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SOME REMARKS OF THE CARATHÉODORY'S INEQUALITY ON THE RIGHT HALF PLANE

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Abstract. In this paper, a boundary version of Carathéodory's inequality on the right half plane for p-valent is investigated. Let $Z(s) = 1+c_p\,(s-1)^p+c_{p+1}\,(s-1)^{p+1}+\cdots$ be an analytic function in the right half plane with $\Re Z(s) \leq A$ $(A>1)$ for $\Re s \geq 0$. We derive inequalities for the modulus of $Z(s)$ function, $|Z'(0)|$, by assuming the $Z(s)$ function is also analytic at the boundary point $s=0$ on the imaginary axis and finally, the sharpness of these inequalities is proved.

1. Introduction

The most classical version of the Schwarz lemma examines the behavior of a bounded, analytic function mapping the origin to the origin in the unit disc $D = \{z : |z| < 1\}$. It is possible to see its effectiveness in the proofs of many important theorems. The Schwarz lemma, which has broad applications and is the direct application of the maximum modulus principle, is given in the most basic form as follows:

Let D be the unit disc in the complex plane $\mathbb{C}$. Let $f : D \to D$ be an analytic function with $f(z) = c_p z^p + c_{p+1} z^{p+1} + \cdots$. Under these conditions, $|f(z)| \leq |z|^p$ for all $z \in D$ and $|c_p| \leq 1$. In addition, if the equality $|f(z)| = |z|^p$ holds for any $z \neq 0$, or $|c_p| = 1$, then f is a rotation; that is $f(z) = z^p e^{i\theta}$, θ real (see [6, p. 329]). The Schwarz lemma is one of the most important results in the classical complex analysis, which has become a crucial theme in many branches of mathematical research for over a hundred years. On the other hand, in the book [7], Sharp Real-Parts Theorems (in particular Carathéodory's inequalities), which are frequently used in the theory of entire functions and analytic function theory, have been studied. Also, a boundary version of the Carathéodory's inequality is considered in unit disc and novel results are obtained in [13–15]. As being assumed that the p-valent of the value

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of 1 at $s=1$ of the function, at first, as in Schwarz lemma, Carathéodory's inequality at right half plane will be presented.

Let $Z(s)=1+c_p\,(s-1)^p+c_p\,(s-1)^p+\cdots$ be an analytic function in the right half plane with $\Re Z(s)\leq A$ $(A>1)$ for $\Re s\geq 0$.

Consider the function

$$f(z)=\frac{Z(s)-1}{Z(s)+1-2A},\ z=\frac{s-1}{s+1}.$$

Here, $f(z)$ is an analytic function in D, $f(0)=0$ and $|f(z)|<1$ for $z\in D$. Now, let us show that $|f(z)|<1$ for $|z|<1$. Since

$$\begin{aligned}\left|Z\left(\frac{1+z}{1-z}\right)-1\right|^2&=\left(Z\left(\frac{1+z}{1-z}\right)-1\right)\left(\overline{Z\left(\frac{1+z}{1-z}\right)}-1\right)\\&=\left|Z\left(\frac{1+z}{1-z}\right)\right|^2-Z\left(\frac{1+z}{1-z}\right)-\overline{Z\left(\frac{1+z}{1-z}\right)}+1\end{aligned}$$

and

$$\begin{aligned}\left|Z\left(\frac{1+z}{1-z}\right)+1-2A\right|^2&=\left(Z\left(\frac{1+z}{1-z}\right)+1-2A\right)\left(\overline{Z\left(\frac{1+z}{1-z}\right)}+1-2A\right)\\&=\left|Z\left(\frac{1+z}{1-z}\right)\right|^2+(1-2A)\,Z\left(\frac{1+z}{1-z}\right)\\&\quad+(1-2A)\,\overline{Z\left(\frac{1+z}{1-z}\right)}+(1-2A)^2,\end{aligned}$$

we obtain

$$\begin{aligned}&\left|Z\left(\frac{1+z}{1-z}\right)-1\right|^2-\left|Z\left(\frac{1+z}{1-z}\right)+1-2A\right|^2\\&=-2\,(1-A)\left(Z\left(\frac{1+z}{1-z}\right)+\overline{Z\left(\frac{1+z}{1-z}\right)}\right)+4A-4A^2\\&=-4\,(1-A)\,\Re Z\left(\frac{1+z}{1-z}\right)+4A-4A^2\\&\leq-4\,(1-A)\,A+4A-4A^2=0.\end{aligned}$$

Therefore, we have $|f(z)|<1$ for $|z|<1$.

Since

$$f(z)=\frac{Z\left(\frac{1+z}{1-z}\right)-1}{Z\left(\frac{1+z}{1-z}\right)+1-2A}=\frac{c_p\left(\frac{2z}{1-z}\right)^p+c_{p+1}\left(\frac{2z}{1-z}\right)^{p+1}+\cdots}{2\,(1-A)+c_p\left(\frac{2z}{1-z}\right)^p+c_{p+1}\left(\frac{2z}{1-z}\right)^{p+1}+\cdots},$$

we obtain

$$\frac{f(z)}{z^p} = \frac{c_p\left(\frac{2}{1-z}\right)^p + c_{p+1}\left(\frac{2}{1-z}\right)^{p+1} z + \cdots}{2\left(1-A\right) + c_p\left(\frac{2z}{1-z}\right)^p + c_{p+1}\left(\frac{2z}{1-z}\right)^{p+1} + \cdots}$$

and from Schwarz lemma

$$|c_p| \leq \frac{A-1}{2^{p-1}}.$$

This result is sharp with the function

$$Z(s) = \frac{(s+1)^p + (s-1)^p\left(1-2A\right)}{(s+1)^p - (s-1)^p}.$$

We thus obtain the following lemma.

