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SOME RESULTS ON CONVERGENCES IN FUZZY METRIC SPACES AND FUZZY NORMED SPACES

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Abstract. In this paper, we introduce the definitions of s_p-convergent sequence in fuzzy metric spaces and fuzzy normed spaces. We investigate relations of convergence, s_p-convergence, s_∞-convergence and st-convergence in fuzzy metric spaces and fuzzy normed spaces. We also study s_p-convergence, s_∞-convergence and st-convergence using the subsequence of convergent sequence in fuzzy metric spaces and fuzzy normed spaces. Stationary fuzzy normed spaces are defined and investigated. We finally define s_p-closed sets, s_∞-closed sets and st-closed sets in fuzzy metric spaces and fuzzy normed spaces and investigate relations of them.

1. Introduction

Various definitions of fuzzy metric space have been investigated by several authors (see [4], [6], [7] and [11]). In this paper, we take the definition of fuzzy metric space introduced by A. George and P. Veeramani [6].

Definition 1. A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (nonempty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$,

(M1) $M(x, y, t) > 0$,
(M2) $M(x, y, t) = 1$ if and only if $x = y$,
(M3) $M(x, y, t) = M(y, x, t)$,
(M4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(M5) $M(x, y, \_) : (0, \infty) \to (0, 1]$ is continuous.

Now we want to give a definition of fuzzy norm in order to investigate some properties of various type convergences studied in fuzzy metric space. Indeed, several authors introduced definitions of fuzzy normed space from different point of view (for example, [1], [5], [12]).

We now consider the definition of fuzzy norm introduced by C. Felbin [5].

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A fuzzy number is a mapping $x : \mathbb{R} \to [0,1]$ over the set $\mathbb{R}$ of all real numbers. A fuzzy number x is convex if $x(t) \geq \min\{x(s), x(r)\}$ where $s \leq t \leq r$. A fuzzy number x is normal if there exists a $t_0 \in \mathbb{R}$ such that $x(t_0) = 1$. For $\alpha \in (0,1]$, α-level set of a fuzzy number x is $[x]_\alpha = \{t \in \mathbb{R} : x(t) \geq \alpha\}$ and α-level set of an upper semi-continuous, convex and normal fuzzy number is a closed interval $[a_\alpha, b_\alpha]$, where $a_\alpha = -\infty$ and $b_\alpha = \infty$ are admissible. When $a_\alpha = -\infty$, $[a_\alpha, b_\alpha]$ means the interval $(-\infty, b_\alpha]$. When $b_\alpha = \infty$, $[a_\alpha, b_\alpha]$ means the interval $[a_\alpha, \infty)$. A fuzzy number x is said to be non-negative if $x(t) = 0$, for all $t < 0$. C. Felbin [5] denoted the set of all convex, normal and upper semi-continuous fuzzy number by $R(I)$ and the set of all non-negative, convex, normal and upper semi-continuous fuzzy number by $R^*(I)$. Since each $r \in \mathbb{R}$ can be considered a fuzzy real number $\tilde{r}$ defined by

$$\tilde{r}(t) = \begin{cases} 1 & \text{if } t = r, \\ 0 & \text{if } t \neq r, \end{cases}$$

$\mathbb{R}$ can be embedded in $R(I)$. A partial ordering $\preceq$ in $R(I)$ is defined by $x \preceq y$ if and only if $a^1_\alpha \leq a^2_\alpha$ and $b^1_\alpha \leq b^2_\alpha$ for all $\alpha \in (0,1]$, where $[x]_\alpha = \left[a^1_\alpha, b^1_\alpha\right]$ and $[y]_\alpha = \left[a^2_\alpha, b^2_\alpha\right]$. The strict inequality in $R(I)$ is defined by $x \prec y$ if and only if $a^1_\alpha < a^2_\alpha$ and $b^1_\alpha < b^2_\alpha$ for all $\alpha \in (0,1]$

Arithmetic operations $\oplus$, $\ominus$, $\odot$ and $\oslash$ on $R(I) \times R(I)$ are defined as in [11]:

$$(x \oplus y)(t) = \sup_{s \in \mathbb{R}} \min\{x(s), y(t-s)\}, \quad t \in \mathbb{R},$$

$$(x \ominus y)(t) = \sup_{s \in \mathbb{R}} \min\{x(s), y(s-t)\}, \quad t \in \mathbb{R},$$

$$(x \odot y)(t) = \sup_{s \in \mathbb{R}, s \neq 0} \min\{x(s), y(t/s)\}, \quad t \in \mathbb{R},$$

$$(x \oslash y)(t) = \sup_{s \in \mathbb{R}} \min\{x(st), y(s)\}, \quad t \in \mathbb{R}.$$

The following definition is found in [5].

Definition 2. Let X be a linear space over a field $\mathbb{R}$. Let $\|\cdot\| : X \to \mathbb{R}$ and let the mappings L, $R : [0,1] \times [0,1] \to [0,1]$ be symmetric, non-decreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$. Write $[\|x\|]_\alpha = \left[\|x\|^1_\alpha, \|x\|^2_\alpha\right]$ for $x \in X$, $\alpha \in (0,1]$ and suppose for all $x \in X$, $x \neq 0$, there exists $\alpha_0 \in (0,1]$ independent of x such that for all $\alpha \leq \alpha_0$,

(A) $\|x\|^2_\alpha < \infty$,

(B) $\inf \|x\|^1_\alpha > 0$.

The quadruple $(X, \|\cdot\|, L, R)$ is called a fuzzy normed liner space and $\|\cdot\|$ a fuzzy norm, if

(i) $\|x\| = \tilde{0}$ if and only if $x = 0$,

(ii) $\|rx\| = |r|\|x\|$, for all $x \in X$ and $r \in \mathbb{R}$

(iii) for all $x, y \in X$,

(a) whenever $s \leq \|x\|^1_1$, $t \leq \|x\|^1_1$ and $s+t \leq \|x+y\|^1_1$, $\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t))$,

(b) whenever $s \geq \|x\|_1^1$, $t \geq \|x\|_1^1$ and $s+t \geq \|x+y\|_1^1$, $\|x+y\|(s+t) \leq R(\|x\|(s), \|y\|(t))$.

