z} 1$, and η is a constant such that for each $f \in A^p_\alpha$,

$$\eta \int_{G_\varepsilon} |f(z)|^p(1-|z|^2)^\beta dA(z) \geq \int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^\beta dA(z).$$

(iv) $\Rightarrow$ (v) and (v) $\Rightarrow$ (iii) are proved in [8].

(ii) $\Rightarrow$ (v) is proved in the proof of the preceding part.

(v) $\Rightarrow$ (i): In the proof of part (i) $\Rightarrow$ (ii) we showed that if $\varphi(\mathbb{D})$ satisfies the reverse Carlson condition, then for any $-1 < \alpha \leq 0$, C_φ has closed range on $\mathcal{D}_\alpha$. □

For any positive constant M, consider the Borel set

$$B_M = \{z \in \mathbb{D} : \ 0 < \frac{N_{\varphi,\alpha}(z)}{(1-|z|^2)^\alpha} \leq M\}.$$

Also, we define $\widetilde{B_M} = B_M \cup (\mathbb{D} \setminus \varphi(\mathbb{D}))$ and $\widetilde{B_M}^c = \mathbb{D} \setminus \widetilde{B_M}$.

Theorem 2.3. *Let $\alpha > -1$ and φ be an analytic self-map of the unit disk. Then the composition operator C_φ is bounded on $\mathcal{D}_\alpha$ if and only if for all f in $\mathcal{D}_\alpha$*

$$\lim_{k\to\infty} \int_{\widetilde{B_k}^c} |f'(z)|^2 N_{\varphi,\alpha}(z) dA(z) = 0. \tag{2.2}$$

Proof. First let C_φ be bounded. We can easily see that the set $\{z \in \mathbb{D} : \frac{N_{\varphi,\alpha}(z)}{(1-|z|^2)^\alpha} = \infty\}$ has zero measure. Therefore, every function in $\mathcal{D}_\alpha$ satisfies in Equation (2.2). Conversely, let Equation (2.2) hold for each f in $\mathcal{D}_\alpha$. Consider f in $\mathcal{D}_\alpha$. By assumption, there are positive constants M and C depending on f such that

$$\int_{\widetilde{B_M}^c} |f'(z)|^p N_{\varphi,\alpha}(z) dA(z) \leq C.$$

Thus

$$\|C_\varphi f\|^2 = \int_{\mathbb{D}} |f'(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^\alpha dA(z)$$

$$
\begin{aligned}
&= \int_{\varphi(\mathbb{D})} |f'(z)|^2 N_{\varphi,\alpha}(z) dA(z) \\
&= \int_{B_M} |f'(z)|^2 N_{\varphi,\alpha}(z) dA(z) + \int_{\widetilde{B_M}^c} |f'(z)|^2 N_{\varphi,\alpha}(z) dA(z) \\
&\leq M \int_{B_M} |f'(z)|^2 (1-|z|^2)^\alpha dA(z) + \int_{\widetilde{B_M}^c} |f'(z)|^2 N_{\varphi,\alpha}(z) dA(z) \\
&\leq M\|f\| + C < \infty.
\end{aligned}
$$

Hence $C_\varphi f \in \mathcal{D}_\alpha$ and by closed graph theorem C_φ is bounded. $\square$

Consider $\alpha > 0$. Kellay and Lefevre in [7] showed that if C_φ is bounded on $\mathcal{D}_\alpha$, then $M = \sup_{z\in\mathbb{D}} \frac{N_{\varphi,\alpha}(z)}{(1-|z|^2)^\alpha} < \infty$. Therefore, for any $k > M$ and $f \in \mathcal{D}_\alpha$, we have

$$\int_{\widetilde{B_k}^c} |f'(z)|^2 N_{\varphi,\alpha}(z) dA(z) = 0.$$

Let C_φ be bounded on $\mathcal{D}_\alpha$, where $-1 < \alpha \leq 0$. Consider the following equation

$$\lim_{k\to\infty} \sup_{\|f\|_\alpha = 1} \int_{\widetilde{B_k}^c} |f'(z)|^2 N_{\varphi,\alpha}(z) dA(z) = 0. \tag{2.3}$$

Clearly if $\sup_{z\in\mathbb{D}} \frac{N_{\varphi,\alpha}(z)}{(1-|z|^2)^\alpha} < \infty$, then Equation (2.3) holds. But non-trivial examples of such operators are compact composition operators.