Lemma 1.1. *Let* $Z(s) = 1 + c_p\left(s-1\right)^p + c_{p+1}\left(s-1\right)^{p+1} + \cdots$ *be an analytic function in the right half plane with* $\Re Z(s) \leq A$ $(A > 1)$ *for* $\Re s \geq 0$. *Then we have the inequality*

$$|c_p| \leq \frac{A-1}{2^{p-1}}. \tag{1.1}$$

This result is sharp and the extremal function is

$$Z(s) = \frac{(s+1)^p + (s-1)^p\left(1-2A\right)}{(s+1)^p - (s-1)^p}.$$

It is an elementary consequence of Schwarz lemma that if f extends continuously to some boundary point b with $|b| = 1$, and if $|f(b)| = 1$ and $f'(b)$ exists, then $|f'(b)| \geq 1$, which is known as the Schwarz lemma on the boundary. In [12], R. Osserman proposed the boundary refinement of the classical Schwarz lemma as follows:

Let $f : D \to D$ be an analytic function with $f(z) = c_p z^p + c_{p+1} z^{p+1} + \cdots$, $p \geq 1$. Assume that there is a $b \in \partial D$ so that f extends continuously to b, $|f(b)| = 1$ and $f'(b)$ exists. Then

$$|f'(b)| \geq p + \frac{1-|c_p|}{1+|c_p|}. \tag{1.2}$$

Thus, by the classical Schwarz lemma, it follows that

$$|f'(b)| \geq p, \tag{1.3}$$

with equality only if f is of the form $f(z) = z^p e^{i\theta}$, θ real. The equality in (1.2) holds if and only if f is of the form $f\left(z\right) = -z^p \frac{\alpha - z}{1-\alpha z}$ for some constant $\alpha \in (-1, 0]$. Inequality (1.3) and its generalizations have important applications in geometric theory of functions and they are still hot topics in the mathematics literature [1–5, 8, 12]. Mercer [9] prove a version of the Schwarz lemma where the images of two points are known. Also, he considers some Schwarz and Carathéodory inequalities at the boundary, as consequences of a lemma due to Rogosinski [10]. In addition, he obtain an new boundary Schwarz lemma, for analytic functions mapping the unit disk to itself [11]. Also, in [16], positive real functions (PRFs) have been considered in terms of electrical engineering as they correspond to driving point impedance functions (DPIFs) of electric circuits. Two inequalities were obtained by considering the derivative of driving point impedance function, $|Z'(0)|$. Accordingly, extremal functions were determined by subjecting these obtained inequalities into sharpness analysis and different circuit structures were synthesized using the obtained driving point impedance functions.

In this paper, we studied "a boundary Carathéodory's inequalities" on the right half plane as analog the Schwarz lemma at the boundary [12]. We present an analytic to understand the behaviour of the derivative of $Z(s)$ function at the zero on the right half plane. In the resulting theorems of the analysis, assuming that $\Re Z(0) = A$, a lower boundary for modulus of the derivative of the $Z(s)$ function at the zero on right half plane, $|Z'(0)|$, are obtained.

Also, we target to find the answer of the question: "What can be said about $Z'(s)$ when it is considered at the boundary?" The answer of the question relies on the boundary analysis of the Schwarz lemma, that is, analysis of $Z(s)$ function at $s = 0$. As a result, in our study, we give a bounded version of Caratheodory inequality on the right half-plane for p-valent. Moreover, by assuming $Z(s)$ is also analytic at the boundary point $s = 0$ on the imaginary axis, we shall give an estimate for $|Z'(0)|$ from below using Taylor expansion coefficients. The sharpness of this inequality is also proved.

2. Main results

In this section, a boundary version of Carathéodory's inequality on the right half plane is investigated. $Z(s)$ is an analytic function defined in the right half of the s-plane. We derive inequalities for the modulus of $Z(s)$ function, $|Z'(0)|$, by assuming the $Z(s)$ function is also analytic at the boundary point $s = 0$ on the imaginary axis and finally, the sharpness of these inequalities is proved. We have following results, which can be offered as the boundary refinement of Carathéodory's inequality on the right half plane. Also, in the following inequalities, p-valent of the value of 1 at point $s = 1$ and the Taylor coefficient that, is different from the first zero, are used.