C. Felbin [5] showed that if we take L as Min and R as Max, then (iii) of Definition 2 is equivalent to $\|x+y\| \preceq \|x\| \oplus \|y\|$ and $\|\cdot\|_\alpha^i$, $i = 1, 2$, are norms on X in the usual sense. In the sequel, we take L =Min and R =Max.

Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in $(X, M, *)$ is said to be convergent to $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$. A sequence $\{x_n\}$ in $(X, M, *)$ is said to be s-convergent to $x \in X$ if $\lim_{n\to\infty} M\left(x_n, x, \frac{1}{n}\right) = 1$ [8]. In [8], the authors showed that s-convergence implies convergence and the converse does not hold. A sequence $\{x_n\}$ in $(X, M, *)$ is said to be strong convergent (briefly st-convergent) to $x \in X$ if for all $\epsilon \in (0, 1)$, there exists $n_\epsilon \in \mathbb{N}$, depending only on ϵ, such that if $n \geq n_\epsilon$, $M(x_n, x, t) > 1 - \epsilon$ for all $t > 0$ [9]. In [9], the authors showed that st-convergence implies s-convergence and the converse is false, in general.

Let $\{x_n\}$ be a sequence in a fuzzy normed space $(X, \|\cdot\|)$ introduced by C. Felbin and $x \in X$. $\{x_n\}$ is said to be convergent to x if, for all $\epsilon > 0$ and $\alpha \in (0, 1]$, there exists $n_{\epsilon,\alpha} \in \mathbb{N}$, depending on ϵ and α, such that $\|x_n - x\|_\alpha^2 < \epsilon$, for $n \geq n_{\epsilon,\alpha}$ [5]. $\{x_n\}$ is said to be s-convergent to x if, for all $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $\|x_n - x\|_{\frac{1}{n}}^2 < \epsilon$ for $n \geq n_\epsilon$ [3]. $\{x_n\}$ is said to be st-convergent to x if, for all $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $\|x_n - x\|_\alpha^2 < \epsilon$ for $n \geq n_\epsilon$ and for all $\alpha \in (0, 1]$ [3]. In [3], the authors proved the following strict implications in fuzzy normed spaces:

$$st\text{-convergence} \Rightarrow s\text{-convergence} \Rightarrow \text{convergence}.$$

In Section 2, we introduce the definitions of s_p-convergent sequence in fuzzy metric spaces and fuzzy normed spaces. We investigate relations of convergence, s_p-convergence, s_∞-convergence and st-convergence in fuzzy metric spaces and fuzzy normed spaces. In Section 3, we study s_p-convergence, s_∞-convergence and st-convergence using the subsequence of convergent sequence in fuzzy metric spaces and fuzzy normed spaces. Stationary fuzzy normed spaces are defined and investigated in Section 4. We finally define s_p-closed sets, s_∞-closed sets and st-closed sets in fuzzy metric spaces and fuzzy normed spaces and investigate relations of them.

2. *st*-convergent sequence and s_p-convergent sequence

The following proposition is found in [9].

Proposition 2.1. *Let $(X, M, *)$ be a fuzzy metric space. Then a sequence (x_n) in $(X, M, *)$ is st-convergent to x_0 if and only if* $\lim_{n,m\to\infty} M(x_n, x_0, \frac{1}{m}) = 1$.

We can get the similar result on fuzzy normed spaces.

Proposition 2.2. *Let $(X, \|\cdot\|)$ be a fuzzy normed space introduced by C. Felbin. Then a sequence (x_n) in $(X, \|\cdot\|)$ is st-convergent to x_0 if and only if* $\lim_{n,m\to\infty} \|x_n - x_0\|_{\frac{1}{m}}^2 = 0$.

Proof. Suppose that (x_n) is an *st*-convergent sequence to x_0 in $(X, \|\cdot\|)$. Let $\epsilon > 0$. Then there exists $n_\epsilon \in \mathbb{N}$ such that if $n \geq n_\epsilon$,

$$\|x_n - x_0\|_\alpha^2 < \epsilon \qquad \text{for all } \alpha \in (0,1].$$

In particular, $\|x_n - x_0\|_{\frac{1}{m}}^2 < \epsilon$ for $n \geq n_\epsilon$ and for all $m \in \mathbb{N}$, i.e., $\lim_{n,m\to\infty} \|x_n - x_0\|_{\frac{1}{m}}^2 = 0$.

Conversely, suppose that $\lim_{n,m\to\infty} \|x_n - x_0\|_{\frac{1}{m}}^2 = 0$. Let $\epsilon > 0$. Then there exists $n_\epsilon \in \mathbb{N}$ such that if $n, m \geq n_\epsilon$, $\|x_n - x_0\|_{\frac{1}{m}}^2 < \epsilon$. Let $\alpha \in (0,1]$. Then there exists $m_\alpha \in \mathbb{N}$ such that $m_\alpha \geq n_\epsilon$ and $\frac{1}{m_\alpha} < \alpha$. For $n \geq n_\epsilon$, $\|x_n - x_0\|_\alpha^2 \leq \|x_n - x_0\|_{\frac{1}{m_\alpha}}^2 < \epsilon$. This means that (x_n) is *st*-convergent to x_0. □

We now introduce the following definitions.

Definition 3. Let $(X, M, *)$ be a fuzzy metric space and $(X, \|\cdot\|)$ a fuzzy normed space introduced by C. Felbin.

