Commun. Korean Math. Soc. **35** (2020), No. 1, pp. 161–184
https://doi.org/10.4134/CKMS.c180458
pISSN: 1225-1763 / eISSN: 2234-3024

ON SOME WEIGHTED HARDY-TYPE INEQUALITIES INVOLVING EXTENDED RIEMANN-LIOUVILLE FRACTIONAL CALCULUS OPERATORS

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Abstract. In this article, we establish some new weighted Hardy-type inequalities involving some variants of extended Riemann-Liouville fractional derivative operators, using convex and increasing functions. As special cases of the main results, we obtain the results of [18,19]. We also prove the boundedness of the k-fractional integral operator on $L_p[a,b]$.

1. Introduction

Fractional calculus deals with the non integer order derivative and integral operators and draws increasing attention due to its applications in many fields see e.g. the books [30,31]. The first application of fractional calculus was due to Abel in his solution to the Tautocrone problem [1]. It also has applications in biophysics, quantum mechanics, wave theory, polymers, continuum mechanics, Lie theory, field theory, spectroscopy and in group theory, among other applications [14–16,25].

Many mathematicians originate the Hardy-type inequalities for different fractional integral and derivative operators. Because of the fundamental importance of such inequalities in technical sciences, over the years much effort and time have been dedicated to the improvement and generalizations of Hardy-type inequalities. But still there are many open questions in this area, see e.g. those pointed out in [24, Section 7.5]. For further details and literature about the rich history of Hardy-type inequalities and fractional calculus, we refer the books [4,13,15] and the papers [2,6,7,10,18,20]. In the present work, we shall introduce some new results concerning Hardy-type inequalities not covered by the literature mentioned above.

Received November 7, 2018; Revised February 19, 2019; Accepted March 11, 2019.

2010 *Mathematics Subject Classification.* 26D15, 26D10, 26A33.

Key words and phrases. Hardy-type inequalities, Riemann-Liouville, fractional integral operator, convex and increasing function.

The research of second author was supported by the Ministry of Education and Science of the Russian Federation (Agreement number No. 02.a03.21.0008).

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2. Preliminaries and basic results

We start with the definition of the *Riemann-Liouville fractional integrals* (see [22]).

Definition 2.1. Let $[a, b]$, $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals $I^{\alpha}_{a_+} f$ and $I^{\alpha}_{b_-} f$ of order $\alpha > 0$ are defined by

$$(I^{\alpha}_{a_+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt, \; (x > a)$$

and

$$(I^{\alpha}_{b_-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1} dt, \; (x < b),$$

respectively. Here Γ is the usual gamma function.

These integrals are called the left-sided and the right-sided fractional integrals and are bounded in $L_p(a, b)$, $1 \leq p \leq \infty$, that is

$$\|I^{\alpha}_{a_+} f\|_p \leq K\|f\|_p, \quad \|I^{\alpha}_{b_-} f\|_p \leq K\|f\|_p, \tag{2.1}$$

where

$$K = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}.$$

Inequalities given in (2.1) were proved by Hardy in one of his paper [12].

Next is the definition of convex function presented in [13].

Definition 2.2. Let I be an interval in $\mathbb{R}$. A function $\Phi : I \to \mathbb{R}$ is called convex if the following inequality

$$\Phi(\lambda x + (1-\lambda)t) \leq \lambda\Phi(x) + (1-\lambda)\Phi(t) \tag{2.2}$$

holds for all points $x, y \in I$ and all $\lambda \in [0, 1]$. The function Φ is strictly convex if inequality (2.2) holds strictly for all distinct points in I and $\lambda \in (0, 1)$.

The generalized L_p space given in [28] defined as follows:

Definition 2.3. A space $L_{p,r}[a, b]$ is defined as a space of continuous real valued function $h(t)$ on $[a, b]$, such that

$$\left(\int_a^b |h(t)|^p t^r dt \right)^{\frac{1}{p}} < \infty,$$

where $1 \leq p < \infty$, $r \geq 0$. It is clearly seems that $L_{p,0}[a, b] = L_p[a, b]$.

Following is the definition of gamma k function defined by Diaz et al. in [9].

Definition 2.4. The Γ_k function is the generalization of the classical Γ function and is defined as follows:

$$\Gamma_k(t) = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{t}{k}-1}}{(t)_{n,k}}, \; k > 0, \; \Re(t) > 0, \tag{2.3}$$

where $(t)_{n,k} = t(t+k)(t+2k)\cdots(t+(n-1)k)$, $n \geq 1$, is called Pochhammer k symbol. The integral representation is given by

$$\Gamma_k(t) = \int_0^\infty x^{t-1} e^{\frac{-x^k}{k}} dx, \ \Re(t) > 0. \tag{2.4}$$

Specially for $k = 1$, $\Gamma_1(t) = \Gamma(t)$.

Next is the well known definition of Riemann-Liouvill fractional derivative (see [22], [32]) of order α defined by

$$\mathfrak{D}_x^\alpha\{f(x)\} = \frac{1}{\Gamma(-\alpha)} \int_0^x f(t)(x-t)^{-\alpha-1} dt, \ \Re(\alpha) < 0.$$

Let $(\Sigma_1, \Omega_1, \mu_1)$ and $(\Sigma_2, \Omega_2, \mu_2)$ be measure spaces with positive σ-finite measures. Let $U(f)$ denote the class of functions $g : \Omega_1 \to \mathbb{R}$ with the representation

$$g(x) := \int_{\Omega_2} k(x,t) f(t) d\mu_2(t)$$

and A_k be an integral operator defined by

$$A_k f(x) := \frac{g(x)}{K(x)} = \frac{1}{K(x)} \int_{\Omega_2} k(x,t) f(t) d\mu_2(t), \tag{2.5}$$

where $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and non-negative kernel, $f : \Omega_2 \to \mathbb{R}$ is a measurable function and

$$0 < K(x) := \int_{\Omega_2} k(x,t) d\mu_2(t), \qquad x \in \Omega_1. \tag{2.6}$$

The following theorem is given in [18].

Theorem 2.5. *Let $(\Sigma_1, \Omega_1, \mu_1)$ and $(\Sigma_2, \Omega_2, \mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, k be a non-negative measurable kernel on $\Omega_1 \times \Omega_2$. Let $0 < p \leq q < \infty$ and the function $x \mapsto u(x)\frac{k(x,t)}{K(x)}$ is integrable on Ω_1. Then for each fixed $t \in \Omega_2$, v is known by*

$$v(t) := \int_{\Omega_1} u(x) \frac{k(x,t)}{K(x)} d\mu_1(x) < \infty.$$

If $\Phi : (0, \infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_{\Omega_1} u(x) \Phi\left(\left|\frac{g(x)}{K(x)}\right|\right) d\mu_1(x) \leq \int_{\Omega_2} v(t) \Phi(|f(t)|) d\mu_2(t)$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$.

Next result is specified in [19].

Theorem 2.6. *Let $f_i : \Omega_2 \to \mathbb{R}$ be measurable functions, $g_i \in U(f_i)$, $(i = 1, 2)$, where $g_2(x) > 0$ for every $x \in \Omega_1$. Let u be a weight function on Ω_1 and*

k a non-negative measurable kernel on $\Omega_1 \times \Omega_2$. Assume that the function $x \mapsto u(x)\frac{f_2(t)k(x,t)}{g_2(x)}$ is integrable on Ω_1 for each fixed $t \in \Omega_2$. Define p on Ω_2 by

$$p(t) := f_2(t) \int_{\Omega_1} \frac{u(x)k(x,t)}{g_2(x)} d\mu_1(x) < \infty.$$

If $\Phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_{\Omega_1} u(x)\Phi\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right) d\mu_1(x) \leq \int_{\Omega_2} p(t)\Phi\left(\left|\frac{f_1(t)}{f_2(t)}\right|\right) d\mu_2(t)$$

holds.

The inequality due to Krulić et al. [23] is given in the following theorem.

Theorem 2.7. *Let the assumptions of Theorem 2.5 be satisfied and w be defined by*

$$w(t) := \left[\int_{\Omega_1} u(x)\left(\frac{k(x,t)}{K(x)}\right)^{\frac{q}{p}} d\mu_1(x)\right]^{\frac{p}{q}} < \infty.$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left[\int_{\Omega_1} u(x)\left(\Phi\left(A_k f(x)\right)\right)^{\frac{q}{p}} d\mu_1(x)\right]^{\frac{1}{q}} \leq \left[\int_{\Omega_2} w(t)\Phi\left(f(t)\right) d\mu_2(t)\right]^{\frac{1}{p}}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ such that $\mathrm{Im} f \subseteq I$.

The following theorem is given in [23].

Theorem 2.8. *Let $g_i \in U(f_i)$, $(i = 1, 2, 3)$, where $g_2(x) > 0$ for every $x \in \Omega_1$. Let u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$. Then r is defined by*

$$r(t) := f_2(t) \int_{\Omega_1} \frac{u(x)k(x,t)}{g_2(x)} dx < \infty.$$

If $\Phi : (0,\infty) \times (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_{\Omega_1} u(x)\Phi\left(\left|\frac{g_1(x)}{g_2(x)}\right|, \left|\frac{g_3(x)}{g_2(x)}\right|\right) d\mu_1(x) \leq \int_{\Omega_2} r(t)\Phi\left(\left|\frac{f_1(t)}{f_2(t)}\right|, \left|\frac{f_3(t)}{f_2(t)}\right|\right) d\mu_2(t)$$

holds true.