Proposition 2.4. *Consider $\alpha > -1$. If C_φ is compact on $\mathcal{D}_\alpha$, then equation* (2.3) *holds.*

Proof. If Equation (2.3) does not hold, then there exist $C > 0$ and a sequence $\{f_k\}$ in the unit ball of $\mathcal{D}_\alpha$ such that for each k

$$\int_{\widetilde{B_k}^c} |f_k'(z)|^2 N_{\varphi,\alpha}(z) dA(z) \geq C.$$

Since $\{f_k\}$ is a normal sequence, there are a subsequence $\{f_{n_k}\}$ of $\{f_k\}$ and an analytic function f on $\mathbb{D}$ such that $\{f_{n_k}\}$ converges to f uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. For convenient let $\{f_k\} := \{f_{n_k}\}$. We can easily see that f is in $\mathcal{D}_\alpha$. By using the compactness of C_φ, the sequence $\{f_k \circ \varphi\}$ converges to $f \circ \varphi$ in the norm of $\mathcal{D}_\alpha$. But

$$
\begin{aligned}
&\int_{\varphi(\mathbb{D})} |f_k'(z) - f'(z)|^2 N_{\varphi,\alpha}(z) dA(z) \\
\geq &\int_{\widetilde{B_k}^c} |f_k'(z) - f'(z)|^2 N_{\varphi,\alpha}(z) dA(z) \\
\geq &\left(\Big(\int_{\widetilde{B_k}^c} |f_k'(z)|^2 N_{\varphi,\alpha}(z) dA(z) \Big)^{\frac{1}{2}} - \Big(\int_{\widetilde{B_k}^c} |f'(z)|^2 N_{\varphi,\alpha}(z) dA(z) \Big)^{\frac{1}{2}} \right)^2.
\end{aligned}
$$

The first integral is greater than C and the second integral converges to zero (by Theorem 2.3) as $k \to \infty$. Hence, the sequence $\{f_k \circ \varphi\}$ cannot converge to $f \circ \varphi$, and this is a contradiction. $\square$

It is easy to see that the converse of the above proposition is not true. Indeed, if φ is a disk automorphism, then Equation (2.3) is true but C_φ is not compact. Zorboska in [13], showed that there is an analytic self map of the unit disk φ such that C_φ is compact on $\mathcal{D}_0$ and $\sup_{z\in\mathbb{D}} N_{\varphi,0}(z) = \infty$. However, Proposition 2.4 shows that Equation (2.3) holds for C_φ. In the following theorem we say that if Equation (2.3) holds, then for all $-1 < \alpha \leq 0$, C_φ has closed range on $\mathcal{D}_\alpha$ if and only if $\varphi(\mathbb{D})$ satisfies the condition $(\star)$. In [6], Jovovic and MacCluer presented an example where C_φ has closed range on $\mathcal{D}_0$ but $\varphi(\mathbb{D})$ does not satisfy the reverse Carleson condition. Thus Equation (2.3) is not true in general case.