(1) A sequence (x_n) in a fuzzy metric space $(X, M, *)$ is said to be s_p-convergent to x_0, for $p \in \mathbb{N}$ if $\lim_{n\to\infty} M\left(x_n, x_0, \frac{1}{n^p}\right) = 1$. A sequence (x_n) in a fuzzy metric space $(X, M, *)$ is said to be s_∞-convergent to x_0 if (x_n) is s_p-convergent to x_0, for all $p \in \mathbb{N}$.

(2) A sequence (x_n) in a fuzzy normed space $(X, \|\cdot\|)$ is said to be s_p-convergent to x_0, for $p \in \mathbb{N}$ if $\lim_{n\to\infty} \|x_n - x_0\|_{\frac{1}{n^p}}^2 = 0$. A sequence (x_n) in $(X, \|\cdot\|)$ is said to be s_∞-convergent to x_0 if (x_n) is s_p-convergent to x_0, for all $p \in \mathbb{N}$.

We note that s_1-convergence coincides to s-convergence in fuzzy metric spaces(or fuzzy normed spaces).

Proposition 2.3. *Let $(X, M, *)$ be a fuzzy metric space and $(X, \|\cdot\|)$ a fuzzy normed space introduced by C. Felbin. Then*

(1) *s_{k+1}-convergence implies s_k-convergence in fuzzy metric spaces (or fuzzy normed spaces), for $k \in \mathbb{N}$,*

(2) *st-convergence implies s_∞-convergence in fuzzy metric spaces (or fuzzy normed spaces).*

Proof. (1) Let (x_n) be a sequence which is s_{k+1}-convergent to $x_0 \in X$ in fuzzy metric space $(X, M, *)$. Then $\lim_{n\to\infty} M\left(x_n, x_0, \frac{1}{n^{k+1}}\right) = 1$. Since

$$\begin{aligned} M\left(x_n, x_0, \frac{1}{n^k}\right) &\geq M\left(x_n, x_0, \frac{1}{n^{k+1}}\right) * M\left(x_0, x_0, \frac{1}{n^k} - \frac{1}{n^{k+1}}\right) \\ &= M\left(x_n, x_0, \frac{1}{n^{k+1}}\right) * 1 \geq M\left(x_n, x_0, \frac{1}{n^{k+1}}\right), \end{aligned}$$

$$\lim_{n\to\infty} M\left(x_n, x_0, \frac{1}{n^k}\right) \geq \lim_{n\to\infty} M\left(x_n, x_0, \frac{1}{n^{k+1}}\right) = 1.$$

This implies that (x_n) is an s_k-convergent sequence in $(X, M, *)$.

Let (x_n) be a sequence which is s_{k+1}-convergent to $x_0 \in X$ in fuzzy normed space $(X, \|\cdot\|)$. Then we can get the result by the following inequality:

$$\|x_n - x_0\|^2_{\frac{1}{n^{k+1}}} \geq \|x_n - x_0\|^2_{\frac{1}{n^k}} .$$

(2) Let (x_n) be a sequence which is st-convergent to $x_0 \in X$ in fuzzy metric space $(X, M, *)$. Then

$$\lim_{n,m\to\infty} M(x_n, x_0, \frac{1}{m}) = 1,$$

where $n, m \in \mathbb{N}$ by Proposition 2.1. Let $p \in \mathbb{N}$. Then for $m = n^p$,

$$\lim_{n\to\infty} M(x_n, x_0, \frac{1}{n^p}) = 1.$$

This implies that (x_n) is an s_∞-convergent sequence in $(X, M, *)$.

Let (x_n) be a sequence which is st-convergent to $x_0 \in X$ in fuzzy normed space $(X, \|\cdot\|)$. Using Proposition 2.2, (x_n) is an s_∞-convergent sequence in $(X, \|\cdot\|)$. $\square$

The converses of Proposition 2.3 are false, in general.

Example 2.4. Let M be a function on $\mathbb{R} \times \mathbb{R} \times (0, \infty)$ defined by

$$M(x, y, t) = \frac{t}{t + |x - y|}.$$

Then $(\mathbb{R}, M, \cdot)$ is a fuzzy metric space [9]. Let $x_n = \frac{1}{n^{k+1}}$ for $k \in \mathbb{N}$. Then

$$M(x_n, 0, \frac{1}{n^k}) = \frac{\frac{1}{n^k}}{\frac{1}{n^k} + \frac{1}{n^{k+1}}} = \frac{1}{1 + \frac{1}{n}}$$

and

$$M(x_n, 0, \frac{1}{n^{k+1}}) = \frac{\frac{1}{n^{k+1}}}{\frac{1}{n^{k+1}} + \frac{1}{n^{k+1}}} = \frac{1}{2}.$$

This means that (x_n) is s_k-convergent to 0 but not an s_{k+1}-convergent sequence to 0 in $(\mathbb{R}, M, \cdot)$.

Let $y_n = \frac{1}{n^n}$ and $k \in \mathbb{N}$. Then

$$M\left(y_n, 0, \frac{1}{n^k}\right) = \frac{\frac{1}{n^k}}{\frac{1}{n^k} + \frac{1}{n^n}} \to 1$$

as $n \to \infty$. This means that (y_n) is an s_∞-convergent sequence in $(\mathbb{R}, M, \cdot)$. We now show that (y_n) is not an st-convergent sequence. Suppose that (y_n) is an st-convergent sequence. Then

$$\lim_{n,m\to\infty} M(y_n, 0, \frac{1}{m}) = 1$$

and in particular, for $m = n^n$,

$$\lim_{n\to\infty} M(y_n, 0, \frac{1}{n^n}) = 1.$$

But $M(y_n, 0, \frac{1}{n^n}) = \frac{\frac{1}{n^n}}{\frac{1}{n^n}+\frac{1}{n^n}} = \frac{1}{2}$. We get the contradiction. This completes the proof.

The following example is found in [2].