3. Weighted Hardy-type integral inequalities for extended Riemann-Liouville fractional derivative operator involving exponential function

This section consists of weighted Hardy-type inequalities for the extended Riemann-Liouville fractional derivative operator established in [5] and is defined as follows:

Definition 3.1. Let $\Re(r) > 0$, $\Re(s) > 0$ and $\Re(\alpha) < 0$. Then the extended Riemann-Liouville fractional derivative $\mathfrak{D}_x^\alpha\{f(x); r, s\}$ of order α, be such that

$$\mathfrak{D}_x^\alpha\{f(x); r, s\} = \frac{1}{\Gamma(-\alpha)} \int_0^x f(t)(x-t)^{-\alpha-1} \exp\left(-\frac{rx}{t} - \frac{sx}{(x-t)}\right) dt. \tag{3.1}$$

Specially for $r = s$ we arrive at the extended Riemann-Liouville fractional derivative of order α given in [3] and is defined by

$$\mathfrak{D}_x^\alpha\{f(x); r\} = \frac{1}{\Gamma(-\alpha)} \int_0^x f(t)(x-t)^{-\alpha-1} \exp\left(-\frac{rx^2}{t(x-t)}\right) dt. \tag{3.2}$$

Example 3.2. Consider the derivative given by (3.1) of x^ν corresponding to $x = 1$, we get

$$\mathfrak{D}_x^\alpha\{x^\nu; r, s\}_{x=1} = \frac{B_{r,s}(\nu+1, -\alpha)}{\Gamma(-\alpha)},$$

where $B_{r,s}(\nu+1, -\alpha)$ is the extended beta functions (see [26]) defined by

$$B_{r,s}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{r}{t} - \frac{s}{1-t}} dt, \tag{3.3}$$

where $x, y, r, s \in \mathbb{C}$, $\Re(r) > 0$, $\Re(s) > 0$. For $r = s$, $B_{r,s}$ becomes B_r and for $r = s = 0$, we get the classical beta function defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \ \Re(x) > 0, \ \Re(y) > 0.$$

In [29] the following inequality is proved for the extended beta function (3.3):

$$B_{r,s}(x, y) \le (2r)^{\frac{2x-1}{2}} (2s)^{\frac{2y-1}{2}} \sqrt{\Gamma(-2x+1, 2r)\Gamma(-2y+1, 2s)},$$

where $r, s > 0$, $0 < x$, $y < \frac{1}{2}$ and $\Gamma(x, y)$ is the incomplete gamma function.

Lemma 3.3. *Let $\Re(r) > 0$, $\Re(s) > 0$ and $\Re(\alpha) < 0$. Then the following relation holds*:

$$\breve{K}(x) = \frac{x^{-\alpha} B_{r,s}(1, -\alpha)}{\Gamma(-\alpha)}. \tag{3.4}$$

Proof. Since

$$\breve{k}(x, t) = \begin{cases} \frac{1}{\Gamma(-\alpha)}(x-t)^{-\alpha-1} \exp\left(-\frac{rx}{t} - \frac{sx}{(x-t)}\right), & 0 \le t \le x\,; \\ 0, & x < t \le b\,. \end{cases} \tag{3.5}$$

Therefore

$$\breve{K}(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha-1} \exp\left(-\frac{rx}{t} - \frac{sx}{(x-t)}\right) dt,$$

which can be written as

$$= \frac{x^{-\alpha-1}}{\Gamma(-\alpha)} \int_0^x (1-\frac{t}{x})^{-\alpha-1} \exp\left(-\frac{rx}{t} - \frac{s}{(1-\frac{t}{x})}\right) dt.$$

By substituting $\frac{t}{x} = y$ and using the simple calculation, we arrive at (3.4). $\square$

The first result for the operator (3.1) is as follows:

Theorem 3.4. *Let $\Re(r) > 0$, $\Re(s) > 0$ and $\Re(\alpha) < 0$. Let $\mathfrak{D}_x^{\alpha}\{f(x); r, s\}$ denote the extension of Riemann-Liouville fractional derivative of order α and let u be a weight function defined on $(0, b)$. For each fixed $t \in (0, b)$, define a function $\breve{v}$ by*

$$\breve{v}(t) = \int_t^b u(x) \frac{(x-t)^{-\alpha-1} \exp\left(-\frac{rx}{t} - \frac{sx}{(x-t)}\right)}{x^{-\alpha} B_{r,s}(1, -\alpha)} dx < \infty.$$

If $\Phi : (0, \infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_0^b u(x)\Phi\left(\left|\frac{\Gamma(-\alpha)\mathfrak{D}_x^{\alpha}\{f(x); r, s\}}{x^{-\alpha} B_{r,s}(1, -\alpha)}\right|\right) dx \le \int_0^b \breve{v}(t)\Phi(|f(t)|)dt \tag{3.6}$$

holds for all measurable functions $f \in L_1(a, b)$.

Proof. Applying Theorem 2.5 with $\Omega_1 = \Omega_2 = (0, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, $\breve{K}(x)$, $\breve{k}(x, t)$ given in (3.4), (3.5), respectively and define

$$\breve{A}_k f(x) = \frac{\Gamma(-\alpha)\mathfrak{D}_x^{\alpha}\{f(x); r, s\}}{x^{-\alpha} B_{r,s}(1, -\alpha)}, \tag{3.7}$$

we get inequality (3.6). □

Theorem 3.5. *Let $\Re(r) > 0$, $\Re(s) > 0$ and $\Re(\alpha) < 0$. Let $\mathfrak{D}_x^{\alpha}\{f(x); r, s\}$ denote the extension of Riemann-Liouville fractional derivative of order α and let u be a weight function defined on $(0, b)$. For each fixed $t \in (0, b)$, define a function*

$$\breve{p}(t) := \frac{f_2(t)}{\Gamma(-\alpha)} \int_t^b u(x) \frac{(x-t)^{-\alpha-1} \exp\left(-\frac{rx}{t} - \frac{sx}{(x-t)}\right)}{\mathfrak{D}_x^{\alpha}\{f_2(x); r, s\}} dx < \infty.$$

If $\Phi : (0, \infty) \to \mathbb{R}$ is a convex and increasing function and $\frac{\mathfrak{D}_x^{\alpha}\{f_1(x);r,s\}}{\mathfrak{D}_x^{\alpha}\{f_2(x);r,s\}}, \frac{f_1(t)}{f_2(t)} \in (0, \infty)$, then the inequality

$$\int_0^b u(x)\Phi\left(\left|\frac{\mathfrak{D}_x^{\alpha}\{f_1(x); r, s\}}{\mathfrak{D}_x^{\alpha}\{f_2(x); r, s\}}\right|\right) dx \le \int_0^b \breve{p}(t)\Phi\left(\left|\frac{f_1(t)}{f_2(t)}\right|\right) dt \tag{3.8}$$

holds for all measurable functions $f_i \in L_1(0, b)$, $(i = 1, 2)$.

Proof. Applying Theorem 2.6 with $\Omega_1 = \Omega_2 = (0, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, $g_1(x) = \mathfrak{D}_x^{\alpha}\{f_1(x); r, s\}$, $g_2(x) = \mathfrak{D}_x^{\alpha}\{f_2(x); r, s\}$ and using $\breve{k}(x, t)$ provided in (3.5), we obtain inequality (3.8). □

The upcoming result is the generalization of Theorem 3.4.

Theorem 3.6. *Let the assumption of Theorem 3.4 be satisfied,* $0 < p \leq q < \infty$ *and the weight function be defined by*

$$\breve{w}(t) = \left[\int_t^b u(x) \left(\frac{(x-t)^{-\alpha-1} \exp\left(-\frac{rx}{t} - \frac{sx}{(x-t)}\right)}{x^{-\alpha} B_{r,s}(1,-\alpha)} \right)^{\frac{q}{p}} dx \right]^{\frac{p}{q}} < \infty.$$

If Φ *is a non negative convex function on* $I \subset \mathbb{R}$*, then the inequality*

$$\left[\int_0^b u(x) \left(\Phi \left(\frac{\Gamma(-\alpha) \mathfrak{D}_x^{\alpha}\{f(x); r, s\}}{x^{-\alpha} B_{r,s}(1,-\alpha)} \right) \right)^{\frac{q}{p}} dx \right]^{\frac{1}{q}} \leq \left[\int_0^b \breve{w}(t) \Phi(f(t)) dt \right]^{\frac{1}{p}} \tag{3.9}$$

holds.