Theorem 2.5. *Let $\alpha > -1$ and φ be an analytic self-map of the unit disk such that satisfies in Equation* (2.3)*. Then C_φ has closed range on $\mathcal{D}_\alpha$ if and only if $\varphi(\mathbb{D})$ satisfies the condition* $(\star)$*.*

Proof. If $\varphi(\mathbb{D})$ does not satisfy the reverse Carleson condition, then there exists a sequence $\{f_k\}$ in $\mathcal{D}_\alpha$ such that $\int_{\mathbb{D}} |f_k'(z)|^2(1-|z|^2)^\alpha dA(z) = 1$ and for any k,

$$\int_{\varphi(\mathbb{D})} |f_k'(z)|^2(1-|z|^2)^\alpha dA(z) < \frac{1}{k^2}.$$

Thus

$$\begin{aligned}
\|C_\varphi f_k\|^2 &= \int_{\mathbb{D}} |f_k'(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^\alpha dA(z)\\
&= \int_{\varphi(\mathbb{D})} |f_k'(z)|^2 N_{\varphi,\alpha}(z) dA(z)\\
&= \int_{B_k} |f_k'(z)|^2 N_{\varphi,\alpha}(z) dA(z) + \int_{\widetilde{B_k}^c} |f_k'(z)|^2 N_{\varphi,\alpha}(z) dA(z)\\
&\leq k\int_{B_k} |f_k'(z)|^2(1-|z|^2)^\alpha dA(z) + \int_{\widetilde{B_k}^c} |f_k'(z)|^2 N_{\varphi,\alpha}(z) dA(z)\\
&\leq \frac{1}{k} + \int_{\widetilde{B_k}^c} |f_k'(z)|^2 N_{\varphi,\alpha}(z) \longrightarrow 0, \quad k \to \infty.
\end{aligned}$$

$\square$

3. Closed range weighted composition operators on A^p_α

For the positive constant ε, we define the sets

$$\Omega_\varepsilon = \{z \in \mathbb{D};\ \frac{|\psi(z)|^p(1-|z|^2)^{\alpha+2}}{(1-|\varphi(z)|^2)^{\alpha+2}} \geq \varepsilon\},\ \text{ and } G_\varepsilon = \varphi(\Omega_\varepsilon).$$

In Theorem 3.1, we show that for the set G_ε, the condition $(\star)$ is a sufficient condition for the closed rangeness of $W_{\varphi,\psi}$. With some additional assumptions, we prove that this condition is also necessary.

Theorem 3.1. *Consider $\alpha > -1$ and $1 \leq p < \infty$. Let ψ be an analytic function on the unit disc and let φ be an analytic self-map of $\mathbb{D}$ such that $W_{\varphi,\psi}$ is a bounded operator on A^p_α. If there exists $\varepsilon > 0$ such that the set G_ε satisfies the condition $(\star)$, then the weighted composition operator $W_{\varphi,\psi}$ has closed range on A^p_α. Moreover with assuming the following conditions, the converse of theorem is also true.*

(i) $\alpha > 0$;
(ii) $\psi \in H^\infty(\mathbb{D})$ *and*
(iii) φ' *be continuous on* $\overline{\mathbb{D}}$ *and* $\varphi'(\zeta) \neq 0$ *for any* $\zeta \in \partial\mathbb{D}$.

Proof. Let $\varepsilon > 0$ be such that the set G_ε satisfies the reverse Carleson condition. By using Schwarz Lemma we have

$$\frac{(1-|z|^2)|\varphi'(z)|}{(1-|\varphi(z)|^2)} \leq 1, \qquad \forall z \in \mathbb{D}.$$

Thus, for any $z \in \Omega_\varepsilon$, we have

$$\frac{|\psi(z)|^p(1-|z|^2)^\alpha}{|\varphi'(z)|^2(1-|\varphi(z)|^2)^\alpha} \geq \frac{|\psi(z)|^p(1-|z|^2)^{\alpha+2}}{(1-|\varphi(z)|^2)^{\alpha+2}} \geq \varepsilon.$$