Example 2.5. Let $X = \mathbb{R}$ and we define a fuzzy norm $\|\cdot\|$ on X by

$$\|x\| = \begin{cases} \frac{|x|}{t} & \text{if } |x| \le t,\ x \ne 0, \\ 1 & \text{if } t = |x| = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(X, \|\cdot\|)$ is a fuzzy normed space introduced by C. Felbin [2] and α-level set of $(X, \|\cdot\|)$ is given by

$$[\|x\|]_\alpha = \left[|x|, \frac{|x|}{\alpha}\right].$$

Let $x_n = \frac{1}{n^{k+1}}$. Then

$$\|x_n\|^2_{\frac{1}{n^k}} = \frac{\frac{1}{n^{k+1}}}{\frac{1}{n^k}} = \frac{1}{n}$$

and

$$\|x_n\|^2_{\frac{1}{n^{k+1}}} = \frac{\frac{1}{n^{k+1}}}{\frac{1}{n^{k+1}}} = 1.$$

These imply that (x_n) is an s_k-convergent and non-s_{k+1} convergent sequence in $(X, \|\cdot\|)$.

Let $y_n = \frac{1}{n^n}$ and $k \in \mathbb{N}$. Then

$$\|y_n\|^2_{\frac{1}{n^k}} = \frac{\frac{1}{n^n}}{\frac{1}{n^k}} \to 0$$

as $n \to \infty$. This means that (y_n) is an s_∞-convergent sequence in $(X, \|\cdot\|)$. We now show that (y_n) is not an st-convergent sequence. Suppose that (y_n) is an st-convergent sequence. Then

$$\lim_{n,m\to\infty} \|y_n\|^2_{\frac{1}{m}} = 0$$

and in particular, for $m = n^n$,

$$\lim_{n\to\infty} \|y_n\|^2_{\frac{1}{n^n}} = 0$$

But $\|y_n\|^2_{\frac{1}{n^n}} = 1$. We get the contradiction. This completes the proof.

By Proposition 2.3, Example 2.4 and Example 2.5, we get the following strict implications in fuzzy metric spaces and fuzzy normed spaces:

$$st\text{-conv.} \Rightarrow s_\infty\text{-conv.} \Rightarrow \cdots \Rightarrow s_{k+1}\text{-conv.} \Rightarrow s_k\text{-conv.} \cdots \Rightarrow s\text{-conv.}$$

3. A subsequence of a convergent sequence in fuzzy metric spaces and fuzzy normed spaces

The following proposition is found in [8].

Proposition 3.1. *Let $(X, M, *)$ be a fuzzy metric space.*

(1) *Each subsequence of an s-convergent sequence in X is s-convergent,*
(2) *Each convergent sequence in X admits an s-convergent subsequence.*

We can get the similar results for s_p-convergence and s_∞-convergence in fuzzy metric spaces and fuzzy normed spaces.

Theorem 3.2. *Let $(X, M, *)$ be a fuzzy metric space.*

(1) *Each subsequence of an s_p-convergent sequence in X is s_p-convergent for all $p \in \mathbb{N}$ and each subsequence of an s_∞-convergent sequence in X is s_∞-convergent,*
(2) *Each convergent sequence in X admits an s_p-convergent subsequence for all $p \in \mathbb{N}$. Moreover, each convergent sequence in X admits an s_∞-convergent subsequence.*

Proof. (1) Let (x_n) be an s_p-convergent sequence to x_0 in X and (y_k) a subsequence of (x_n), where $y_k = x_{n_k}$ and $n_k \geq k$. Since (x_n) is an s_p-convergent sequence to x_0 in X, for all $\epsilon \in (0,1)$, there exists $n_\epsilon \in \mathbb{N}$ such that $M\left(x_n, x_0, \frac{1}{n^p}\right) > 1 - \epsilon$, for $n \geq n_\epsilon$. We can find $k_0 \in \mathbb{N}$ such that $n_{k_0} \geq n_\epsilon$. If $k \geq k_0$,

$$M\left(y_k, x_0, \frac{1}{k^p}\right) = M\left(x_{n_k}, x_0, \frac{1}{k^p}\right) \geq M\left(x_{n_k}, x_0, \frac{1}{(n_k)^p}\right) > 1 - \epsilon,$$

since $n_k \geq n_{k_0} \geq n_\epsilon$ and $n_k \geq k$. This implies that each subsequence of an s_p-convergent sequence in X is s_p-convergent for all $p \in \mathbb{N}$. It is same with this proof to show each subsequence of an s_∞-convergent sequence in X is s_∞-convergent.

(2) Since s_∞-convergence implies s_p-convergence, it suffices to show that each convergent sequence in X admits an s_∞-convergent subsequence. Suppose that (x_n) is a convergent sequence to x_0 in X. Then for all $r \in (0,1)$ and $t > 0$, there exists $n_{r,t} \in \mathbb{N}$ such that $M(x_n, x_0, t) > 1 - r$ for $n \geq n_{r,t}$. Let $x_{n_1} = x_1$. Since $x_n \to x_0$, there exists $n_2 (\geq 2) \in \mathbb{N}$ such that $M\left(x_{n_2}, x_0, \frac{1}{2^2}\right) > 1 - \frac{1}{2^2}$. Since $x_n \to x_0$, there exists $n_3\,(\geq \max\{3, n_2 + 1\}) \in \mathbb{N}$ such that $M\left(x_{n_3}, x_0, \frac{1}{3^3}\right) > 1 - \frac{1}{3^3}$.

$$\vdots$$

Continuing this process, we get a subsequence (x_{n_k}) of (x_n) such that

$$M\left(x_{n_k}, x_0, \frac{1}{k^k}\right) > 1 - \frac{1}{k^k} \qquad \text{for all } k \in \mathbb{N}.$$