Proof. Applying Theorem 2.7 with $\Omega_1 = \Omega_2 = (0, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, $\breve{K}(x)$, $\breve{k}(x, t)$ and $\breve{A}_k f(x)$ are given in (3.4), (3.5) and (3.7), respectively, we get inequality (3.9). □

Theorem 3.7. *Let* $\Re(r) > 0$*,* $\Re(s) > 0$ *and* $\Re(\alpha) < 0$*. Let* $\mathfrak{D}_x^{\alpha}\{f(x); r, s\}$ *denote the extension of Riemann-Liouville fractional derivative of order* α *and* $\mathfrak{D}_x^{\alpha}\{f_2(x); r, s\} > 0$ *for every* $x \in (a, b)$*. Let* u *be a weight function defined on* (a, b)*. For each fixed* $t \in (0, b)$*, define a function*

$$\breve{r}(t) := \frac{f_2(t)}{\Gamma(-\alpha)} \int_t^b u(x) \frac{(x-t)^{-\alpha-1} \exp\left(-\frac{rx}{t} - \frac{sx}{(x-t)}\right)}{\mathfrak{D}_x^{\alpha}\{f_2(x); r, s\}} dx < \infty.$$

If $\Phi : (0, \infty) \times (0, \infty) \to \mathbb{R}$ *is a convex and increasing function and* $\frac{\mathfrak{D}_x^{\alpha}\{f_1(x);r,s\}}{\mathfrak{D}_x^{\alpha}\{f_2(x);r,s\}}$, $\frac{\mathfrak{D}_x^{\alpha}\{f_3(x);r,s\}}{\mathfrak{D}_x^{\alpha}\{f_4(x);r,s\}}$, $\frac{f_1(t)}{f_2(t)} \in (0, \infty)$*, then the inequality*

$$\begin{aligned} &\int_0^b u(x) \Phi \left(\left| \frac{\mathfrak{D}_x^{\alpha}\{f_1(x); r, s\}}{\mathfrak{D}_x^{\alpha}\{f_2(x); r, s\}} \right|, \left| \frac{\mathfrak{D}_x^{\alpha}\{f_3(x); r, s\}}{\mathfrak{D}_x^{\alpha}\{f_2(x); r, s\}} \right| \right) dx \\ \leq &\int_0^b \breve{r}(t) \Phi \left(\left| \frac{f_1(t)}{f_2(t)} \right|, \left| \frac{f_3(t)}{f_2(t)} \right| \right) dt \end{aligned} \tag{3.10}$$

holds for all measurable functions $f_i \in L_1(0, b)$, $(i = 1, 2, 3)$.

Proof. Applying Theorem 2.8 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, $\breve{k}(x, t)$, is offered in (3.5), $g_1(x) = \mathfrak{D}_x^{\alpha}\{f_1(x); r, s\}$, $g_2(x) = \mathfrak{D}_x^{\alpha}\{f_2(x); r, s\}$ and $g_3(x) = \mathfrak{D}_x^{\alpha}\{f_3(x); r, s\}$, we get inequality (3.10). □

Remark 3.8. If we choose $r = s$ in inequalities (3.6), (3.8), (3.9) and (3.10), we get the results for the extended Riemann-Liouville fractional derivative given in (3.2).

Remark 3.9. If we choose $r = s = 0$ in inequalities (3.6), (3.8), (3.9) and (3.10), we get the results for the classical Riemann-Liouville fractional derivative.

4. Weighted Hardy-type integral inequalities for extended Riemann-Liouville fractional derivative

This section includes weighted Hardy-type integral inequalities for more general extended Riemann-Liouville fractional derivative given in [29] and is defined as follows:

Definition 4.1. Let $\Re(\mu) < 0$, $\Re(p) > 0$ and $\Re(\alpha) > 0$. Then the more general extended Riemann-Liouville fractional derivative $\mathfrak{D}_{x;p}^{\mu;\alpha} f$ of order α is given by

$$\mathfrak{D}_{x;p}^{\mu;\alpha}\{f(x)\} = \frac{1}{\Gamma(-\mu)} \int_0^x f(t)(x-t)^{-\mu-1} E_\alpha \left(-\frac{px^2}{t(x-t)}\right) dt, \tag{4.1}$$

where E_α is the Mittag-Leffler function was introduced and studied by Mittag-Leffler in the year (1903) and is given by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \ \alpha \in \mathbb{C}. \tag{4.2}$$

In particular if we choose $\alpha = 1$ in (4.1), we get extended Riemann-Liouville fractional derivative given by (3.2). Similarly, if $\alpha = 1, p = 0$, it reduces to classical Riemann-Liouville fractional derivative.

Lemma 4.2. *Let $\Re(\mu) < 0$, $\Re(p) > 0$ and $\Re(\alpha) > 0$. Then the following equation holds*:

$$\widetilde{K}(x) = \frac{x^{-\mu} B_p^\alpha(1,-\mu)}{\Gamma(-\mu)}, \tag{4.3}$$

where

$$B_p^\alpha(\delta_1,\delta_2) = \int_0^1 t^{\delta_1-1}(1-t)^{\delta_2-1} E_\alpha\left(-\frac{p}{t(1-t)}\right) dt, \ \Re(\delta_1), \ \Re(\delta_2) > 0$$

is the modified extension of beta function presented in [33].

Proof. Since

$$\widetilde{k}(x,t) = \begin{cases} \frac{1}{\Gamma(-\mu)}(x-t)^{-\mu-1} E_\alpha\left(-\frac{px^2}{t(x-t)}\right), & 0 \le t \le x\,; \\ 0, & x < t \le b\,. \end{cases} \tag{4.4}$$

Therefore

$$\widetilde{K}(x) = \frac{1}{\Gamma(-\mu)} \int_0^x (x-t)^{-\mu-1} E_\alpha\left(-\frac{px^2}{t(x-t)}\right) dt,$$

which can be written as

$$= \frac{x^{-\mu-1}}{\Gamma(-\mu)} \int_0^x (1-\frac{t}{x})^{-\mu-1} E_\alpha\left(-\frac{p}{\frac{t}{x}(1-\frac{t}{x})}\right) dt.$$

By substituting $\frac{t}{x} = y$ and using the simple calculation, we arrive at (4.3). $\square$

Theorem 4.3. *Let $\Re(\mu) < 0$, $\Re(p) > 0$ and $\Re(\alpha) > 0$. Let $\mathfrak{D}^{\mu;\alpha}_{x;p} f$ denote the extension of Riemann-Liouville fractional derivative of order α and let u be a weight function defined on $(0, b)$. For each fixed $t \in (0, b)$, define a function $\widetilde{v}$ by*

$$\widetilde{v}(t) = \int_t^b u(x) \frac{(x-t)^{-\mu-1} E_\alpha\left(-\frac{px^2}{t(x-t)}\right)}{x^{-\mu} B^\alpha_p(1, -\mu)} dx < \infty.$$

If $\Phi : (0, \infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_0^b u(x) \Phi\left(\left|\frac{\Gamma(-\mu)\mathfrak{D}^{\mu;\alpha}_{x;p}\{f(x)\}}{x^{-\mu} B^\alpha_p(1, -\mu)}\right|\right) dx \leq \int_0^b \widetilde{v}(t)\Phi(|f(t)|)dt \tag{4.5}$$

holds for all measurable functions $f \in L_1(0, b)$.

Proof. Applying Theorem 2.5 with $\Omega_1 = \Omega_2 = (0, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, $\widetilde{K}(x)$, $\widetilde{k}(x, t)$ given in (4.3), (4.4), respectively and define

$$\widetilde{A}_k f(x) = \frac{\Gamma(-\mu)\mathfrak{D}^{\mu;\alpha}_{x;p}\{f(x)\}}{x^{-\mu} B^\alpha_p(1, -\mu)}, \tag{4.6}$$

we get inequality (4.5). □

Theorem 4.4. *Let $\Re(\mu) < 0$, $\Re(p) > 0$ and $\Re(\alpha) > 0$. Let $\mathfrak{D}^{\mu;\alpha}_{x;p} f$ denote the extension of Riemann-Liouville fractional derivative of order α and let u be a weight function defined on $(0, b)$. For each fixed $t \in (0, b)$, define a function*

$$\widetilde{p}(t) := \frac{f_2(t)}{\Gamma(-\mu)} \int_t^b u(x) \frac{(x-t)^{-\mu-1} E_\alpha\left(-\frac{px^2}{t(x-t)}\right)}{\mathfrak{D}^{\mu,\alpha}_{x,p}\{f_2(x)\}} dx < \infty.$$

If $\Phi : (0, \infty) \to \mathbb{R}$ is a convex and increasing function and $\frac{\mathfrak{D}^{\mu;\alpha}_{x:p}\{f_1(x)\}}{\mathfrak{D}^{\mu;\alpha}_{x:p}\{f_2(x)\}}, \frac{f_1(t)}{f_2(t)} \in (0, \infty)$, then the inequality

$$\int_0^b u(x) \Phi\left(\left|\frac{\mathfrak{D}^{\mu,\alpha}_{x:p}\{f_1(x)\}}{\mathfrak{D}^{\mu;\alpha}_{x:p}\{f_2(x)\}}\right|\right) dx \leq \int_0^b \widetilde{p}(t)\Phi\left(\left|\frac{f_1(t)}{f_2(t)}\right|\right) dt \tag{4.7}$$

holds for all measurable functions $f_i \in L_1(0, b)$, $(i = 1, 2)$.

Proof. Applying Theorem 2.6 with $\Omega_1 = \Omega_2 = (0, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, $g_1(x) = \mathfrak{D}^{\mu;\alpha}_{x:p}\{f_1(x)\}$, $g_2(x) = \mathfrak{D}^{\mu;\alpha}_{x:p}\{f_2(x)\}$ and using $\widetilde{k}(x, t)$ provided in (4.4), we obtain inequality (4.7). □

The upcoming result is the generalization of Theorem 4.3.

Theorem 4.5. *Let the assumption of Theorem 4.3 be satisfied, $0 < r \leq s < \infty$ and the weight function be defined by*

$$\widetilde{w}(t) = \left[\int_t^b u(x) \left(\frac{(x-t)^{-\alpha-1} E_\alpha\left(-\frac{px^2}{t(x-t)}\right)}{x^{-\mu} B^\alpha_p(1, -\mu)}\right)^{\frac{s}{r}} dx\right]^{\frac{r}{s}} < \infty.$$

If ϕ is a non-negative convex function on $I \subset \mathbb{R}$, then the inequality

$$\left[\int_0^b u(x)\left(\Phi\left(\frac{\Gamma(-\mu)\mathfrak{D}_{x:p}^{\mu,\alpha}\{f(x)\}}{x^{-\mu}B_p^q(1,-\mu)}\right)\right)^{\frac{s}{r}} dx\right]^{\frac{1}{s}} \leq \left[\int_0^b \widetilde{w}(t)\Phi(f(t))dt\right]^{\frac{1}{r}} \tag{4.8}$$

holds.