Theorem 2.1. *Let $Z(s) = 1 + c_p (s-1)^p + c_{p+1} (s-1)^{p+1} + \cdots$ be an analytic function in the right half plane with $\Re Z(s) \leq A$ for $\Re s \geq 0$. Suppose that $Z(s)$ is analytic at the point $s = 0$ of the imaginary axis with $\Re Z(0) = A$. Then*

$$|Z'(0)| \geq (A-1)\, p. \tag{2.1}$$

Moreover, the equality in (2.1) *occurs for the function*

$$Z(s) = \begin{cases} \frac{(s+1)^p + (s-1)^p (1-2A)}{(s+1)^p - (s-1)^p}, & p = 1, 3, 5, \ldots, n, \\ \frac{(s+1)^p - (s-1)^p (1-2A)}{(s+1)^p + (s-1)^p}, & p = 2, 4, \ldots, n. \end{cases}$$

Proof. Let

$$f(z)=\frac{Z\left(\frac{1+z}{1-z}\right)-1}{Z\left(\frac{1+z}{1-z}\right)+1-2A}.$$

$f(z)$ is an analytic function in D, $f(0)=0$ and $|f(z)|<1$ for $z\in D$. Also, for $s=0$ (at the point $s=0$ of the imaginary axis), we take $|f(-1)|=1$ for $b=-1\in\partial D$ and $\Re Z(0)=A$. From the definition of $f(z)$, with the simple calculations, we get

$$f'(z)=\frac{\frac{4}{(1-z)^2}Z'\left(\frac{1+z}{1-z}\right)(1-A)}{\left(Z\left(\frac{1+z}{1-z}\right)+1-2A\right)^2}.$$

Therefore, from (1.3), we obtain

$$p\leq|f'(-1)|=\left|\frac{Z'(0)(1-A)}{(Z(0)+1-2A)^2}\right|.$$

Since

$$\begin{aligned}|Z(0)+1-2A|^2&\geq(\Re(Z(0)+1-2A))^2\\&=(\Re Z(0)+1-2A)^2=(A+1-2A)^2=(1-A)^2,\end{aligned}$$

we take

$$p\leq\frac{|Z'(0)|\,|1-A|}{|1-A|^2}=\frac{|Z'(0)|}{|1-A|}$$

and

$$|Z'(0)|\geq(A-1)\,p.$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$Z\left(\frac{1+z}{1-z}\right)=\frac{1+z^p(1-2A)}{1-z^p}.$$

Then

$$\frac{2}{(1-z)^2}Z'\left(\frac{1+z}{1-z}\right)=\frac{pz^{p-1}(1-2A)(1-z^p)+pz^{p-1}(1+z^p(1-2A))}{(1-z^p)^2},$$

$$\frac{1}{2}Z'(0)=\frac{p(-1)^{p-1}(1-2A)(1-(-1)^p)+p(-1)^{p-1}(1+(-1)^p(1-2A))}{(1-(-1)^p)^2},$$

$$\begin{aligned}Z'(0)&=2\frac{p(-1)^{p-1}(1-2A)(1-(-1)^p)+p(-1)^{p-1}(1+(-1)^p(1-2A))}{(1-(-1)^p)^2}\\&=-2p(-1)^p\,\frac{2(1-A)}{(1-(-1)^p)^2}\end{aligned}$$

and

$$|Z'(0)| = \frac{4p\,(A-1)}{|1-(-1)^p|^2}.$$

Therefore, since $p = 1, 3, 5, \ldots, n$, (2.1) is satisfied with equality. Similarly, let

$$Z\left(\frac{1+z}{1-z}\right) = \frac{1 - z^p\,(1-2A)}{1+z^p}.$$

Then

$$\frac{2}{(1-z)^2} Z'\left(\frac{1+z}{1-z}\right) = \frac{-pz^{p-1}\,(1-2A)\,(1+z^p) - pz^{p-1}\,(1 - z^p\,(1-2A))}{(1+z^p)^2},$$

$$\begin{aligned} Z'(0) &= 2\frac{-p\,(-1)^{p-1}\,(1-2A)\,(1+(-1)^p) - p\,(-1)^{p-1}\,(1-(-1)^p\,(1-2A))}{(1+(-1)^p)^2} \\ &= 2p\,(-1)^p\,\frac{2\,(1-A)}{(1+(-1)^p)^2} \end{aligned}$$

and

$$|Z'(0)| = \frac{4p\,(A-1)}{|1+(-1)^p|^2}.$$

Thus, since $p = 2, 4, \ldots, n$, (2.1) is satisfied with equality. □

The inequality (2.1) can be strengthened as below by taking into account c_p which is the first coefficient in the expansion of the function $Z(s)$.

Theorem 2.2. *Let $Z(s) = 1 + c_p\,(s-1)^p + c_{p+1}\,(s-1)^{p+1} + \cdots$ be an analytic function in the right half plane with $\Re Z(s) \le A$ for $\Re s \ge 0$. Suppose that $Z(s)$ is analytic at the point $s = 0$ of the imaginary axis with $\Re Z(0) = A$. Then*

$$|Z'\,(0)| \ge (A-1)\left(p + \frac{2\,(A-1) - 2^p\,|c_p|}{2\,(A-1) + 2^p\,|c_p|}\right). \tag{2.2}$$

Moreover, the equality in (2.2) occurs for the function

$$Z(s) = \begin{cases} \dfrac{1-a\left(\frac{s-1}{s+1}\right)+(1-2A)\left(\left(\frac{s-1}{s+1}\right)^{p+1}-a\left(\frac{s-1}{s+1}\right)^p\right)}{1-a\left(\frac{s-1}{s+1}\right)-\left(\left(\frac{s-1}{s+1}\right)^{p+1}-a\left(\frac{s-1}{s+1}\right)^p\right)}, & p = 2, 4, 6, \ldots, n, \\ \dfrac{1-a\left(\frac{s-1}{s+1}\right)-(1-2A)\left(\left(\frac{s-1}{s+1}\right)^{p+1}-a\left(\frac{s-1}{s+1}\right)^p\right)}{1-a\left(\frac{s-1}{s+1}\right)+\left(\left(\frac{s-1}{s+1}\right)^{p+1}-a\left(\frac{s-1}{s+1}\right)^p\right)}, & p = 1, 3, 4, \ldots, n, \end{cases}$$

where $a = \frac{2^{p-1}|c_p|}{A-1}$ is an arbitrary number from $[0,1]$ (see (1.1)).

