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ESTIMATION OF A MODIFIED INTEGRAL ASSOCIATED WITH A SPECIAL FUNCTION KERNEL OF FOX'S *H*-FUNCTION TYPE

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Abstract. In this article, we discuss classes of generalized functions for certain modified integral operator of Bessel-type involving Fox's H-function kernel. We employ a known differentiation formula of Fox's H-function to obtain the definition and properties of the distributional modified Bessel-type integral. Further, we derive a smoothness theorem for its kernel in a complete countably multi-normed space. On the other hand, using an appropriate class of convolution products, we derive axioms and establish spaces of modified Boehmians which are generalized distributions. On the defined spaces, we introduce addition, convolution, differentiation and scalar multiplication and further properties of the extended integral.

1. Introduction and preliminaries

Generalized functions or distributions ([12, 19]), a relatively recent mathematical approach to classical Fourier analysis, are continuous linear forms on a space of infinitely differentiable functions. The utility of distributions arises from the fact that they are generalized functions, which allows operations of differentiation and convolution to act on objects that fail to be functions. Distributions not only opened up a new area of research in physics and engineering but also helped to promote the development of mathematical disciplines in partial differential equations in addition to being very useful in operational calculus, transformation theory and functional analysis. In order to get an adequate theory of generalized functions, we perform linear operations on testing functions without emerging from the testing function class. Moreover, we assign some concept of convergence to the given testing function classes.

Regular operators form a subalgebra of Mikusiński operators. They include only such functions whose supports are bounded from the left and, at the same time, do not have any restriction on their support. In contrast to the space of

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Schwartz distributions, the construction of a Boehmian space is algebraic and similar to the construction of the field of quotients.

Boehmians are equipped with a topology defined in a canonical way, but the properties of their assigned topology can differ significantly for different spaces of Boehmians. The structure necessary for constructing a Boehmian space consists of a vector space a, a semigroup $(b, \bullet)$ and an operation $\star$ such that for each $x, y \in a$ and $z_1, z_2, \in b, \alpha \in \mathbb{C}$, we have $x \star (z_1 \bullet z_2) = (x \star z_1) \star z_2$, $(x + y) \star z_1 = x \star z_1 + y \star z_1$ and $\alpha (x \bullet z_1) = (\alpha x \bullet z_1)$. Further, for $n \in N$, N is the set of natural numbers, there should be a family of delta sequences $\Delta \subset b$ such that $(z_n \bullet t_n) \in \Delta$ for every $(z_n), (t_n) \in \Delta$. Let

$$Q = \{(x_n, z_n) : x_n \star z_m = x_m \star z_n, x_n \in a, (z_n) \in \Delta \ \ (\forall m, n \in N)\}.$$

In Q, the quotients (x_n, z_n) and (y_n, t_n) are said to be equivalent if $x_n \star t_m = y_m \star z_n$, $\forall m, n \in N$. The relation $\sim$ between (x_n, z_n) and (y_n, t_n) is an equivalence relation which splits Q into equivalence classes. The space of all equivalence classes in Q is denoted by $\beta(\star, \bullet)$. For the convenience of the reader, a typical element of $\beta(\star, \bullet)$ will be written as $\frac{x_n}{z_n}$. Elements of $\beta(\star, \bullet)$ are called Boehmians.

A canonical embedding between a and $\beta(\star, \bullet)$ can be expressed as $x \to \frac{x \star z_n}{z_n}$. If $\frac{x_n}{z_n} \in \beta(\star, \bullet)$ and $z \in b$, then $\frac{x_n}{z_n} \star z = \frac{x_n \star z}{z_n}$. Further $\frac{x_n}{z_n} \star x = \frac{x_n \star x}{z_n}$. Using such idea of generalization of Boehmians, various Boehmian spaces have been considered in [1–8, 14, 15, 17], to mention but a few.

In [10], Fox introduces the H-function as a generalization of the MacRobert's E-function, Wright's generalized hypergeometric function and the Meijer's G-function as well. He also investigates the most generalized Fourier kernel associated with the H-function and establishes many special cases of its kernel function. Fox's H-functions are defined in terms of the Mellin-Barnes type integral in the following manner:

$$H^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} (a_1, \hat{a}_1), \ldots, (a_p, \hat{a}_p) \\ \left(b_1, \hat{b}_1\right), \ldots, \left(b_q, \hat{b}_q\right) \end{matrix}\right) = \frac{1}{2\pi i}\int_L X(\xi)\, z^{-\xi} \lambda\xi, \tag{1}$$