Proof. Applying Theorem 2.7 with $\Omega_1 = \Omega_2 = (0,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, $\widetilde{K}(x)$, $\widetilde{k}(x,t)$ and $\widetilde{A}_k f(x)$ are given in (4.3), (4.4) and (4.6), respectively, we get inequality (4.8). □

Theorem 4.6. *Let $\Re(\mu) < 0$, $\Re(p) > 0$ and $\Re(\alpha) > 0$. Let $\mathfrak{D}_{x;p}^{\mu;\alpha} f$ denote the extension of Riemann-Liouville fractional derivative of order α and $\mathfrak{D}_{x:p}^{\mu;\alpha}\{f_2(x)\} > 0$ for every $x \in (0,b)$. Let u be a weight function defined on (o,b). For each fixed $t \in (0,b)$, define a function*

$$\widetilde{r}(t) := \frac{f_2(t)}{\Gamma(-\mu)}\int_t^b u(x)\frac{(x-t)^{-\mu-1}E_\alpha\left(-\frac{px^2}{t(x-t)}\right)}{\mathfrak{D}_{x:p}^{\mu;\alpha}\{f_2(x)\}}dx < \infty.$$

If $\Phi : (0,\infty) \times (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\begin{aligned} &\int_0^b u(x)\Phi\left(\left|\frac{\mathfrak{D}_{x:p}^{\mu,\alpha}\{f_1(x)\}}{\mathfrak{D}_{x:p}^{\mu,\alpha}\{f_2(x)\}}\right|, \left|\frac{\mathfrak{D}_{x:p}^{\mu,\alpha}\{f_3(x)\}}{\mathfrak{D}_{x:p}^{\mu,\alpha}\{f_2(x)\}}\right|\right) dx \\ \leq &\int_0^b \widetilde{r}(t)\Phi\left(\left|\frac{f_1(t)}{f_2(t)}\right|, \left|\frac{f_3(t)}{f_2(t)}\right|\right) dt \end{aligned} \tag{4.9}$$

holds for all measurable functions $f_i \in L_1(0,b)$, $(i = 1,2,3)$.

Proof. Applying Theorem 2.8 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, $\widetilde{k}(x,t)$, is offered in (4.4), $g_1(x) = \mathfrak{D}_{x:p}^{\mu;\alpha}\{f_1(x)\}$, $g_2(x) = \mathfrak{D}_{x:p}^{\mu;\alpha}\{f_2(x)\}$ and $g_3(x) = \mathfrak{D}_{x:p}^{\mu;\alpha}\{f_3(x)\}$, we get inequality (4.9). □

Remark 4.7. If we choose $\alpha = 1$ in inequalities (4.5), (4.7), (4.8) and (4.9), we get the results for the extended Riemann-Liouville fractional derivative given in (3.2).

Remark 4.8. If we choose $\alpha = 1$, $p = 0$ in inequalities (4.5), (4.7), (4.8) and (4.9), we get the results for the classical Riemann-Liouville fractional derivative operator.

5. Hardy-type inequalities for generalized fractional integral operator involving Gauss hypergeometric function

This section deals with Hardy-type inequalities for generalized fractional integral operator involving Guass hypergeometric function in its kernel. We first give the definition of generalized fractional integral operator offered in [8].

Definition 5.1. Let $\alpha > 0$, $\mu > -1$, $\beta, \eta \in \mathbb{R}$. Then the generalized fractional integral $I_{a,x}^{\alpha,\beta,\eta,\mu} f$ of order α, for a real-valued continuous function f is defined by:

$$\begin{aligned} I_{a,x}^{\alpha,\beta,\eta,\mu} f(x) = \frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)} \int_a^x & t^{\mu}(x-t)^{\alpha-1} \\ & \times {}_2F_1\left(\alpha+\beta+\mu, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)dt, \quad x \in [a,b], \end{aligned} \tag{5.1}$$

where the function ${}_2F_1(\cdot,\cdot;\cdot)$ appearing in the kernel of above operator is the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} t^n$$

and $(a)_n$ is the Pochhammer symbol defined as $(a)_n = a(a+1)\cdots(a+n-1)$, $(a)_0 = 1$. The operator (5.1) includes the Saigo, the Riemann-Liouville and the Erdélyi-Kober fractional integral operators, i.e.,

$$I_{a,x}^{\alpha,\beta,\eta,0} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta-\eta; \alpha; 1-\frac{t}{x}\right) f(t)dt, \ x \in [a,b],$$

$$R^{\alpha} f(x) = I_{a,x}^{\alpha,-\alpha,\eta,0} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ x \in [a,b]$$

and

$$I^{\alpha,\eta} f(x) = I_{a,x}^{\alpha,0,\eta,0} f(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} t^{\eta} f(t)dt, \ x \in [a,b].$$

The upcoming lemma includes the calculation of $K(x)$ given in (2.6) for the generalized integral operator (5.1), which we used to derive our main results of this section.

Lemma 5.2. *Let $\alpha > 0$, $\mu > -1$, $\beta, \eta \in \mathbb{R}$. Then the following relation exists*:

$$\tilde{K}(x) = \frac{x^{-\mu-\beta}\Gamma(\mu+1)\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(\alpha+\mu+1+\eta)}. \tag{5.2}$$

Proof. Since

$$\tilde{k}(x,t) = \begin{cases} \frac{x^{-\alpha-\beta-2\mu} t^{\mu} (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta+\mu, -\eta; \alpha; 1-\frac{t}{x}\right)}{\Gamma(\alpha)}, & 0 \le t \le x\,; \\ 0, & x < t \le b\,, \end{cases} \tag{5.3}$$

so that

$$\tilde{K}(x) = \int_0^x \frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)} {}_2F_1\left(\alpha+\beta+\mu, -\eta, \alpha; 1-\frac{t}{x}\right) t^{\mu}(x-t)^{\alpha-1} dt,$$

substituting $1-\frac{t}{x} = y$ and using formula given in ([11], page 813), i.e.,

$$\int_0^1 x^{\gamma-1}(1-x)^{\rho-1} {}_2F_1(\alpha,\beta;\gamma;x)dx = \frac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\Gamma(\rho+\gamma-\beta)},$$

we get (5.2). □

Our first main result is given in next theorem.

Theorem 5.3. *Let $\alpha > 0$, $\mu > -1$, $\beta, \eta \in \mathbb{R}$, $I_{a,x}^{\alpha,\beta,\eta,\mu} f$ denote the generalized fractional integral of order α and u be a weight function defined on $(0,b)$. Moreover, for each fixed $t \in (0,b)$, we define $\tilde{v}$ by*

$$\tilde{v}(t) = \frac{\Gamma(1-\beta)\Gamma(\alpha+\mu+1+\eta)}{\Gamma(\mu+1)\Gamma(1-\beta+\eta)} \int_t^b u(x) x^{-\alpha-\mu} \times {}_2F_1\left(\alpha+\beta+\mu, -\eta, \alpha; 1-\frac{t}{x}\right) t^{\mu}(x-t)^{\alpha-1} dx.$$

If $\Phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_0^b u(x)\Phi\left(\left|\frac{\Gamma(1-\beta)\Gamma(\alpha+\mu+1+\eta)}{x^{-\mu-\beta}\Gamma(\mu+1)\Gamma(1-\beta+\eta)} I_{0,x}^{\alpha,\beta,\eta,\mu} f(x)\right|\right) dx \leq \int_0^b \tilde{v}(t)\Phi(|f(t)|)dt \tag{5.4}$$

holds for all measurable functions $f \in L_1(0,b)$.

Proof. Applying Theorem 2.5 with $\Omega_1 = \Omega_2 = (0,b)$, $d\mu_1(x) = dx, d\mu_2(t) = dt$, $\tilde{K}(x)$, $\tilde{k}(x,t)$ are given in (5.2), (5.3), respectively and define

$$\tilde{A}_k f(x) = \frac{I_{0,x}^{\alpha,\beta,\eta,\mu} f(x)\Gamma(1-\beta)\Gamma(\alpha+\mu+1+\eta)}{x^{-\mu-\beta}\Gamma(\mu+1)\Gamma(1-\beta+\eta)}, \tag{5.5}$$

we get inequality (5.4). □

Theorem 5.4. *Let $\alpha > 0$, $\mu > -1$, β, $\eta \in \mathbb{R}$, $I_{a,x}^{\alpha,\beta,\eta,\mu} f$ denote the generalized fractional integral of order α and $I_{a,x}^{\alpha,\beta,\eta,\mu} f_2(x) > 0$. Define $\tilde{p}$ on (a,b) by*

$$\tilde{p}(t) := \frac{f_2(t)}{\Gamma(\alpha)} \int_t^b \frac{u(x)x^{-\alpha-\beta-2\mu} {}_2F_1(\alpha+\beta+\mu, -\eta, \alpha; 1-\frac{t}{x}) t^{\mu}(x-t)^{\alpha-1}}{I_{a,x}^{\alpha,\beta,\eta,\mu} f_2(x)}.$$

If $\Phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function and $\frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f_1(x)}{I_{a,x}^{\alpha,\beta,\eta,\mu} f_2(x)}, \frac{f_1(t)}{f_2(t)} \in (0,\infty)$, then the inequality

$$\int_a^b u(x)\Phi\left(\left|\frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f_1(x)}{I_{a,x}^{\alpha,\beta,\eta,\mu} f_2(x)}\right|\right) dx \leq \int_a^b \tilde{p}(t)\Phi\left(\left|\frac{f_1(t)}{f_2(t)}\right|\right) dt \tag{5.6}$$

holds for all measurable functions $f_i \in L_1(a,b)$, $(i = 1,2)$.