Proof. Let $f(z)$ be the same as in the proof of Theorem 2.1. Therefore, from (1.2), we obtain

$$p + \frac{1-|a_p|}{1+|a_p|} \le |f'(-1)| = \left|\frac{Z'\,(0)\,(1-A)}{(Z\,(0)+1-2A)^2}\right|,$$

$$|a_p| = \frac{\left|f^{(p)}(0)\right|}{p!} = \frac{\left|Z^{(p)}(1)\right|}{p!} = \frac{2^{p-1}\,|c_p|}{A-1}.$$

Therefore, we take

$$p+\frac{1-\frac{2^{p-1}|c_p|}{A-1}}{1+\frac{2^{p-1}|c_p|}{A-1}}\leq\frac{|Z'(0)|}{A-1}$$

and

$$|Z'(0)|\geq(A-1)\left(p+\frac{A-1-2^{p-1}|c_p|}{A-1+2^{p-1}|c_p|}\right).$$

Now, we shall show that the inequality (2.2) is sharp. Let

$$Z(s)=\frac{1-a\left(\frac{s-1}{s+1}\right)+(1-2A)\left(\left(\frac{s-1}{s+1}\right)^{p+1}-a\left(\frac{s-1}{s+1}\right)^{p}\right)}{1-a\left(\frac{s-1}{s+1}\right)-\left(\left(\frac{s-1}{s+1}\right)^{p+1}-a\left(\frac{s-1}{s+1}\right)^{p}\right)}.$$

Then

$$Z'(s)=\frac{4\left(\frac{s-1}{s+1}\right)^p(A-1)\left(p+a^2p-ps^2+a^2s^2+2ap-a^2-s^2+2aps^2-a^2ps^2+1\right)}{(s-1)(s+1)\left(a+s+a\left(\frac{s-1}{s+1}\right)^p-s\left(\frac{s-1}{s+1}\right)^p+\left(\frac{s-1}{s+1}\right)^p-as+as\left(\frac{s-1}{s+1}\right)^p+1\right)^2}$$

and

$$\begin{aligned}|Z'(0)|&=\frac{4(A-1)\left(p+a^2p+2ap-a^2+1\right)}{|a+a(-1)^p+(-1)^p+1|^2}\\&=\frac{4(A-1)(p(1+a)+1-a)}{(1+a)|1+(-1)^p|^2}\\&=\frac{4(A-1)}{|1+(-1)^p|^2}\left(p+\frac{1-a}{1+a}\right).\end{aligned}$$

Since $a=\frac{2^{p-1}|c_p|}{A-1}$ and $p=2,4,\ldots$, (2.2) is satisfied with equality.

Similarly, for $p=1,3,\ldots$, we take the equality (2.2). □

In the following theorem, inequality (2.2) has been strenghened by adding the consecutive terms c_p and c_{p+1} of $Z(s)$.

Theorem 2.3. *Let $Z(s)=1+c_p(s-1)^p+c_{p+1}(s-1)^{p+1}+\cdots$ be an analytic function in the right half plane with $\Re Z(s)\leq A$ for $\Re s\geq 0$. Suppose that $Z(s)$ is analytic at the point $s=0$ of the imaginary axis with $\Re Z(0)=A$. Then*

(2.3)

$$|Z'(0)|\geq(A-1)\left(p+\frac{2\left(2(A-1)-2^p|c_p|\right)^2}{4(A-1)^2-2^{2p}|c_p|^2+4(A-1)2^{p-1}|pc_p+2c_{p+1}|}\right).$$

The inequality (2.3) *is sharp with extremal function*

$$Z(s)=\begin{cases}\frac{(s+1)^{p+1}+(s-1)^{p+1}(1-2A)}{(s+1)^{p+1}-(s-1)^{p+1}}, & p=2,4,6,\ldots,\\ \frac{(s+1)^{p+1}-(s-1)^{p+1}(1-2A)}{(s+1)^{p+1}+(s-1)^{p+1}}, & p=1,3,5,\ldots.\end{cases}$$

Proof. Let $f(z)$ be the same as in the proof of Theorem 2.1. Let us consider the function

$$h(z)=\frac{f(z)}{B(z)},$$

where $B(z)=z^p$. The function $h(z)$ is analytic in D. According to the maximum principle, we have $|h(z)|<1$ for $z\in D$. In particular, we take

$$\begin{aligned}
h(z) &= \frac{Z\left(\frac{1+z}{1-z}\right)-1}{\left(Z\left(\frac{1+z}{1-z}\right)+1-2A\right)z^p}\\
&= \frac{c_p\left(\frac{2}{1-z}\right)^p+c_{p+1}\left(\frac{2}{1-z}\right)^{p+1}z+\cdots}{2\left(1-A\right)+c_p\left(\frac{2z}{1-z}\right)^p+c_{p+1}\left(\frac{2z}{1-z}\right)^{p+1}+\cdots}\\
&= \frac{c_p2^p}{2\left(1-A\right)}+\frac{\left(c_p2^pp+2^{p+1}c_{p+1}\right)}{2\left(1-A\right)}z+\cdots,\\
|h(0)| &= \frac{2^{p-1}\left|c_p\right|}{A-1}\leq 1
\end{aligned}$$