Let there exist $\varepsilon > 0$ such that the set G_ε satisfies the reverse Carleson condition. So there is a constant $\eta > 0$ such that $\eta \int_{G_\varepsilon} |f(z)|^p(1-|z|^2)^\alpha dA(z) \geq \int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^\alpha dA(z)$ for each $f \in A^p_\alpha$. We show that $W_{\varphi,\psi}$ is bounded below. Let $f \in A^p_\alpha$,

$$\begin{aligned}
&\int_{\mathbb{D}} |\psi(z)f(\varphi(z))|^p(1-|z|^2)^\alpha dA(z) \\
&\geq \int_{\Omega_\varepsilon} |\psi(z)f(\varphi(z))|^p(1-|z|^2)^\alpha dA(z) \\
&\geq \varepsilon \int_{\Omega_\varepsilon} |f(\varphi(z))|^p|\varphi'(z)|^2(1-|\varphi(z)|^2)^\alpha dA(z) \\
&\geq \varepsilon \int_{G_\varepsilon} |f(z)|^p \eta_\varphi(z)(1-|z|^2)^\alpha dA(z) \\
&\geq \varepsilon \int_{G_\varepsilon} |f(z)|^p(1-|z|^2)^\alpha dA(z) \\
&\geq \frac{\varepsilon}{\eta} \int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^\alpha dA(z).
\end{aligned}$$

Conversely, let $W_{\varphi,\psi}$ be a closed range operator and conditions (i), (ii) and (iii) hold. Suppose that there is no $\varepsilon > 0$ such that the set G_ε satisfies the reverse Carleson condition. Thus, for each $k \in \mathbb{N}$, there is a function $f_k \in A^p_\alpha$ such that $\int_{\mathbb{D}} |f_k(z)|^p(1-|z|^2)^\alpha dA(z) = 1$ and

$$\int_{G_k} |f_k(z)|^p(1-|z|^2)^\alpha dA(z) \longrightarrow 0, \quad k \to \infty,$$

where $G_k = \varphi(\Omega_k) = \varphi(\{z \in \mathbb{D};\ \frac{|\psi(z)|^p(1-|z|^2)^{\alpha+2}}{(1-|\varphi(z)|^2)^{\alpha+2}} \geq \frac{1}{k}\})$. Since $\{f_k\}$ is a normal sequence, there is a function f analytic on $\mathbb{D}$ and a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ uniformly on compact subsets of $\mathbb{D}$. For convenience, we use $\{f_k\}$ instead of $\{f_{n_k}\}$.

Now, we claim that $f \equiv 0$ on $\mathbb{D}$. Indeed, let $0 < r < 1$, rD must have an open subset G of positive measure such that $|\psi(z)|^p > \delta$ on G for some $\delta > 0$. For any $z \in G$, we have

$$\frac{|\psi(z)|^p(1-|z|)^{\alpha+2}}{(1-|\varphi(z)|)^{\alpha+2}} \geq \delta(1-r)^{\alpha+2},$$

thus $\varphi(G) \subseteq G_{\delta(1-r)^{\alpha+2}}$ and so there exists a positive integer k_0 such that $A(G_{k_0}) > 0$. Consider the Banach space $A^p_\alpha(G_K)$ with the norm

$$\|f\|_k^p = \int_{G_k} |f(z)|^p(1-|z|^2)^\alpha dA(z).$$

But $G_{k_0} \subseteq G_k$ for every $k \geq k_0$. Thus, if $w \in G_{k_0}$ and K_w denotes the point evaluation at w on $A^p_\alpha(G_{k_0})$, then

$$\begin{aligned}|f_k(w)|^p &\leq \|K_w\|_{k_0}^p \int_{G_{k_0}} |f_k(z)|^p(1-|z|^2)^\alpha dA(z)\\ &\leq \|K_w\|_{k_0}^p \int_{G_k} |f_k(z)|^p(1-|z|^2)^\alpha dA(z) \to 0\end{aligned}$$

as $k \to \infty$.