Proof. Applying Theorem 2.6 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(t) = dt$ and $\tilde{k}(x,t)$ is given in (5.3), we get inequality (5.6). □

Next theorem is the generalization of Theorem 5.3.

Theorem 5.5. *Let $\alpha > 0$, $\mu > -1$, $\beta, \eta \in \mathbb{R}$ and $I_{a,x}^{\alpha,\beta,\eta,\mu} f$ denote the generalized fractional integral of order α and u be a weight function. Let $0 < p \leq q < \infty$ and*

$$\tilde{w}(t) = \frac{\Gamma(1-\beta)\Gamma(\alpha+\mu+1+\eta)}{\Gamma(\mu+1)\Gamma(1-\beta+\eta)} \times \left(\int_t^b u(x) \left(x^{-\alpha-\mu} {}_2F_1 \left(\alpha+\beta+\mu, -\eta, \alpha; 1-\frac{t}{x} \right) t^{\mu} (x-t)^{\alpha-1} \right)^{\frac{q}{p}} dx \right)^{\frac{p}{q}}.$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$, then the following inequality

$$\left[\int_0^b u(x) \left(\Phi \left(\frac{\Gamma(1-\beta)\Gamma(\alpha+\mu+1+\eta)}{x^{-\mu-\beta}\Gamma(\mu+1)\Gamma(1-\beta+\eta)} I_{0,x}^{\alpha,\beta,\eta,\mu} f(x) \right) \right)^{\frac{q}{p}} dx \right]^{\frac{1}{q}} \leq \left[\int_0^b \tilde{w}(t) \Phi(f(t)) dt \right]^{\frac{1}{p}} \tag{5.7}$$

holds true.

Proof. Applying Theorem 2.7 with $\Omega_1 = \Omega_2 = (0, b)$, $d\mu_1(x) = dx, d\mu_2(t) = dt$, $\tilde{K}(x)$ $\tilde{k}(x,t)$ and $\tilde{A}_k f(x)$ are provided in (5.2), (5.3) and (5.5), respectively, we get inequality (5.7). □

Theorem 5.6. *Let $\alpha > 0$, $\mu > -1$, $\beta, \eta \in \mathbb{R}$, $I_{a,x}^{\alpha,\beta,\eta,\mu} f$ denote the generalized fractional integral of order α and $I_{a,x}^{\alpha,\beta,\eta,\mu} f_2(x) > 0$ for every $x \in (a,b)$. Let u be a weight function on (a,b). Then for each fixed $t \in (a,b)$, we define $\tilde{r}$ by*

$$\tilde{r}(t) := \frac{f_2(t)}{\Gamma(\alpha)} \int_t^b u(x) \frac{x^{-\alpha-\beta-2\mu} {}_2F_1(\alpha+\beta+\mu, -\eta, \alpha; 1-\frac{t}{x}) t^{\mu} (x-t)^{\alpha-1}}{I_{a,x}^{\alpha,\beta,\eta,\mu} f_2(x)} dx < \infty.$$

If $\Phi : (0,\infty) \times (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the following inequality

$$\int_a^b u(x) \Phi \left(\left| \frac{(I_{a,x}^{\alpha,\beta,\eta,\mu} f_1)(x)}{(I_{a,x}^{\alpha,\beta,\eta,\mu} f_2(x)} \right|, \left| \frac{(I_{a,x}^{\alpha,\beta,\eta,\mu} f_3)(x)}{(I_{a,x}^{\alpha,\beta,\eta,\mu} f_2)(x)} \right| \right) dx \leq \int_a^b \tilde{r}(t) \Phi \left(\left| \frac{f_1(t)}{f_2(t)} \right|, \left| \frac{f_3(t)}{f_2(t)} \right| \right) dt \tag{5.8}$$

holds for all measurable functions $f_i \in L_1(a,b)$, $(i = 1, 2, 3)$.

Proof. Applying Theorem 2.8 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(t) = dt$, $\tilde{k}(x,t)$ is offered in (5.3), we get inequality (5.8). □

Remark 5.7. If we take $\mu = 0$, in inequalities (5.4), (5.6), (5.7) and (5.8), we get the inequalities for the Saigo fractional integral.

Remark 5.8. If along $\mu = 0$, we take $\beta = -\alpha$, in inequalities (5.4), (5.6), (5.7) and (5.8), we get the inequalities for the Riemann-Liouvill fractional integral.

Remark 5.9. If we take $\beta = 0$ and $\mu = 0$, in inequalities (5.4), (5.6), (5.7) and (5.8), we get the inequalities for the Erdélyi-Kober fractional integral.

6. Hardy-type inequalities for generalized Riemann-Liouville fractional integral operator

Let us recall the definition of generalized Riemann-Liouville fractional integral operator specified in [21].

Definition 6.1. Let $\alpha > 0$, $a \geq 0$ and $r \neq -1$, a real number and let $f \in L_{1,r}[a,b]$. Then the generalized Riemann-Liouville fractional integral $I_a^{\alpha,r} f$ is defined by

$$I_a^{\alpha,r} f(x) = \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt, \ x \in (a,b). \tag{6.1}$$

We note that if $r \to -1^+$ the integral operator (6.1) reduces to the famous Hadamard fractional integral:

$$I_a^{\alpha,-1^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t} dt. \tag{6.2}$$

The first result for the generalized Riemann-Liouville fractional integral operator is as follows:

Theorem 6.2. *Let $f \in L_{1,r}[a,b]$, $\alpha \geq 0$ and $r \neq -1$ and let u be a weight function on (a,b). Assume that the function $x \mapsto u(x) \frac{\alpha(r+1)(x^{r+1}-t^{r+1})^{\alpha-1}t^r}{(x^{r+1}-a^{r+1})^\alpha}$ is integrable on (a,b). Then for each fixed $t \in (a,b)$, we define $\bar{v}$ by*

$$\bar{v}(t) := \alpha(r+1) \int_t^b u(x) \frac{(x^{r+1} - t^{r+1})^{\alpha-1} t^r}{(x^{r+1} - a^{r+1})^\alpha} dx < \infty.$$

If $\Phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\begin{aligned} &\int_a^b u(x) \Phi \left(\left| \frac{\alpha(r+1)}{(x^{r+1} - a^{r+1})^\alpha} \int_a^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right| \right) dx \\ &\leq \int_a^b \bar{v}(t) \Phi(|f(t)|) dt \end{aligned} \tag{6.3}$$

holds for all measurable functions $f : (a,b) \to \mathbb{R}$.

Proof. Applying Theorem 2.5 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(t) = dt$,

$$\bar{k}(x,t) = \begin{cases} \frac{(r+1)^{1-\alpha}(x^{r+1}-t^{r+1})^{\alpha-1}t^r}{\Gamma(\alpha)}, & a \leq t \leq x\,; \\ 0, & x < t \leq b\,, \end{cases} \tag{6.4}$$

$$\bar{K}(x) = \frac{(x^{r+1} - a^{r+1})^\alpha}{\alpha \Gamma(\alpha)(r+1)^\alpha} \tag{6.5}$$

and

$$\bar{A}_k f(x) = \frac{\alpha(r+1)}{(x^{r+1}-a^{r+1})^{\alpha}} \int_a^x (x^{r+1}-t^{r+1})^{\alpha-1} t^r f(t)dt, \tag{6.6}$$

we get inequality (6.3). □

Remark 6.3. In particular if we choose $r = 0$ in inequality 6.3, we get [18, Corollary 2.2].

Corollary 6.4. *In particular if $r \to -1$ and $a > 0$ in Theorem 6.2, we get*

$$\bar{s}(t) := \frac{\alpha}{t} \int_t^b u(x) \frac{\left(\log \frac{x}{t}\right)^{\alpha-1}}{\left(\log \frac{x}{a}\right)^{\alpha}} dx$$

and the inequality (6.3) *reduces to*

$$\int_a^b u(x)\Phi\left(\left|\alpha \int_a^x \frac{\left(\log \frac{x}{t}\right)^{\alpha-1}}{\left(\log \frac{x}{a}\right)^{\alpha}} \frac{f(t)}{t} dt\right|\right) dx \le \int_a^b \tilde{s}(t)\Phi(|f(t)|)dt.$$

Corollary 6.5. *In particular for the weight function $u(x) = x^r(x^{r+1}-a^{r+1})^{\alpha}$, $x \in (a,b)$, in Theorem 6.2, we obtain $\bar{v}(t) = t^r(b^{r+1}-t^{r+1})^{\alpha}$, then the inequality* (6.3) *takes the form*

$$\int_a^b x^r(x^{r+1}-a^{r+1})^{\alpha}\Phi\left(\left|\frac{\alpha(r+1)}{(x^{r+1}-a^{r+1})^{\alpha}} \int_a^x (x^{r+1}-t^{r+1})^{\alpha-1} t^r f(t)dt\right|\right) dx$$

$$\le \int_a^b t^r(b^{r+1}-t^{r+1})^{\alpha}\Phi(|f(t)|)dt. \tag{6.7}$$

Although (6.3) *holds for all convex and increasing functions, but we shall consider a power function which is of our interest. Let $q > 1$ and function $\Phi : \mathbb{R} \to \mathbb{R}$ be define by $\Phi(x) = x^q$. Then* (6.7) *reduces to*

$$(\alpha\Gamma(\alpha)(r+1)^{\alpha})^q \int_a^b x^r(x^{r+1}-a^{r+1})^{\alpha(1-q)} \left|I_a^{\alpha,r} f(x)\right| dx$$

$$\le \int_a^b t^r \left(b^{r+1}-t^{r+1}\right)^{\alpha} |f(t)|^q dt. \tag{6.8}$$