provided

$$X(\xi) = \frac{\prod_{j=1}^{m} \Gamma\left(b_j + \hat{b}_j \xi\right) \prod_{i=1}^{n} \Gamma(1 - a_j - \hat{a}_j \xi)}{\prod_{j=m+1}^{q} \Gamma\left(1 - b_j - \hat{b}_j \xi\right) \prod_{i=n+1}^{p} \Gamma(a_j + \hat{a}_j \xi)},$$

where an empty product is always interpreted as unity: $m, n, p, q \in N_0, N_0 = N \cup \{0\}$ with $0 \le n \le p, 1 \le m \le q, \hat{b}_j \in N_+, N_+ = N \cup \{\infty\}, a_i, b_j \in N$ or $C\,(i = 1, \ldots, p; j = 1, \ldots, q)$ and that $\hat{a}_i (b_j + k) \neq \hat{b}_j (a_i - \ell - 1), k, \ell \in N_0; i = 1, \ldots, n; j = 1, \ldots, m$, L is a suitable contour which separates all poles of $\Gamma\left(b_j + \hat{b}_j \xi\right) (j = 1, \ldots, m)$ from those of $\Gamma(1 - a_j - \hat{a}_j \xi)$, $i = 1, \ldots, n$, C being the set of complex numbers. The integral (1) converges absolutely and defines an analytic function in the sector

$$|\arg(z)| < \frac{1}{2}\pi\varphi,$$

where

$$\varphi = \sum_{i=1}^{n} \hat{a}_i - \sum_{i=n+1}^{p} \hat{a}_i + \sum_{j=1}^{m} \hat{b}_j - \sum_{j=m+1}^{q} > 0,$$

$z = 0$ being excluded.

It is not out of place to be mentioned here that the importance of the H-functions has significantly been increased for researchers in science and engineering topics, especially in the area of fractional reaction, fractional diffusion and anomalous diffusion problems in complex systems. On the other-hand, the trends of research in many branches of applied mathematics and technology have increased toward applications of some integral transforms of H-function kernels.

Some properties associated with the Fox's H-function can be given as follows.

(i) A change of variable: Let c be a positive constant, then

$$H_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} (a_1, \hat{a}_1), \ldots, (a_p, \hat{a}_p) \\ \left(b_1, \hat{b}_1\right), \ldots, \left(b_q, \hat{b}_q\right) \end{matrix}\right) = cH_{p,q}^{m,n}\left(x^c \,\middle|\, \begin{matrix} (a_1, \hat{a}_1), \ldots, (a_p, \hat{a}_p) \\ \left(b_1, \hat{b}_1\right), \ldots, \left(b_q, \hat{b}_q\right) \end{matrix}\right).$$

(ii) Reciprocity.

$$H_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} (a_1, \hat{a}_1), \ldots, (a_p, \hat{a}_p) \\ \left(b_1, \hat{b}_1\right), \ldots, \left(b_q, \hat{b}_q\right) \end{matrix}\right) = H_{p,q}^{m,n}\left(\frac{1}{x} \,\middle|\, \begin{matrix} \left(1-b_1, \hat{b}_1\right), \ldots, \left(1-b_q, \hat{b}_q\right) \\ (1-a_1, \hat{a}_1), \ldots, (1-a_p, \hat{a}_p) \end{matrix}\right).$$

(iii) For small x,

$$H_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} (a_1, \hat{a}_1), \ldots, (a_p, \hat{a}_p) \\ \left(b_1, \hat{b}_1\right), \ldots, \left(b_q, \hat{b}_q\right) \end{matrix}\right) = O\left(|x|^p\right),$$

$p = \min\limits_{1\le j\le m} \mathrm{Re}\frac{b_j}{\hat{b}_j}$, and for large x,

$$H_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} (a_1, \hat{a}_1), \ldots, (a_p, \hat{a}_p) \\ \left(b_1, \hat{b}_1\right), \ldots, \left(b_q, \hat{b}_q\right) \end{matrix}\right) = O\left(|x|^Q\right),$$

where $Q = \max\limits_{1\le j\le n} \mathrm{Re}\frac{a_j-1}{\hat{a}_j}$. We refer to [1, 9–11, 13, 16, 18] for more details.

This article consists of four sections. In Section 2, we show that the kernel function of the modified Bessel-type integral belongs to a Fréchet space of integrable functions and define the distributional integral as an analytic function. In Section 3 we introduce convolutions, approximating identities and prove various axioms for establishing the generalized distribution spaces. In Section 4 we extend our cited integral and get some elementary properties.