We now show that (x_{n_k}) is s_∞-convergent to x_0. Let $\epsilon \in (0,1)$ and $p \in \mathbb{N}$. Then there exists $k_0 \in \mathbb{N}$ such that $\frac{1}{{k_0}^p} \leq \epsilon$. If $k \geq \max\{k_0, p\}$, $M\left(x_{n_k}, x_0, \frac{1}{k^p}\right) \geq$

$M\left(x_{n_k}, x_0, \frac{1}{k^k}\right) > 1 - \frac{1}{k^k} \geq 1 - \frac{1}{k_0{}^p} \geq 1 - \epsilon$. This means that (x_{n_k}) is s_p-convergent. Since $p \in \mathbb{N}$ is arbitrary, (x_{n_k}) is s_∞-convergent. □

Theorem 3.3. *Let $(X, \|\cdot\|)$ be a fuzzy normed space introduced by C. Felbin.*

(1) *Each subsequence of an s_p-convergent sequence in X is s_p-convergent for all $p \in \mathbb{N}$ and each subsequence of an s_∞-convergent sequence in X is s_∞-convergent,*
(2) *Each convergent sequence in X admits an s_p-convergent subsequence for all $p \in \mathbb{N}$. Moreover, each convergent sequence in X admits an s_∞-convergent subsequence.*

Proof. (1) Let (x_n) be an s_p-convergent sequence to x_0 in $(X, \|\cdot\|)$ and (y_k) a subsequence of (x_n), where $y_k = x_{n_k}$ and $n_k \geq k$. Since (x_n) is an s_p-convergent sequence to x_0 in X, for all $\epsilon \in (0,1)$, there exists n_ϵ such that $\|x_n - x_0\|^2_{\frac{1}{n^p}} < \epsilon$, for $n \geq n_\epsilon$. We can find $k_0 \in \mathbb{N}$ such that $n_{k_0} \geq n_\epsilon$. If $k \geq k_0$,

$$\|y_k - x_0\|^2_{\frac{1}{k^p}} = \|x_{n_k} - x_0\|^2_{\frac{1}{k^p}} \leq \|x_{n_k} - x_0\|^2_{\frac{1}{n_k{}^p}} < \epsilon,$$

since $n_k \geq n_{k_0} \geq n_\epsilon$ and $n_k \geq k$. This implies that each subsequence of an s_p-convergent sequence in X is s_p-convergent for all $p \in \mathbb{N}$. It is same with this proof to show each subsequence of an s_∞-convergent sequence in X is s_∞-convergent.

(2) Suppose that (x_n) is a convergent sequence to x_0 in $(X, \|\cdot\|)$. Then for all $\epsilon > 0$ and $\alpha \in (0,1]$, there exists $n_{\epsilon,\alpha} \in \mathbb{N}$ such that $\|x_n - x_0\|^2_\alpha < \epsilon$ for $n \geq n_{\epsilon,\alpha}$. Let $x_{n_1} = x_1$. Since $x_n \to x_0$, there exists $n_2(\geq 2) \in \mathbb{N}$ such that $\|x_{n_2} - x_0\|^2_{\frac{1}{2^2}} < \frac{1}{2^2}$. Since $x_n \to x_0$, there exists $n_3\,(\geq \max\{3, n_2+1\}) \in \mathbb{N}$ such that $\|x_{n_3} - x_0\|^2_{\frac{1}{3^3}} < \frac{1}{3^3}$.

$$\vdots$$

Continuing this process, we get a subsequence (x_{n_k}) of (x_n) such that

$$\|x_{n_k} - x_0\|^2_{\frac{1}{k^k}} < \frac{1}{k^k} \qquad \text{for all } k \in \mathbb{N}.$$

We now show that (x_{n_k}) is s_∞-convergent to x_0. Let $\epsilon > 0$ and $p \in \mathbb{N}$. Then there exists $k_0 \in \mathbb{N}$ such that $\frac{1}{k_0{}^p} \leq \epsilon$. If $k \geq \max\{k_0, p\}$, $\|x_{n_k} - x_0\|^2_{\frac{1}{k^p}} \leq \|x_{n_k} - x_0\|^2_{\frac{1}{k^k}} < \frac{1}{k^k} \leq \frac{1}{k_0{}^p} \leq \epsilon$. This means that (x_{n_k}) is s_p-convergent. Since $p \in \mathbb{N}$ is arbitrary, (x_{n_k}) is s_∞-convergent. □

We note that s_p-convergence does not imply s_{p+1}-convergence in fuzzy metric spaces and fuzzy normed spaces [Example 2.4 and 2.5]. However, we can get the following corollary by (2) of Theorem 3.2 and (2) of Theorem 3.3, since s_p-convergence and s_∞-convergence in fuzzy metric spaces and fuzzy normed spaces implies convergence for all $p \in \mathbb{N}$.

Corollary 3.4. *Any s_p-convergent sequence in fuzzy metric spaces or fuzzy normed spaces admits s_{p+1}-convergent subsequence, for all $p \in \mathbb{N}$ and s_∞-convergent subsequence.*

By Proposition 2.3 and Corollary 3.4, it is natural to consider the following question.

Question. Does an s_p-convergent sequence or an s_∞-convergent sequence admit st-subsequence in fuzzy metric spaces or fuzzy normed spaces?

The answer is negative. It suffices to show that an s_∞-convergent sequence does not admit st-subsequence in fuzzy metric spaces or fuzzy normed spaces in general, since an s_∞-convergent sequence is an s_p-convergent sequence.