Since $x \in (a,b)$ and $\alpha(1-q) < 0$, then from the left hand side of (6.8)*, we can have*

$$(\alpha\Gamma(\alpha)(r+1)^{\alpha})^q \int_a^b x^r(x^{r+1}-a^{r+1})^{\alpha(1-q)} \left|I_a^{\alpha,r} f(x)\right| dx$$

$$\ge (\alpha\Gamma(\alpha)(r+1)^{\alpha})^q\, a^r(b^{r+1}-a^{r+1})^{\alpha(1-q)} \int_a^b |I_a^{\alpha,r} f(x)|^q dx \tag{6.9}$$

and the right hand side of (6.8) *can be estimated as*

$$\int_a^b t^r(b^{r+1}-t^{r+1})^{\alpha}|f(t)|^q dt \le b^r(b^{r+1}-a^{r+1})^{\alpha} \int_a^b |f(t)|^q dt. \tag{6.10}$$

Combining (6.9) *and* (6.10)*, we get*

$$\begin{aligned}&(\alpha\Gamma(\alpha)(r+1)^{\alpha})^{q}\, a^{r}(b^{r+1}-a^{r+1})^{\alpha(1-q)}\int_a^b |I_a^{\alpha,r}f(x)|^q dx\\ &\leq b^{r}(b^{r+1}-a^{r+1})^{\alpha}\int_a^b |f(t)|^q dt,\end{aligned}$$

that is

$$\int_a^b |I_a^{\alpha,r}f(x)|^q dx \leq \frac{b^r}{a^r}\left(\frac{\left(b^{r+1}-a^{r+1}\right)^{\alpha}}{\alpha\Gamma(\alpha)(r+1)^{\alpha}}\right)^{q}\int_a^b |f(t)|^q dt. \tag{6.11}$$

Taking power $\frac{1}{q}$ *on both sides of inequality* (6.11)*, we can have*

$$\|I_a^{\alpha,r}f\|_q \leq M\|f\|_q, \tag{6.12}$$

where

$$M=\left(\frac{b}{a}\right)^{\frac{r}{q}}\frac{(b^{r+1}-a^{r+1})^{\alpha}}{\Gamma(\alpha+1)(r+1)^{\alpha}}.$$

Remark 6.6. If in particular we choose $r=0$ in inequality (6.12), we get [18, Remark 2.5].

Theorem 6.7. *Let u be a weight function on (a,b), $I_a^{\alpha,r}f$ be the generalized Riemann-Liouville fractional integral of order $\alpha>0$, $r\neq 0$ and $I_a^{\alpha,r}f_2(x)>0$. Assume that the function $x\mapsto u(x)\frac{f_2(t)(x^{r+1}-t^{r+1})^{\alpha-1}t^r}{I_a^{\alpha,r}f_2(x)}$ is integrable on (a,b). Define $\bar{p}$ on (a,b) by*

$$\bar{p}(t):=\frac{(r+1)^{1-\alpha}f_2(t)}{\Gamma(\alpha)}\int_t^b \frac{u(x)(x^{r+1}-t^{r+1})^{\alpha-1}t^r}{I_a^{\alpha,r}f_2(x)}dx<\infty. \tag{6.13}$$

If $\Phi:(0,\infty)\to\mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\Phi\left(\left|\frac{I_a^{\alpha,r}f_1(x)}{I_a^{\alpha,r}f_2(x)}\right|\right)dx \leq \int_a^b \bar{p}(t)\Phi\left(\left|\frac{f_1(t)}{f_2(t)}\right|\right)dt \tag{6.14}$$

holds for all measurable functions $f_i\in L_1(a,b)$, $(i=1,2)$.

Proof. Applying Theorem 2.6 with $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx, d\mu_2(t)=dt$ and $\bar{k}(x,t)$ is given by (6.4), we get inequality (6.14). □

Corollary 6.8. *For the particular choice of $r\to -1^{+}$, in Theorem 6.7 the weight function* (6.13) *becomes*

$$\bar{q}(t):=\frac{f_2(t)}{t\Gamma(\alpha)}\int_t^b u(x)\frac{\left(\log\frac{x}{t}\right)^{\alpha-1}}{I_a^{\alpha,-1}f_2(x)}dx<\infty$$

and the inequality (6.14) *takes the form*

$$\int_a^b u(x)\Phi\left(\left|\frac{I_a^{\alpha,-1}f_1(x)}{I_a^{\alpha,-1}f_2(x)}\right|\right)dx \leq \int_a^b \bar{q}(t)\Phi\left(\left|\frac{f_1(t)}{f_2(t)}\right|\right)dt,$$

which is [19, Corollary 2.6].

Next theorem is the generalized version of Theorem 6.2.

Theorem 6.9. *Let $0 < p \leq q < \infty$ and $f \in L_{1,r}[a,b]$. Let $I_a^{\alpha,r}f$ denote the generalized Riemann-Liouville fractional integral of order $\alpha > 0$ and $r \neq -1$. Suppose u is a weight function and $x \mapsto u(x)\left(\frac{(x^{r+1}-t^{r+1})^{\alpha-1}t^r}{(x^{r+1}-a^{r+1})^{\alpha}}\right)^{\frac{q}{p}}$ is integrable on (a,b). Then for each fixed $t \in (a,b)$, the weight function $\bar{w}$ is defined by*

$$\bar{w}(t) := \alpha(r+1)\left[\int_t^b u(x)\left(\frac{(x^{r+1}-t^{r+1})^{\alpha-1}t^r}{(x^{r+1}-a^{r+1})^{\alpha}}\right)^{\frac{q}{p}} dx\right]^{\frac{p}{q}} < \infty. \tag{6.15}$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left[\int_a^b \left(\Phi\left(\frac{\alpha\Gamma(\alpha)(r+1)^{\alpha} I_a^{\alpha,r}f(x)}{(x^{r+1}-a^{r+1})^{\alpha}}\right)\right)^{\frac{q}{p}} dx\right]^{\frac{1}{q}} \leq \left[\int_a^b \bar{w}(t)\Phi\left(f(t)\right)dt\right]^{\frac{1}{p}} \tag{6.16}$$

holds for all measurable functions $f : (a,b) \to \mathbb{R}$.

Proof. Applying Theorem 2.7 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(t) = dt$, $\bar{k}(x,t)$, $\bar{K}(x)$ and $\bar{A}_k f(x)$, are given by (6.4), (6.5) and (6.6), respectively, we get inequality (6.16). □

The upcoming corollary is a special case of Theorem 6.9, which involve results for the Hadamard fractional integral operator.

Corollary 6.10. *In particular when $r \to -1^+$, then the weight function* (6.15) *can be written as*:

$$\bar{\lambda}(t) := \alpha\left[\int_t^b u(x)\left(\frac{(\log\frac{x}{t})^{\alpha-1}}{t(\log\frac{x}{a})^{\alpha}}\right)^{\frac{q}{p}} dx\right]^{\frac{p}{q}}$$

and the inequality (6.16) *becomes*

$$\left[\int_a^b \left(\Phi\left(\frac{\Gamma(\alpha+1)}{(\log\frac{x}{a})^{\alpha}} I_a^{\alpha,-1}f(x)\right)\right)^{\frac{q}{p}} dx\right]^{\frac{1}{q}} \leq \left[\int_a^b \bar{\lambda}(t)(t)\Phi\left(f(t)\right)dt\right]^{\frac{1}{p}}.$$

Remark 6.11. If we choose $r = 0$ in inequality (6.16), we get [17, Corollary 2.4].

Theorem 6.12. *Let $f \in L_{1,r}[a,b]$, $I_a^{\alpha,r}f$ be the generalized Riemann-Liouville fractional integral of order $\alpha > 0$, $r \neq -1$ and $I_a^{\alpha,r}f_2(x) > 0$ for every $x \in (a,b)$. Let u be a weight function on (a,b). Then $\bar{r}$ is defined by*

$$\bar{r}(t) := \frac{(r+1)^{1-\alpha}f_2(t)}{\Gamma(\alpha)}\int_t^b \frac{u(x)(x^{r+1}-t^{r+1})^{\alpha-1}t^r}{I_a^{\alpha,r}f_2(x)}dx < \infty. \tag{6.17}$$

If $\Phi : (0,\infty)\times(0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\Phi\left(\left|\frac{(I_a^{\alpha,r}f_1)(x)}{I_a^{\alpha,r}f_2(x)}\right|, \left|\frac{(I_a^{\alpha,r}f_3)(x)}{(I_a^{\alpha,r}f_2)(x)}\right|\right)dx \tag{6.18}$$

$$\leq \int_a^b \bar{r}(t)\Phi\left(\left|\frac{f_1(t)}{f_2(t)}\right|, \left|\frac{f_3(t)}{f_2(t)}\right|\right) dt$$

holds for all measurable functions $f_i \in L_1(a,b)$, $(i = 1,2,3)$.

Proof. Applying Theorem 2.8 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(t) = dt$ and $\bar{k}(x,t)$ is given in (6.4), we get inequality (6.18). $\square$

As special case of Theorem 6.12, we next present the result for the Hadamard fractional integral operator.