and

$$|h'(0)|=\frac{2^{p-1}}{A-1}\left|pc_p+2c_{p+1}\right|.$$

If $|h(0)|=1$, then by the maximum principle we have $\frac{f(z)}{B(z)}=e^{i\theta}$ and

$$Z\left(\frac{1+z}{1-z}\right)=\frac{1+\left(1-2A\right)z^pe^{i\theta}}{1-z^pe^{i\theta}}.$$

Therefore, we may assume that

$$Z\left(\frac{1+z}{1-z}\right)\not\equiv\frac{1+\left(1-2A\right)z^pe^{i\theta}}{1-z^pe^{i\theta}},$$

and so $|h(0)|<1$. Moreover, since the expression $\frac{bf'(b)}{f(b)}$ is a real number greater than or equal to 1 ([2]) and $\Re Z(0)=A$ yields $|f(b)|=1$ for $-1=b\in\partial D$, we take

$$\frac{bf'(b)}{f(b)}=\left|\frac{bf'(b)}{f(b)}\right|=\left|f'(b)\right|.$$

Also, since $|f(z)|\leq|B(z)|$, we get

$$\frac{1-|f(z)|}{1-|z|}\geq\frac{1-|B(z)|}{1-|z|}.$$

Passing to limit in the last inequality yields

$$\left|f'(b)\right|\geq\left|B'(b)\right|.$$

Thus, we obtain

$$\frac{bf'(b)}{f(b)}=\left|f'(b)\right|\geq\left|B'(b)\right|=\frac{bB'(b)}{B(b)}.$$

The composite function

$$T(z) = \frac{h(z) - h(0)}{1 - \overline{h(0)}h(z)}$$

is analytic in the unit disc D, $|T(z)| < 1$, $T(0) = 0$ and $|T(-1)| = 1$ for $-1 \in \partial D$.

For $p = 1$, from (1.3), we obtain

$$\begin{aligned}
\frac{2}{1 + |T'(0)|} &\leq |T'(-1)| \\
&= \frac{1 - |h(0)|^2}{\left|1 - \overline{h(0)}h(-1)\right|^2} |h'(-1)| \\
&= \frac{1 - |h(0)|^2}{\left|1 - \overline{h(0)}h(-1)\right|^2} \left|\frac{f'(-1)}{B(-1)} - \frac{f(-1)B'(-1)}{B^2(-1)}\right| \\
&= \frac{1 - |h(0)|^2}{\left|1 - \overline{h(0)}h(-1)\right|^2} \left|\frac{(-1)\, f'(-1)}{f(-1)} - \frac{(-1)\, B'(-1)}{B(-1)}\right| \\
&\leq \frac{1 + |h(0)|}{1 - |h(0)|} \left\{|f'(-1)| - |B'(-1)|\right\}
\end{aligned}$$

and

$$\frac{2}{1 + |T'(0)|} \leq \frac{1 + |h(0)|}{1 - |h(0)|} \left\{|f'(-1)| - |B'(-1)|\right\}. \tag{2.4}$$

It can be seen that

$$\begin{aligned}
T'(z) &= \frac{1 - |h(0)|^2}{\left(1 - \overline{h(0)}h(z)\right)^2} h'(z), \\
T'(0) &= \frac{1 - |h(0)|^2}{\left(1 - |h(0)|^2\right)^2} h'(0) \\
&= \frac{h'(0)}{1 - |h(0)|^2}
\end{aligned}$$

and

$$|T'(0)| = \frac{|h'(0)|}{1 - |h(0)|^2} = \frac{\frac{2^{p-1}}{A-1} |pc_p + 2c_{p+1}|}{1 - \left(\frac{2^{p-1}|c_p|}{A-1}\right)^2} = \frac{4\,(A-1)\,2^{p-1}\,|pc_p + 2c_{p+1}|}{4\,(A-1)^2 - 2^{2p}\,|c_p|^2}.$$

Also, we have $|B'(-1)| = p$ for $-1 \in \partial D$. Let's substitute the values of $|T'(0)|$, $|f'(-1)|$, $|B'(-1)|$ and $|h(0)|$ into (2.4). Therefore, we obtain

$$\frac{2}{1+\frac{4(A-1)2^{p-1}|pc_p+2c_{p+1}|}{4(A-1)^2-2^{2p}|c_p|^2}} \leq \frac{1+\frac{2^{p-1}|c_p|}{A-1}}{1-\frac{2^{p-1}|c_p|}{A-1}}\left\{\frac{|Z'(0)|}{A-1}-p\right\}$$
$$= \frac{A-1+2^{p-1}|c_p|}{A-1-2^{p-1}|c_p|}\left\{\frac{|Z'(0)|}{A-1}-p\right\},$$

$$\frac{2\left(4(A-1)^2-2^{2p}|c_p|^2\right)}{4(A-1)^2-2^{2p}|c_p|^2+4(A-1)2^{p-1}|pc_p+2c_{p+1}|}\frac{A-1-2^{p-1}|c_p|}{A-1+2^{p-1}|c_p|} \leq \frac{|Z'(0)|}{A-1}-p$$

and

$$\frac{2\left(2(A-1)-2^{p}|c_p|\right)^2}{4(A-1)^2-2^{2p}|c_p|^2+4(A-1)2^{p-1}|pc_p+2c_{p+1}|}+p \leq \frac{|Z'(0)|}{A-1}.$$

So, we take the inequality (2.3).