Example 3.5. Let M be a function on $\mathbb{R} \times \mathbb{R} \times (0, \infty)$ defined by

$$M(x, y, t) = \frac{t}{t + |x - y|}.$$

Then $(\mathbb{R}, M, \cdot)$ is a fuzzy metric space [9]. Let $y_n = \frac{1}{n^n}$. Then $\{y_n\}$ is s_∞-convergent to 0 by Example 2.4. We now show that (y_n) does not have an st-convergent subsequence. Suppose that there exists a subsequence (y_{n_k}) of (y_n) which is st-convergent. Then

$$\lim_{k,m\to\infty} M\left(y_{n_k}, 0, \frac{1}{m}\right) = 1$$

and in particular, for $m = n_k^{n_k}$,

$$\lim_{k\to\infty} M\left(y_{n_k}, 0, \frac{1}{n_k^{n_k}}\right) = 1.$$

But $M(y_{n_k}, 0, \frac{1}{n_k^{n_k}}) = \frac{1}{2}$. We get the contradiction. This completes the proof.

The following example is found in [2].

Example 3.6. Let $X = \mathbb{R}$ and we define a fuzzy norm $\|\cdot\|$ on X by

$$\|x\| = \begin{cases} \frac{|x|}{t} & \text{if } |x| \leq t,\ x \neq 0, \\ 1 & \text{if } t = |x| = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(X, \|\cdot\|)$ is a fuzzy normed space introduced by C. Felbin [2] and α-level set of $(X, \|\cdot\|)$ is given by

$$[\|x\|]_\alpha = \left[|x|, \frac{|x|}{\alpha}\right].$$

Let $y_n = \frac{1}{n^n}$. Then $\{y_n\}$ is s_∞-convergent to 0 (Example 2.5). We now show that (y_n) has not st-convergent subsequence. Suppose that there exists a subsequence (y_{n_k}) of (y_n) which is st-convergent. Then

$$\lim_{k,m\to\infty} \|y_{n_k}\|^2_{\frac{1}{m}} = 0$$

and in particular, for $m = n_k^{n_k}$,

$$\lim_{k\to\infty} \|y_{n_k}\|^2_{\frac{1}{n_k^{n_k}}} = 0.$$

But $\|y_{n_k}\|^2_{\frac{1}{n_k^{n_k}}} = 1$. We get the contradiction. This completes our proof.

The following theorem is found in [9].

Theorem 3.7. *Every convergent sequence in a fuzzy metric space $(X, M, *)$ is st-convergent if and only if every convergent sequence in X is s-convergent.*

We can get the same result on fuzzy normed spaces.

Theorem 3.8. *Every convergent sequence in a fuzzy normed space $(X, \|\cdot\|)$ introduced by C. Felbin is st-convergent if and only if every convergent sequence in $(X, \|\cdot\|)$ is s-convergent.*

Proof. If every convergent sequence in $(X, \|\cdot\|)$ is *st*-convergent, then every convergent sequence in $(X, \|\cdot\|)$ is *s*-convergent, since *st*-convergence implies *s*-convergence.

Conversely, suppose that every convergent sequence in $(X, \|\cdot\|)$ is *s*-convergent. Assume the assertion were false; there exists a convergent sequence (x_n) to x_0 in $(X, \|\cdot\|)$ which is not *st*-convergent. Then there exists $\delta > 0$ such that for all $k \in \mathbb{N}$, $\|x_{n(k)} - x_0\|^2_{\alpha(k)} \geq \delta$, for some $n(k)(\geq k) \in \mathbb{N}$ and $\alpha(k) \in (0, 1]$. We may take $n_1 \in \mathbb{N}$ such that $n_1 \geq \max\left\{\frac{1}{\alpha(1)}, n(1) + 1\right\}$ and let

$$y_1 = y_2 = \cdots = y_{n_1} = x_{n(1)}.$$

We may take $n_2 \in \mathbb{N}$ such that $n_2 \geq \max\left\{\frac{1}{\alpha(n_1)}, n(n_1) + 1\right\}$ and let

$$y_{n_1+1} = y_{n_1+2} = \cdots = y_{n_2} = x_{n(n_1)}.$$

$$\vdots$$

Continuing this process, we get a sequence (y_j) such that

$$n_k(\in \mathbb{N}) \geq \max\left\{\frac{1}{\alpha(n_{k-1})}, n(n_{k-1}) + 1\right\},\ k \in \mathbb{N},\ n_0 = 1$$

and

$$y_{n_k+1} = y_{n_k+2} = \cdots = y_{n_{k+1}} = x_{n(n_k)}.$$

We now show that (y_j) converges to x_0 in $(X, \|\cdot\|)$. Since (x_n) converges to x_0, for all $\epsilon > 0$ and $\alpha \in (0, 1]$, there exists $n_{\epsilon,\alpha} \in \mathbb{N}$ such that $\|x_n - x_0\| < \epsilon$, for $n \geq n_{\epsilon,\alpha}$. Since $n(n_k)$ is increasing, there exists $k_0 \in \mathbb{N}$ such that $n(n_{k_0}) \geq n_{\epsilon,\alpha}$. If $j \geq n_{k_0} + 1$, $y_j = x_{n(n_k)}$ for some $k \geq k_0$. Then if $j \geq n_{k_0} + 1$,

$$\|y_j - x_0\|^2_\alpha = \|x_{n(n_k)} - x_0\|^2_\alpha < \epsilon,$$

since $n(n_k) \geq n(n_{k_0}) \geq n_{\epsilon,\alpha}$, $y_j = x_{n(n_k)}$ and $k \geq k_0$. We finally show that (y_j) is not s-convergent to x_0. For $k \in \mathbb{N}$,

$$\begin{aligned}\|y_{n_{k+1}} - x_0\|^2_{\frac{1}{n_{k+1}}} &= \|x_{n(n_k)} - x_0\|^2_{\frac{1}{n_{k+1}}} \\ &\geq \|x_{n(n_k)} - x_0\|^2_{\alpha(n_k)} \geq \delta.\end{aligned}$$

This completes the proof. □

It is clear that if a convergent sequence in fuzzy metric spaces (or fuzzy normed spaces) is not s-converges, then it is not st-converges, since st-convergence implies s-convergence in fuzzy metric spaces (or fuzzy normed spaces). The converse is not true, in general. There exist convergent non-st-convergence sequences examples which are s-convergent in fuzzy metric spaces (or fuzzy normed spaces) [9] and [3]. However, we can get the following corollary, by the proofs of Theorem 3.7 and Theorem 3.8.