Corollary 6.13. *If we choose* $r \to -1^+$, *then the weight function* (6.17) *becomes*

$$\bar{\lambda}(t) := \frac{f_2(t)}{\Gamma(\alpha)} \int_t^b u(x) \frac{(\log \frac{x}{t})^{\alpha-1}}{t I_a^{\alpha,-1} f_2(x)} dx < \infty$$

and the inequality (6.18) *for the Hadamard fractional integral turn into*

$$\int_a^b u(x)\Phi\left(\left|\frac{(I_a^{\alpha,-1}f_1)(x)}{I_a^{\alpha,-1}f_2(x)}\right|, \left|\frac{(I_a^{\alpha,-1}f_3)(x)}{(I_a^{\alpha,-1}f_2)(x)}\right|\right) dx$$

$$\leq \int_a^b \bar{\lambda}(t)\Phi\left(\left|\frac{f_1(t)}{f_2(t)}\right|, \left|\frac{f_3(t)}{f_2(t)}\right|\right) dt.$$

7. Hardy-type inequalities for the Riemann-Liouville *k*-fractional integral operator

The definition and notation of generalized Riemann-Liouville k-fractional integral operator presented in [27] is defined as follows:

Definition 7.1. Let $f \in L_1[a,b]$. Then the Riemann-Liouville k-fractional integral $I_{a,k}^{\alpha}f$ of order $\alpha > 0$ and $k > 0$, is given by

$$I_{a,k}^{\alpha}f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \ x \in (a,b), \tag{7.1}$$

where Γ_k is defined by (2.4). Moreover, if we choose $k = 1$, the integral operator (7.1) represents the left sided Riemann-Liouville fractional integral.

Theorem 7.2. *Let* $f \in L_1[a,b]$. *Then the Riemann-Liouville* k*-fractional integral* $I_{a,k}^{\alpha}f$ *of order* $\alpha > 0$ *and* $k > 0$, *is bounded, i.e.,*

$$\|I_{a,k}^{\alpha}f\|_q \leq A\|f\|_q,$$

where $A = \frac{k^{\frac{1}{q}-1}(b-a)^{\frac{\alpha}{k}}}{(q\alpha)^{\frac{1}{q}}(\Gamma_k(\alpha))\left(p(\frac{\alpha}{k}-\frac{1}{q})\right)^{\frac{1}{p}}}$.

Proof. Since we have

$$|I_{a,k}^{\alpha}f(x)| \leq \frac{1}{k\Gamma_k(\alpha)} \int_a^x |f(t)|(x-t)^{\frac{\alpha}{k}-1}dt.$$

Using Hölder's inequality on the right hand side of the above inequality, we have

$$\begin{aligned}|I^{\alpha}_{a,k}f(x)| &\leq \frac{1}{k\Gamma_k(\alpha)}\left(\int_a^x (x-t)^{p(\frac{\alpha}{k}-1)}dt\right)^{\frac{1}{p}}\left(\int_a^x |f(t)|^q dt\right)^{\frac{1}{q}}\\ &= \frac{(x-a)^{\frac{\alpha}{k}-\frac{1}{q}}}{k\Gamma_k(\alpha)\left((p(\frac{\alpha}{k}-\frac{1}{q}))\right)^{\frac{1}{p}}}\left(\int_a^x |f(t)|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

Consequently, we find

$$\|I^{\alpha}_{a,k}f(x)\|_q \leq \frac{k^{\frac{1}{q}-1}(b-a)^{\frac{\alpha}{k}}}{(q\alpha)^{\frac{1}{q}}\left(\Gamma_k(\alpha)\right)\left(p(\frac{\alpha}{k}-\frac{1}{q})\right)^{\frac{1}{p}}}\|f(t)\|_q.$$

This completes the proof. □

Remark 7.3. If we take $k = 1$, in above result, we arrive at [18, Theorem 2.6].

Theorem 7.4. *Let $f \in L_1[a,b]$, $\alpha > 0$ and let u is a weight function, $x \mapsto u(x)\frac{\alpha(x-t)^{\frac{\alpha}{k}-1}}{k(x-a)^{\frac{\alpha}{k}}}$ is integrable on (a,b). Then for each fixed $t \in (a,b)$, we define a function $\hat{v}$ by*

$$\hat{v}(t) := \frac{\alpha}{k}\int_t^b \frac{u(x)(x-t)^{\frac{\alpha}{k}-1}}{(x-a)^{\frac{\alpha}{k}}}dx < \infty.$$

If $\Phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\Phi\left(\left|\frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}}\int_a^x (x-t)^{\frac{\alpha}{k}-1}f(t)dt\right|\right)dx \leq \int_a^b \hat{v}(t)\Phi(|f(t)|)dt \tag{7.2}$$

holds for all measurable functions $f : (a,b) \to \mathbb{R}$.

Proof. Applying Theorem 2.5 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(t) = dt$,

$$\hat{k}(x,t) = \begin{cases} \frac{(x-t)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}, & a \leq t \leq x\,;\\ 0, & x < t \leq b\,,\end{cases} \tag{7.3}$$

$\hat{K}(x)$ is defined by

$$\hat{K}(x) = \frac{1}{\alpha\Gamma_k(\alpha)}(x-a)^{\frac{a}{k}} \tag{7.4}$$

and

$$\hat{A}_k f(x) = \frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}}\int_a^x (x-t)^{\frac{a}{k}-1}f(t)dt, \tag{7.5}$$

we get inequality (7.2). □

Corollary 7.5. *Let $I^{\alpha}_{a,k}f$ be the Riemann-Liouville k-fractional integral of order $\alpha > 0$ and $k > 0$. Choose a particular weight function $u(x) = (x-a)^{\frac{\alpha}{k}}$, $x \in (a,b)$ in Theorem 7.4, we obtain $\hat{v}(t) = (b-t)^{\frac{\alpha}{k}}$. Then the inequality* (7.2) *takes the form*

$$\int_a^b (x-a)^{\frac{\alpha}{k}} \Phi\left(\left|\frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt\right|\right) dx$$

$$\leq \int_a^b (b-t)^{\frac{\alpha}{k}} \Phi(|f(t)|)dt. \tag{7.6}$$

Let $q > 1$ and function $\Phi : (0,\infty) \to \mathbb{R}$ be define by $\Phi(x) = x^q$. Then (7.6) *becomes*

$$\int_a^b (x-a)^{\frac{\alpha}{k}} \left(\frac{\alpha\Gamma_k(\alpha)|I^{\alpha}_{a,k}f(x)|}{(x-a)^{\frac{\alpha}{k}}}\right)^q dx \leq \int_a^b (b-t)^{\frac{\alpha}{k}} |f(t)|^q dt.$$

Since $x \in (a,b)$ and $\frac{\alpha}{k}(1-q) < 0$, then after some calculations, we have

$$\|I^{\alpha}_{a,k}\|_q \leq N\|f\|_q,$$

where

$$N = \frac{(b-a)^{\frac{\alpha}{k}}}{\alpha\Gamma_k(\alpha)}.$$

Remark 7.6. Particularly for $k = 1$, we get [18, Remark 2.3].

Theorem 7.7. *Let u be a weight function on (a,b), $I^{\alpha}_{a,k}f$ be the Riemann-Liouville k-fractional integral of order $\alpha > 0$ and $I^{\alpha}_{a,k}f_2(x) > 0$ for every $x \in (a,b)$. Assume that the function $x \mapsto u(x)\frac{f_2(t)(x-t)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)I^{\alpha}_{a,k}f_2(x)}$ is integrable on (a,b). Then define $\hat{p}$ on (a,b) by*

$$\hat{p}(t) := \frac{f_2(t)}{k\Gamma_k(\alpha)} \int_t^b \frac{u(x)(x-t)^{\frac{\alpha}{k}-1}}{I^{\alpha}_{a,k}f_2(x)} dx < \infty.$$

If $\Phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\Phi\left(\left|\frac{I^{\alpha}_{a,k}f_1(x)}{I^{\alpha}_{a,k}f_2(x)}\right|\right) dx \leq \int_a^b \hat{p}(t)\Phi\left(\left|\frac{f_1(t)}{f_2(t)}\right|\right) dt \tag{7.7}$$

holds for all measurable functions $f_i \in L_1(a,b)$, $(i = 1,2)$.

Proof. Applying Theorem 2.6 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(t) = dt$ and $\hat{k}(x,t)$, is given in (7.3), we get inequality (7.7). □

Theorem 7.8. *Let $\alpha > 0$, $0 < p \leq q < \infty$, $f \in L_1[a,b]$, let u be a weight function on (a,b) and $x \mapsto u(x)\left(\frac{(x-t)^{\frac{\alpha}{k}-1}}{(x-a)^{\frac{\alpha}{k}}}\right)^{\frac{q}{p}}$ is integrable on (a,b). Then for each fixed $t \in (a,b)$, $\hat{w}$ is defined by*

$$\hat{w}(t) := \frac{\alpha}{k}\left[\int_t^b u(x)\left(\frac{(x-t)^{\frac{\alpha}{k}-1}}{(x-a)^{\frac{\alpha}{k}}}\right)^{\frac{q}{p}} dx\right]^{\frac{p}{q}} < \infty.$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left[\int_a^b u(x)\left(\Phi\left(\frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}}\int_a^x (x-t)^{\frac{\alpha}{k}-1}f(t)dt\right)\right)^{\frac{q}{p}}dx\right]^{\frac{1}{q}}$$

$$\le \left[\int_a^b \hat{w}(t)\Phi\left(f(t)\right)dt\right]^{\frac{1}{p}} \tag{7.8}$$

holds for all measurable functions $f:(a,b)\to\mathbb{R}$ such that $Imf \subseteq I$.