Now, we shall show that the inequality (2.3) is sharp. For $p = 2, 4, \ldots$, let

$$Z(s) = \frac{(s+1)^{p+1}+(s-1)^{p+1}(1-2A)}{(s+1)^{p+1}-(s-1)^{p+1}}.$$

Then

$$Z'(s) = -\frac{4(A-1)(p+1)(s-1)^p(s+1)^p}{\left((s-1)^{p+1}-(s+1)^{p+1}\right)^2},$$

$$|Z'(0)| = \frac{4(A-1)(p+1)}{|1+(-1)^p|^2}$$

and

$$|Z'(0)| = (A-1)(p+1).$$

Since $c_p = 0$ and $c_{p+1} = \frac{1-A}{2^p}$, we obtain

$$(A-1)\left(p+\frac{2\left(2(A-1)-2^{p}|c_p|\right)^2}{4(A-1)^2-2^{2p}|c_p|^2+4(A-1)2^{p-1}|pc_p+2c_{p+1}|}\right)$$
$$= (A-1)\left(p+\frac{8(A-1)^2}{4(A-1)^2+4(A-1)2^{p-1}\left|2\frac{1-A}{2^p}\right|}\right)$$
$$= (A-1)\left(p+\frac{8(A-1)^2}{8(A-1)^2}\right)$$
$$= (A-1)(p+1).$$

Similarly, for $p = 1, 3, \ldots$, we take the equality (2.3). □

In the following theorem, we shall give an estimate below $|Z'(0)|$ according to the first nonzero Taylor coefficient of about two points, namely $s=1$ and $s=s_1\neq 0$. Motivated by the results of the work presented in [1], the following result has been obtained.

Theorem 2.4. *Let $Z(s)$ be an analytic function in the right half plane with $\Re Z(s)\leq A$ for $\Re s\geq 0$. Assume that, for positive integers p and m, $Z(s)$ have expansions $Z(s)=1+c_p\,(s-1)^p+c_{p+1}\,(s-1)^{p+1}+\cdots$, $c_p\neq 0$ and $Z(s)=1+a_m\,(s-s_1)^m+a_{m+1}\,(s-s_1)^{m+1}+\cdots$, $a_m\neq 0$, about the points $s=1$ and $s=s_1$, respectively. Suppose that $Z(s)$ is analytic at the point $s=0$ of the imaginary axis with $\Re Z(0)=A$. Then*

$$(1)\quad |Z'(0)|\geq (A-1)\left\{p+m\frac{\Re s_1}{|s_1|^2}+\frac{2\,(A-1)\,|s_1-1|^m-2^p\,|c_p|\,|s_1+1|^m}{2\,(A-1)\,|s_1-1|^m+2^p\,|c_p|\,|s_1+1|^m}\right.$$
$$\left.\times\left[1+\frac{\Phi\,(s_1)-\Psi\,(s_1)}{\Phi\,(s_1)+\Psi\,(s_1)}\frac{\Re s_1}{|s_1|^2}\right]\right\},$$

where

$$\Phi\,(s_1)=4\,(A-1)^2\left|\frac{s_1-1}{s_1+1}\right|^{m+p}\left(\frac{4}{|s_1+1|^2}\right)^m+2^{m+p}\,|c_p|\,|a_m|\left(\frac{4\Re s_1}{|s_1+1|^2}\right)^m,$$

$$\Psi\,(s_1)=2^{m+1}\,(A-1)\left|\frac{s_1-1}{s_1+1}\right|^{m-1}|a_m|\left(\frac{4\Re s_1}{|s_1+1|^2}\right)^m$$
$$-2^{p+1}\,(A-1)\left|\frac{s_1-1}{s_1+1}\right|^{p-1}\left(\frac{4}{|s_1+1|^2}\right)^m.$$

The inequality (2.5) is sharp, with equality for each possible values $|c_p|$ and $|a_m|$.

Proof. Consider the function

$$q(z)=\frac{z-a}{1-\overline{a}z}.$$

In addition, let $g:D\to D$ be an analytic function and a point $a\in D$. From the Schwarz-pick lemma, we take

$$\left|\frac{g(z)-g(z_1)}{1-\overline{g(z_1)}g(z)}\right|\leq\left|\frac{z-a}{1-\overline{a}z}\right|=|q(z)|$$

and

$$(2.6)\qquad |g(z)|\leq\frac{|g(z_1)|+|q(z)|}{1+|g(z_1)|\,|q(z)|}.$$

If $w : D \to D$ is an analytic function and $0 < |a| < 1$, letting

$$g(z) = \frac{w(z) - w(0)}{z\left(1 - \overline{w(0)}w(z)\right)}$$

in (2.6), we obtain

$$\left| \frac{w(z) - w(0)}{z\left(1 - \overline{w(0)}w(z)\right)} \right| \leq \frac{\left| \frac{w(a)-w(0)}{a\left(1-\overline{w(0)}w(a)\right)} \right| + |q(z)|}{1 + \left| \frac{w(a)-w(0)}{a\left(1-\overline{w(0)}w(a)\right)} \right| |q(z)|}$$