2. The modified Bessel-type integral on Zemanian spaces

Let $\beta > 0, x > 0, \sigma \in R; \mathrm{Re}(\gamma) > \frac{1}{\beta} - 1, \mathrm{Re}(z) > 0, N_0 =: N \cup \{0\}$ and $\mu \in C$. Then Glaeske et al. [14] have investigated the Bessel-type integral

$$\left(l_{\gamma,\sigma}^{(\beta)}\varphi\right)(x) = \int_0^\infty \lambda_{\gamma,\sigma}^{(\beta)}(xt)\,\varphi(t)\,dt, \tag{2}$$

on the Fréchet spaces

$$F_{\rho,\mu} = \left\{\varphi \in C_0^\infty(R_+) : x^k \frac{d^k}{dx^k}\left(x^{-\mu}\varphi(x)\right) \in L^\rho\left(R^+\right), 1 \le \rho < \infty, k \in N_0\right\},$$

$$F_{\infty,\mu} = \left\{\varphi \in C_0^\infty(R_+) : x^k \frac{d^k}{dx^k}\left(x^{-\mu}\varphi(x)\right) \to 0 \text{ as } x \to 0 \text{ and } x \to \infty, k \in N_0\right\},$$

where

$$\lambda_{\gamma,\sigma}^{(\beta)}(z) = \frac{\beta}{\Gamma\left(\gamma + 1 - \frac{1}{\beta}\right)} \int_1^\infty \left(t^\beta - 1\right)^{\gamma - \frac{1}{\beta}} t^\sigma e^{-zt}\,dt \tag{3}$$

is the generalized Macdonald function $K_{-\gamma}(z)$. They have further derived composition formulas of the transform $l_{\gamma,\sigma}^{(\beta)}$ with the left-side and right-side Liouville fractional integrals.

In this article, we motivate the $l_{\gamma,\sigma}^{(\beta)}$ integral operator in terms of the Fox's H function as follows.

$$\left(l_{\gamma,\sigma}^{(\beta)} f\right)(x) = \int_0^\infty H_{1,2}^{2,0}\left(xt \,\middle|\, \begin{matrix} \left(1 - \frac{(\sigma+1)}{\beta}\right), \frac{1}{\beta} \\ (0,1), \left(-\gamma - \frac{\sigma}{\beta}, \frac{1}{\beta}\right) \end{matrix}\right) f(t)\,dt. \tag{4}$$

F_c^d will denote the countably complete multinormed space of all smooth functions $\varphi(t)$ $(0 < t < \infty)$ such that the set of seminorms

$$\bar{\delta}_k(\varphi) = \sup_{0<t<\infty} \left|\lambda_{c,d}(\log t)\,(tD_t)^k \sqrt{t}\varphi(t)\right|$$

is finite, where

$$\lambda_{c,d}(\log t) = \begin{cases} t^c, & 1 \le t \le \infty, \\ t^d, & 0 < t < 1. \end{cases}$$

The dual space of F_c^d is denoted by $\acute{F}_c^d$.

Theorem 1. *Let $c + \frac{1}{2} + Q_1 < 0$ and $d + \frac{1}{2} + P_1 > 0$. Then we have*

$$\lambda_{\gamma,\sigma}^{(\beta)}(xt) = H_{1,2}^{2,0}\left(xt \,\middle|\, \begin{matrix} \left(1 - \frac{(\sigma+1)}{\beta}\right), \frac{1}{\beta} \\ (0,1), \left(-\gamma - \frac{\sigma}{\beta}, \frac{1}{\beta}\right) \end{matrix}\right)$$

belongs to F_c^d for every $\beta > 0, \mathrm{Re}(\gamma) > \frac{1}{\beta} - 1; \sigma \in R$.

Proof. Let the hypothesis of the theorem be satisfied for some β, γ, β and σ. Then, we write

$$(tD_t)^k \sqrt{t}\lambda_{\gamma,\sigma}^{(\beta)}(xt) = \sum_{r=0}^{k} D^{k-r}\sqrt{t}D^r H_{1,2}^{2,0}\left(xt \left| \begin{array}{l} \left(1-\frac{(\sigma+1)}{\beta}\right), \frac{1}{\beta} \\ (0,1), \left(-\gamma-\frac{\sigma}{\beta}, \frac{1}{\beta}\right) \end{array} \right.\right).$$

Hence, by Leibniz's rule we get

$$(5) \quad (tD_t)^k \sqrt{t}\lambda_{\gamma,\sigma}^{(\beta)}(xt) = \sum_{r=0}^{k} A_r t^{\frac{1}{2}+r} D_t^r H_{1,2}^{2,0}\left(xt \left| \begin{array}{l} \left(1-\frac{(\sigma+1)}{\beta}\right), \frac{1}{\beta} \\ (0,1), \left(-\gamma-\frac{\sigma}{\beta}, \frac{1}{\beta}\right) \end{array} \right.\right).$$