Proof. Applying Theorem 2.7 with $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx, d\mu_2(t)=dt$, $\hat{k}(x,t)$, $\hat{K}(x)$ and $\hat{A}_k f(x)$ are given by (7.3), (7.4) and (7.5), respectively, we get inequality (7.8). □

Corollary 7.9. *Let $0<p\le q<\infty$, $f\in L_1[a,b]$ and $I^{\alpha}_{a,k}f$ be the Riemann-Liouville k-fractional integral of order $\alpha>0$. Suppose $\Phi(x)=x^s$, $s>0$, is a convex function and $u(x)=(x-a)^{\frac{q\alpha}{pk}}$ a particular weight function. Then for each fixed $t\in(a,b)$, $\hat{\Upsilon}(t)$ is defined by*

$$\hat{\Upsilon}(t)=\frac{\alpha}{k\left(\frac{q}{p}(\frac{\alpha}{k}-1)+1\right)^{\frac{p}{q}}}(b-t)^{\frac{\alpha}{k}-1+\frac{p}{q}}.$$

Substituting these values in (7.8) and after some calculations we get the inequality

$$\left(\int_a^b \left(I^{\alpha}_{a,k}f(x)\right)^{\frac{qs}{p}}dx\right)^{\frac{1}{q}} \le \frac{(\alpha)^{\frac{1}{p}}(b-a)^{\frac{\alpha s}{pk}+\frac{1}{q}-\frac{1}{p}}}{(k(\Gamma_k(\alpha+k))^s)^{\frac{1}{p}}\left(\frac{q}{p}(\frac{\alpha}{k}-1)+1\right)^{\frac{1}{q}}}\left(\int_a^b f^s(t)dt\right)^{\frac{1}{p}}.$$

Theorem 7.10. *Let $I^{\alpha}_{a,k}f$ be the Riemann-Liouville k-fractional integral of order $\alpha>0$ and let $I^{\alpha}_{a,k}f_2(x)>0$ for every $x\in(a,b)$, u is a weight function on (a,b). Then $\hat{r}$ is defined by*

$$\hat{r}(t):=\frac{f_2(t)}{k\Gamma_k(\alpha)}\int_t^b\frac{u(x)(x-t)^{\frac{\alpha}{k}-1}}{I^{\alpha}_{a,k}f_2(x)}dx<\infty.$$

If $\Phi:(0,\infty)\times(0,\infty)\to\mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\Phi\left(\left|\frac{I^{\alpha}_{a,k}f_1(x)}{I^{\alpha}_{a,k}f_2(x)}\right|,\left|\frac{I^{\alpha}_{a,k}f_3(x)}{I^{\alpha}_{a,k}f_2(x)}\right|\right)dx\le\int_a^b\hat{r}(t)\Phi\left(\left|\frac{f_1(t)}{f_2(t)}\right|,\left|\frac{f_3(t)}{f_2(t)}\right|\right)dt \tag{7.9}$$

holds for measurable functions $f_i\in L_1(a,b)$, $(i=1,2,3)$.

Proof. Applying Theorem 2.8 with $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx, d\mu_2(t)=dt$ and $\hat{k}(x,t)$, is given by (7.3), we get inequality (7.9). □

Remark 7.11. If we choose $k=1$, in inequalities (7.2), (7.7), (7.8) and (7.9), we get results for the Riemann-Liouville fractional integral operator.

References

[1] N. Abel, *Solution de quelques problèmes à l'aide d'intégrales définies*, In L. Sylow & S. Lie (Eds.), Oeuvres complètes de Niels Henrik Abel: Nouvelle édition (Cambridge Library Collection - Mathematics, pp. 11–27). Cambridge: Cambridge University Press, 2012. https://doi.org/10.1017/CBO9781139245807.003

[2] E. Adeleke, A. Čižmešija, J. Oguntuase, L. E. Persson, and D. Pokaz, *On a new class of Hardy-type inequalities*, J. Inequal. Appl. **2012** (2012), 259, 19 pp. https://doi.org/10.1186/1029-242X-2012-259

[3] M. Ali Özarslan and E. Özergin, *Some generating relations for extended hypergeometric functions via generalized fractional derivative operator*, Math. Comput. Modelling **52** (2010), no. 9-10, 1825–1833. https://doi.org/10.1016/j.mcm.2010.07.011

[4] G. A. Anastassiou, *Fractional differentiation inequalities*, Springer, Dordrecht, 2009. https://doi.org/10.1007/978-0-387-98128-4

[5] D. Baleanu, P. Agarwal, R. K. Parmar, M. M. Alquarashi, and S. Salahshour, *Extension of the fractional derivative operator of the Riemann-Liouville*, J. Nonlinear Sci. Appl. **10** (2017), no. 6, 2914–2924. https://doi.org/10.22436/jnsa.010.06.06

[6] A. Čižmešija, S. Hussain, and J. Pečarić, *Some new refinement of Hardy and Pólya-Knopp's inequalities*, Banach. J. Math. Anal. **4** (2010), 122–145.

[7] A. Čižmešija, K. Krulić, and J. Pečarić, *Some new refined Hardy-type inequalities with kernels*, J. Math. Inequal. **4** (2010), no. 4, 481–503. https://doi.org/10.7153/jmi-04-44

[8] L. Curiel and L. Galué, *A generalization of the integral operators involving the Gauss' hypergeometric function*, Rev. Técn. Fac. Ingr. Univ. Zulia **19** (1996), no. 1, 17–22.

[9] R. Diaz and E. Pariguan, *On hypergeometric functions and Pochhammer k-symbol*, Divulg. Mat. **15** (2007), no. 2, 179–192.

[10] G. Farid, K. Krulić, and J. Pečarić, *On refinement of Hardy type inequalities via superquadratic functions*, Sarajevo J. Math. **7** (2011), no. 2, 163–175.

[11] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, translated from the Russian, translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, seventh edition, Elsevier/Academic Press, Amsterdam, 2007.

[12] G. H. Hardy, *Notes on some points in the integral calculus* (LX): *An inequality between integrals*, Messenger Math. **54** (1925), 150–156.

[13] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, reprint of the 1952 edition, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1988.

[14] R. Herrmann, *Fractional Calculus*, second edition, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2014. https://doi.org/10.1142/8934

[15] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Publishing Co., Inc., River Edge, NJ, 2000. https://doi.org/10.1142/9789812817747

[16] ———, *Threefold introduction to fractional derivatives*, Anomalous transport: Foundations and Applications (2008), 17–73.

[17] S. Iqbal, K. K. Himmelreich, and J. Pečarić, *On an inequality of G. H. Hardy for convex function with fractional integrals and fractional derivatives*, Tbil. Math. J. **6** (2013), 1–12.

[18] S. Iqbal, K. Krulić, and J. Pečarić, *On an inequality of H. G. Hardy*, J. Inequal. Appl. **2010** (2010), Art. ID 264347, 23 pp. https://doi.org/10.1155/2010/264347

[19] ———, *On an inequality for convex functions with some applications on fractional derivatives and fractional integrals*, J. Math. Inequal. **5** (2011), no. 2, 219–230. https://doi.org/10.7153/jmi-05-20

[20] S. Iqbal, J. Pečarić, M. Samraiz, and Ž. Tomovski, *Hardy-type inequalities for genaralized feactional integral opeartor*, Tib. Math. J. **10** (2017), no. 1, 75–90.

[21] U. N. Katugampola, *New approach to a generalized fractional integral*, Appl. Math. Comput. **218** (2011), no. 3, 860–865. https://doi.org/10.1016/j.amc.2011.03.062

[22] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, **204**, Elsevier Science B.V., Amsterdam, 2006.
[23] K. Krulić, J. Pečarić, and L.-E. Persson, *Some new Hardy type inequalities with general kernels*, Math. Inequal. Appl. **12** (2009), no. 3, 473–485. https://doi.org/10.7153/mia-12-36
[24] A. Kufner, L.-E. Persson, and N. Samko, *Weighted Inequalities of Hardy Type*, second edition, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017. https://doi.org/10.1142/10052
[25] J. A. Machado, *And i say to myself: what a fractional world!*, Fractional Calculus and Applied Analysis **14** (2011), no. 4, 635–654.
[26] K. Mehrez and Z. Tomovski, *On a new* (p, q)*-Mathieu type power series and its applications*, submitted.
[27] S. Mubeen and G. M. Habibullah, k*-fractional integrals and application*, Int. J. Contemp. Math. Sci. **7** (2012), no. 1-4, 89–94.
[28] S. Mubeen and S. Iqbal, *Grüss type integral inequalities for generalized Riemann-Liouville* k*-fractional integrals*, J. Inequal. Appl. **2016** (2016), Paper No. 109, 13 pp. https://doi.org/10.1186/s13660-016-1052-x
[29] K. S. Nisar, G. Rehman, and Z. Tomovski, *On a certain extension of the Riemann-Liouville fractional derivative operator*, Commun. Korean. Math. Soc. (Accepted: https://doi.org/10.4134/CKMS.c180140)
[30] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, 1974.
[31] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering, **198**, Academic Press, Inc., San Diego, CA, 1999.
[32] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional integrals and derivatives*, translated from the 1987 Russian original, Gordon and Breach Science Publishers, Yverdon, 1993.
[33] M. Shadab, S. Jabeen, and J. Choi, *An extension of beta function and its application*, Far East J. Math. Sci. **103** (2018), no. 1, 235–251.

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