and

$$|w(z)| \leq \frac{|w(0)| + |z| \frac{|\mathtt{M}|+|q(z)|}{1+|\mathtt{M}||q(z)|}}{1 + |w(0)|\,|z| \frac{|\mathtt{M}|+|q(z)|}{1+|\mathtt{M}||q(z)|}}, \tag{2.7}$$

where

$$\mathtt{M} = \frac{w(a) - w(0)}{a\left(1 - \overline{w(0)}w(a)\right)}.$$

If we take

$$w(z) = \frac{f(z)}{z^p\left(\frac{z-a}{1-\overline{a}z}\right)^m},$$

we have

$$w(0) = (-1)^m \frac{2^{p-1}c_p}{(1-A)\,a^m}, \; w(a) = \frac{2^{m-1}a_m\left(1 - |a|^2\right)^m}{(1-A)\,(1-a)^{2m}\,a^p}.$$

Let $\alpha = \frac{2^{p-1}|c_p|}{(A-1)|a|^m}$ and

$$\mathtt{M}_1 = \frac{\left| \frac{2^{m-1}a_m\left(1-|a|^2\right)^m}{(1-A)(1-a)^{2m}a^p} \right| + \left| \frac{2^{p-1}c_p}{(1-A)a^m} \right|}{|a|\left(1 + \left| \frac{2^{p-1}c_p}{(1-A)a^m} \right| \left| \frac{2^{m-1}a_m\left(1-|a|^2\right)^m}{(1-A)(1-a)^{2m}a^p} \right|\right)}.$$

From (2.7), we obtain

$$|f(z)| \leq |z|^p\,|q(z)|^m \frac{\alpha + |z| \frac{\mathtt{M}_1+|q(z)|}{1+\mathtt{M}_1|q(z)|}}{1 + \alpha\,|z| \frac{\mathtt{M}_1+|q(z)|}{1+\mathtt{M}_1|q(z)|}}$$

and

$$\frac{1 - |f(z)|}{1 - |z|} \geq \frac{1 + \alpha\,|z| \frac{\mathtt{M}_1+|q(z)|}{1+\mathtt{M}_1|q(z)|} - \alpha\,|z|^p\,|q(z)|^m - |z|^{p+1}\,|q(z)|^m \frac{\mathtt{M}_1+|q(z)|}{1+\mathtt{M}_1|q(z)|}}{(1 - |z|)\left(1 + \alpha\,|z| \frac{\mathtt{M}_1+|q(z)|}{1+\mathtt{M}_1|q(z)|}\right)}.$$

Let

$$\xi(z) = 1 + \alpha\,|z| \frac{\mathtt{M}_1 + |q(z)|}{1 + \mathtt{M}_1\,|q(z)|}$$

and

$$\sigma(z) = 1 + \mathtt{M}_1 \left| q(z) \right|.$$

Thus, we have

$$\frac{1 - |f(z)|}{1 - |z|} \geq \frac{1}{\xi(z)\sigma(z)} \left\{ \frac{1 - |z|^{p+1} \left| q(z) \right|^{m+1}}{1 - |z|} + \mathtt{M}_1 \left| q(z) \right| \frac{1 - |z|^{p+1} \left| q(z) \right|^{m-1}}{1 - |z|} \right.$$
$$\left. + \alpha \left| z \right| \left| q(z) \right| \frac{1 - |z|^{p-1} \left| q(z) \right|^{m-1}}{1 - |z|} + \alpha \left| z \right| \mathtt{M}_1 \frac{1 - |z|^{p-1} \left| q(z) \right|^{m-1}}{1 - |z|} \right\}.$$

Since

$$\lim_{z \to -1} \xi(z) = \lim_{z \to -1} 1 + \alpha \left| z \right| \frac{\mathtt{M}_1 + \left| q(z) \right|}{1 + \mathtt{M}_1 \left| q(z) \right|} = 1 + \alpha,$$
$$\lim_{z \to -1} \sigma(z) = \lim_{z \to -1} 1 + \mathtt{M}_1 \left| q(z) \right| = 1 + \mathtt{M}_1$$

and

$$\lim_{z \to -1} \frac{1 - |z|^i \left| q(z) \right|^j}{1 - |z|} = i + j \frac{1 - |a|^2}{|1 + a|^2},$$

for nonnegative integers i and j, passing to the angular limit in the last inequality yields

$$|f'(-1)| \geq p + m \frac{1 - |a|^2}{|1 + a|^2} + \frac{1 - \alpha}{1 + \alpha} \left[1 + \frac{1 - \mathtt{M}_1}{1 + \mathtt{M}_1} \frac{1 - |a|^2}{|1 + a|^2} \right].$$