Now we show that there are constants $0 < r < 1$ and $\delta > 0$ such that $|\varphi'(z)| > \delta$ for any $z \in \mathbb{D} \setminus r\mathbb{D}$. To see this, let (r_n) be a sequence in $(0,1)$ such that $r_n \uparrow 1$ and for each n there is a point $z_n \in \mathbb{D} \setminus r_n\mathbb{D}$ such that $|\varphi'(z_n)| \leq \frac{1}{n}$. The sequence (z_n) must have a limit point in $\partial\mathbb{D}$. By the continuity of φ' there is some $\zeta \in \partial\mathbb{D}$ such that $\varphi'(\zeta) = 0$. This is a contradiction and the claim is proved.

Now we prove the theorem. For each $f \in A^p_\alpha$,

$$\begin{aligned}&\int_{\mathbb{D}} |\psi(z)f_k(\varphi(z))|^p(1-|z|^2)^\alpha dA(z)\\ =& \int_{r\mathbb{D}} |\psi(z)f_k(\varphi(z))|^p(1-|z|^2)^\alpha dA(z)\\ &+ \int_{(\mathbb{D}\setminus r\mathbb{D})\cap\Omega_k} |\psi(z)f_k(\varphi(z))|^p(1-|z|^2)^\alpha dA(z)\\ &+ \int_{(\mathbb{D}\setminus r\mathbb{D})\cap\Omega_k^c} |\psi(z)f_k(\varphi(z))|^p(1-|z|^2)^\alpha dA(z).\end{aligned}$$

Since $\{f_k\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$, the first integral tends to zero as $k \to \infty$. We address the second integral. The above results show that there are positive constants C_1 and C_2 such that

$$\begin{aligned}
&\int_{(\mathbb{D}\setminus r\mathbb{D})\cap\Omega_k} |\psi(z)f_k(\varphi(z))|^p(1-|z|^2)^\alpha dA(z)\\
&\le C_1\int_{\Omega_k} |f_k(\varphi(z))|^p|\varphi'(z)|^2(1-|z|^2)^\alpha dA(z)\\
&= C_1\int_{G_k} |f_k(z)|^p N_{\varphi,\alpha}(z)dA(z)\\
&\le C_2\int_{G_k} |f_k(z)|^p(1-|z|^2)^\alpha dA(z)\to 0
\end{aligned}$$

as $k\to\infty$. Now we show that the third integral tends to 0, consider $0<\beta<1$ such that $(\alpha+2)\beta<\alpha$. Again we can find positive constants C_1 and C_2 such that

$$\begin{aligned}
&\int_{(\mathbb{D}\setminus r\mathbb{D})\cap\Omega_k^c} |\psi(z)f_k(\varphi(z))|^p(1-|z|^2)^\alpha dA(z)\\
&\le C_1\int_{(\mathbb{D}\setminus r\mathbb{D})\cap\Omega_k^c} |f_k(\varphi(z))|^p|\psi(z)|^{p\beta}|\varphi'(z)|^2(1-|z|^2)^{(\alpha+2)\beta}(1-|z|^2)^{\alpha-(\alpha+2)\beta}dA(z)\\
&\le \frac{C_1}{k^\beta}\int_{\Omega_k^c} |f_k(\varphi(z))|^p|\varphi'(z)|^2(1-|\varphi(z)|^2)^{(\alpha+2)\beta}(1-|z|^2)^{\alpha-(\alpha+2)\beta}dA(z)\\
&= \frac{C_1}{k^\beta}\int_{\Omega_k^c} |f_k(z)|^p(1-|z|^2)^{(\alpha+2)\beta}N_{\varphi,(\alpha-(\alpha+2)\beta)}(z)dA(z)\\
&\le \frac{C_2}{k^\beta}\int_{\mathbb{D}} |f_k(z)|^p(1-|z|^2)^\alpha dA(z) = \frac{C_2}{k^\beta},
\end{aligned}$$

where this last quantity converges to zero, as $k\to\infty$. □

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