Corollary 3.9. *Let $(X, M, *)$ be a fuzzy metric space and $(X, \|\cdot\|)$ a fuzzy normed space introduced by C. Felbin.*

(1) *If (x_n) is convergent to x_0 and non-st-convergent sequence in a fuzzy metric space $(X, M, *)$, we can construct a sequence (y_n) which is convergent to x_0 and non-s-convergent sequence in $(X, M, *)$.*
(2) *If (x_n) is convergent to x_0 and non-st-convergent sequence in a fuzzy normed space $(X, \|\cdot\|)$, we can construct a sequence (y_n) which is convergent to x_0 and non-s-convergent sequence in $(X, \|\cdot\|)$.*

4. Stationary fuzzy normed spaces and s_p-closed sets

A fuzzy metric M on X is said to be stationary [9] if M does not depend on t, i.e. if for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant. In this case we write $M(x, y)$ instead of $M(x, y, t)$. In similar way, we can define the stationary fuzzy normed spaces.

Definition 4. A fuzzy normed space $(X, \|\cdot\|)$ introduced by C. Felbin is said to be stationary if for each $x \in X$, $\|x\|(t)$ is constant for all $t \in \mathbb{R}$.

Non-trivial examples of stationary fuzzy metric spaces are found in [10], [13]. Stationary fuzzy normed spaces, however, are trivial.

Proposition 4.1. *If $(X, \|\cdot\|)$ is a stationary fuzzy normed space introduced by C. Felbin, $\|x\|(t) = 1$ for all $x \in X$ and $t \in \mathbb{R}$.*

Proof. Let $x \in X$. Then for all $t \in \mathbb{R}$, $\|x\|(t)$ is constant. Since $\|x\|$ is normal, $\|x\|(t) = 1$ for all $t \in \mathbb{R}$. □

We now introduce the following definitions.

Definition 5. Let $(X, M, *)$ be a fuzzy metric space and $(X, \|\cdot\|)$ a fuzzy normed space introduced by C. Felbin.

(1) A subset C of X is said to be closed in $(X, M, *)$ if for any sequence (x_n) in C converges to x, i.e., $\lim_{n\to\infty} M(x_n, x, t) = 1$ for $t > 0$, $x \in C$. A subset C of X is said to be s_p-closed if for any sequence (x_n) in C s_p-converges to x, i.e., $\lim_{n\to\infty} M\left(x_n, x, \frac{1}{n^p}\right) = 1$, $x \in C$. A subset C of X is said to be st-closed if for any sequence (x_n) in C st-converges to x, i.e., $\lim_{n,m\to\infty} M\left(x_n, x, \frac{1}{m}\right) = 1$, $x \in C$.

(2) A subset C of X is said to be closed in $(X, \|\cdot\|)$ if for any sequence (x_n) in C converges to x, i.e., $\lim_{n\to\infty} \|x_n - x\|_\alpha^2 = 0$ for $\alpha \in (0, 1]$, $x \in C$. A subset C of X is said to be s_p-closed if for any sequence (x_n) in C s_p-converges to x, i.e., $\lim_{n\to\infty} \|x_n - x\|^2_{\frac{1}{n^p}} = 0$, $x \in C$. A subset C of X is said to be st-closed if for any sequence (x_n) in C st-converges to x, i.e., $\lim_{n,m\to\infty} N\|x_n - x\|^2_{\frac{1}{m}} = 0$, $x \in C$.

A subset C of X is said to be s-closed if it is s_1-closed in X and a subset C of X is said to be s_∞-closed if it is s_p-closed in X for all $p \in \mathbb{N}$.

Since st-conv. $\Rightarrow$ s_∞-conv.$\Rightarrow \cdots \Rightarrow$ s_p-conv. $\Rightarrow$ s-conv.$\cdots \Rightarrow$ conv., we get the following implication for a subset C in fuzzy metric spaces and fuzzy normed space:

closed $\Rightarrow$ s-closed $\Rightarrow \cdots \Rightarrow$ s_p-closed $\Rightarrow$ s_∞-closed $\Rightarrow$ st-closed.

Proposition 4.2. *Closedness, s_p-closedness and s_∞-closedness are coincide in fuzzy metric spaces and fuzzy normed spaces.*

Proof. It suffices to show that s_∞-closedness implies closedness. Let X be a fuzzy metric space. Let C be an s_∞-closed subset of X. Let (x_n) be a convergent sequence to x in X. Then there exists a subsequence (y_n) of (x_n) such that (y_n) is s_∞-convergent to x, by Theorem 3.2 and Theorem 3.3. Since C is s_∞-closed, $x \in C$. This completes the proof. □

We now consider relation of st-closedness and closedness.

Lemma 4.3. *Let $X = \mathbb{R}$ and we define a function M on $X \times X \times (0, \infty)$ by*

$$M(x, y, t) = \frac{t}{t + |x - y|}.$$

Then $(X, M, \cdot)$ is a fuzzy metric space (Example 2.4). A sequence (x_n) in X is st-convergent to x_0 if and only if there exists $N \in \mathbb{N}$ such that $x_n = x_0$ for all $n \geq N$ (i.e., (x_n) is eventually constant).

Proof. It is clear that if (x_n) is eventually constant, then (x_n) in X is st-convergent.