Also, since

$$|f'(-1)| \leq \frac{|Z'(0)|}{A - 1},$$

$$\frac{1 - |a|^2}{|1 + a|^2} = \frac{1 - \left| \frac{s_1 - 1}{s_1 + 1} \right|^2}{\left| 1 + \frac{s_1 - 1}{s_1 + 1} \right|^2} = \frac{\frac{4 \Re s_1}{|s_1 + 1|^2}}{\frac{4 s_1^2}{|s_1 + 1|^2}} = \frac{\Re s_1}{s_1^2},$$

$$\frac{1 - \alpha}{1 + \alpha} = \frac{1 - \frac{2^{p-1} |c_p|}{(A-1)|a|^m}}{1 + \frac{2^{p-1} |c_p|}{(A-1)|a|^m}} = \frac{2 (A - 1) |a|^m - 2^p |c_p|}{2 (A - 1) |a|^m + 2^p |c_p|},$$

$$\frac{1 - \mathtt{M}_1}{1 + \mathtt{M}_1} = \frac{1 - \frac{\left| \frac{2^{m-1} a_m \left(1 - |a|^2\right)^m}{(1-A)(1-a)^{2m} a^p} \right| + \left| \frac{2^{p-1} c_p}{(1-A) a^m} \right|}{|a| \left(1 + \left| \frac{2^{p-1} c_p}{(1-A) a^m} \right| \left| \frac{2^{m-1} a_m \left(1 - |a|^2\right)^m}{(1-A)(1-a)^{2m} a^p} \right| \right)}}{1 + \frac{\left| \frac{2^{m-1} a_m \left(1 - |a|^2\right)^m}{(1-A)(1-a)^{2m} a^p} \right| + \left| \frac{2^{p-1} c_p}{(1-A) a^m} \right|}{|a| \left(1 + \left| \frac{2^{p-1} c_p}{(1-A) a^m} \right| \left| \frac{2^{m-1} a_m \left(1 - |a|^2\right)^m}{(1-A)(1-a)^{2m} a^p} \right| \right)}}$$

and

$$\frac{1 - \mathtt{M}_1}{1 + \mathtt{M}_1} = \frac{\Phi(s_1) - \Psi(s_1)}{\Phi(s_1) + \Psi(s_1)},$$

where

$$\Phi(s_1) = 4(A-1)^2 \left|\frac{s_1-1}{s_1+1}\right|^{m+p} \left(\frac{4}{|s_1+1|^2}\right)^m + 2^{m+p}|c_p||a_m|\left(\frac{4\Re s_1}{|s_1+1|^2}\right)^m,$$

$$\Psi(s_1) = 2^{m+1}(A-1)\left|\frac{s_1-1}{s_1+1}\right|^{m-1}|a_m|\left(\frac{4\Re s_1}{|s_1+1|^2}\right)^m$$
$$-2^{p+1}(A-1)\left|\frac{s_1-1}{s_1+1}\right|^{p-1}\left(\frac{4}{|s_1+1|^2}\right)^m,$$

we obtain

$$\frac{|Z'(0)|}{A-1} \geq p + m\frac{\Re s_1}{|s_1|^2} + \frac{2(A-1)\left|\frac{s_1-1}{s_1+1}\right|^m - 2^p|c_p|}{2(A-1)\left|\frac{s_1-1}{s_1+1}\right|^m + 2^p|c_p|}$$
$$\times\left[1+\frac{\Phi(s_1)-\Psi(s_1)}{\Phi(s_1)+\Psi(s_1)}\frac{\Re s_1}{|s_1|^2}\right].$$

Thus, we obtain the inequality (2.5).

In order to show that the inequality is sharp, choose arbitrary real numbers a, x and y such that $0 < x < |a|^m$, $0 < y < \frac{|a|^p}{(1-|a|^2)^m}$.

Let

$$\mathtt{K} = \frac{\frac{y}{a^p}\left(1-|a|^2\right)^m + (-1)^{m-1}\frac{x}{a^m}}{a\left(1+(-1)^{m-1}\frac{y}{a^p}\left(1-|a|^2\right)^m\frac{x}{a^m}\right)}$$

and

$$f(z) = z^p\left(\frac{z-a}{1-\overline{a}z}\right)^m \frac{(-1)^m\frac{x}{a^m} + z\frac{\mathtt{K}+\frac{z-a}{1-\overline{a}z}}{1+\mathtt{K}\frac{z-a}{1-\overline{a}z}}}{1+(-1)^m\frac{x}{a^m}z\frac{\mathtt{K}+\frac{z-a}{1-\overline{a}z}}{1+\mathtt{K}\frac{z-a}{1-\overline{a}z}}}. \tag{2.8}$$

From (2.8), with the simple calculations, we obtain $\left|\frac{f^{(p)}(0)}{p!}\right| = x$ and $\left|\frac{f^{(m)}(a)}{m!}\right| = y$ and

$$f'(-1) = -\left[(-1)^{p+m}\left(p+m\frac{1-|a|^2}{(1+a)^2}\right) + \frac{a^m-(-1)^m x}{a^m+(-1)^m x}\times\left(1+\frac{1+\mathtt{K}}{1-\mathtt{K}}\frac{1-|a|^2}{(1+a)^2}\right)\right]$$

and

$$|f'(-1)| = p + m\frac{1-|a|^2}{(1+a)^2} + \frac{a^m-(-1)^m x}{a^m+(-1)^m x}\times\left(1+\frac{1+\mathtt{K}}{1-\mathtt{K}}\frac{1-|a|^2}{(1+a)^2}\right). \tag{2.9}$$

Choosing suitable signs of the numbers $a = \frac{s_1-1}{s_1+1}$, x and y, we conclude from (2.9) that the inequality (2.5) is sharp. □

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