Conversely, suppose that (x_n) in X is st-convergent to x_0. Then for all $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$M(x_n, x_0, \frac{1}{m}) = \frac{\frac{1}{m}}{\frac{1}{m} + |x_n - x_0|} > 1 - \epsilon \qquad \text{for } n, m \geq N_\epsilon.$$

For $\epsilon = \frac{1}{2}$,

$$M(x_n, x_0, \frac{1}{m}) = \frac{\frac{1}{m}}{\frac{1}{m} + |x_n - x_0|} > \frac{1}{2}, \qquad \text{for } n, m \geq N_{\frac{1}{2}}.$$

This implies that

$$|x_n - x_0| < \frac{1}{m} \qquad \text{for } n, m \geq N_{\frac{1}{2}}.$$

Let $n \geq N_{\frac{1}{2}}$. Then

$$|x_n - x_0| = \lim_{m\to\infty} |x_n - x_0| \leq \lim_{m\to\infty} \frac{1}{m} = 0.$$

This implies that $x_n = x_0$ for $n \geq N_{\frac{1}{2}}$. This completes the proof. □

Example 4.4. Let $C = \{1, \frac{1}{2^2}, \frac{1}{3^3}, \ldots, \frac{1}{n^n}, \ldots\}$ be a set of $\mathbb{R}$ in Example 2.4. Then C is not s_p-closed and not s_∞-closed, for all $p \in \mathbb{N}$, since $\left(\frac{1}{n^n}\right)$ is s_∞-convergent to 0 by Example 2.4 and $0 \notin C$. C is *st*-closed, since any *st*-convergent sequence in X is eventually constant by Lemma 4.3.

We can also get the same result in fuzzy normed spaces introduced by C. Felbin.

Lemma 4.5. *Let $X = \mathbb{R}$ and we define a fuzzy norm $\|\cdot\|$ on X by*

$$\|x\| = \begin{cases} \frac{|x|}{t} & \text{if } |x| \leq t,\ x \neq 0, \\ 1 & \text{if } t = |x| = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(X, \|\cdot\|)$ is a fuzzy normed space introduced by C. Felbin [2]*. A sequence (x_n) in X is st-convergent to x_0 if and only if there exists $N \in \mathbb{N}$ such that $x_n = x_0$ for all $n \geq N$ (i.e., (x_n) is eventually constant).*

Proof. It is clear that if (x_n) is eventually constant, then (x_n) in X is *st*-convergent.

Conversely, suppose that (x_n) in X is *st*-convergent to x_0. Then for all $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\|x_n - x_0\|^2_{\frac{1}{m}} = m\,|x_n - x_0| < \epsilon \qquad \text{for } n, m \geq N_\epsilon.$$

For $\epsilon = \frac{1}{2}$,

$$\|x_n - x_0\|^2_{\frac{1}{m}} = m\,|x_n - x_0| < \frac{1}{2} \qquad \text{for } n, m \geq N_{\frac{1}{2}}.$$

This implies that

$$|x_n - x_0| < \frac{1}{2m} \qquad \text{for } n, m \geq N_{\frac{1}{2}}.$$

Let $n \geq N_{\frac{1}{2}}$. Then

$$|x_n - x_0| = \lim_{m\to\infty} |x_n - x_0| \leq \lim_{m\to\infty} \frac{1}{2m} = 0.$$

This implies that $x_n = x_0$ for $n \geq N_{\frac{1}{2}}$. This completes the proof. □

Example 4.6. Let $C = \{1, \frac{1}{2^2}, \frac{1}{3^3}, \dots, \frac{1}{n^n}, \dots\}$ be a set of X in Example 2.5. Then C is not s_p-closed and not s_∞-closed, for all $p \in \mathbb{N}$, since $\left(\frac{1}{n^n}\right)$ is s_∞-convergent to 0 in $(X, \|\cdot\|)$ by Example 2.5 and $0 \notin C$. C is st-closed set, since any st-convergent sequence in $(X, \|\cdot\|)$ is eventually constant by Lemma 4.5.

By Proposition 4.2, Example 4.4 and Example 4.6, we get the following diagram in fuzzy metric spaces and normed spaces:

$$\text{closedness} \Leftrightarrow s_p\text{-closedness} \Leftrightarrow s_\infty\text{-closedness} \Rightarrow st\text{-closedness}.$$

We finally introduce another definition of fuzzy normed space suggested by R. Saadati and S. M. Vaezpour [12].

Definition 6. The triple $(X, N, *)$ is said to be a fuzzy normed space if X is a vector space and $*$ is a continuous t-norm and N is a fuzzy set on $X \times (0, \infty)$ satisfying the following conditions, for all $x, y \in X$, $s, t > 0$,

(N1) $N(x,t) > 0$,
(N2) $N(x,t) = 1$ if and only if $x = 0$,
(N3) $N(\alpha x, t) = N\left(x, \frac{t}{|\alpha|}\right)$ for any $\alpha \neq 0$,
(N4) $N(x,s) * N(y,t) \leq N(x+y, s+t)$,
(N5) $N(x, \_) : (0,\infty) \to [0,1]$ is continuous.
(N6) $\lim_{t\to\infty} N(x,t) = 1$.

The following is found in [12].

Lemma 4.7. *Let $(X, N, *)$ be a fuzzy normed space. If we define a fuzzy set M on $X \times X \times (0,\infty)$ by*

$$M(x,y,t) = N(x-y,t),$$

then M is a fuzzy metric on X, which is called the fuzzy metric induced by the fuzzy norm N.

By Lemma 4.7, if topological property is satisfied in fuzzy metric spaces, it is also satisfied in fuzzy normed spaces introduced by R. Saadati and S. M. Vaezpour. All convergence properties in fuzzy metric spaces in this paper are also satisfied in fuzzy normed spaces introduced by R. Saadati and S. M. Vaezpour. Lemma 4.7 shows that the fuzzy norm in fuzzy normed spaces introduced by R. Saadati and S. M. Vaezpour could induce fuzzy metric. However, it is not easy to see whether or not the fuzzy norm in fuzzy normed spaces introduce by C. Felbin could induce a fuzzy metric. This will be the subject of further study.

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