Making use of the differentiation formula [11, (2.2.5), p. 31]

$$\begin{aligned} &\left(\frac{d}{dz}\right)^k H_{p,q}^{m,n}\left((cz+d)^\sigma \left| \begin{array}{l} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{array} \right.\right) \\ &= \frac{c^k}{(cz+d)^k} H_{p,q+1}^{m,n+1}\left((cz+d)^\sigma \left| \begin{array}{l} (0,\sigma)\,(a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q}, (k,\sigma) \end{array} \right.\right). \end{aligned}$$

Eq. (5) can be written as

$$\begin{aligned} (tD_t)^k \sqrt{t}\lambda_{\gamma,\sigma}^{(\beta)}(xt) &= \sum_{r=0}^{k} t^k A_r t^{\frac{1}{2}-(k-r)} D_t^r H_{1,2}^{2,0}\left(xt \left| \begin{array}{l} \left(1-\frac{(\sigma+1)}{\beta}\right), \frac{1}{\beta} \\ (0,1), \left(-\gamma-\frac{\sigma}{\beta}, \frac{1}{\beta}\right) \end{array} \right.\right) \\ &= \sum_{r=0}^{k} t^k A_r \sqrt{t} H_{2,3}^{2,1}\left(xt \left| \begin{array}{l} (0,1)\left(1-\frac{(\sigma+1)}{\beta}\right), \frac{1}{\beta} \\ (0,1), \left(-\gamma-\frac{\sigma}{\beta}, \frac{1}{\beta}\right), (r,1) \end{array} \right.\right). \end{aligned}$$

Setting

$$\widehat{H}_{2,3}^{2,1}(xt) = H_{2,3}^{2,1}\left(xt \left| \begin{array}{l} (0,1)\left(1-\frac{(\sigma+1)}{\beta}\right), \frac{1}{\beta} \\ (0,1), \left(-\gamma-\frac{\sigma}{\beta}, \frac{1}{\beta}\right), (r,1) \end{array} \right.\right),$$

yields

(6)

$$\sup_{0<t<\infty}\left|\lambda_{c,d}(\log t)(tD_t)^k\left(\sqrt{t}\lambda_{\gamma,\sigma}^{(\beta)}(xt)\right)\right| = \sup_{0<t<\infty}\left|\lambda_{c,d}(\log t)B_r\sqrt{t}\widehat{H}_{2,3}^{2,1}(xt)\right|.$$

Therefore, for large values of t, where $c+\frac{1}{2}+Q<0$, Eq. (6) indeed gives

$$(7) \qquad \sup_{1\le t<\infty}\left|t^c B_r\sqrt{t}\widehat{H}_{2,3}^{2,1}(xt)\right| < \infty$$

and, for small values of t, where $d+\frac{1}{2}+P>0$, Eq. (6) gives

$$(8) \qquad \sup_{0\le t<1}\left|t^d B_r\sqrt{t}\widehat{H}_{2,3}^{2,1}(xt)\right| < \infty.$$

Hence, the proof is finished. □

On account of Theorem 1, the distributional $l_{\gamma,\sigma}^{(\beta)}$ transform of $f \in \acute{F}_c^d$ is given as follows.

Definition 2. If $f \in \acute{F}_c^d$, then the generalized $\widetilde{l}_{\gamma,\sigma}^{(\beta)}$ transform can be extended to $\acute{F}_c^d$ as

$$\left(\widetilde{l}_{\gamma,\sigma}^{(\beta)} f\right)(x) = \left\langle f(t), \lambda_{\gamma,\sigma}^{(\beta)}(xt)\right\rangle. \tag{9}$$

Definition 2 is well-defined by Theorem 1. Linearity and analyticity of $\widetilde{l}_{\gamma,\sigma}^{(\beta)}$ can be read as follows.

Remark 3. If $f, g \in \acute{F}_c^d, \alpha \in C$, then we have

$$\left(\widetilde{l}_{\gamma,\sigma}^{(\beta)} \alpha (f+g)\right)(x) = \alpha \left\langle f(t), \lambda_{\gamma,\sigma}^{(\beta)}(xt)\right\rangle + \alpha \left\langle g(t), \lambda_{\gamma,\sigma}^{(\beta)}(xt)\right\rangle,$$
$$D_x \left(\widetilde{l}_{\gamma,\sigma}^{(\beta)} f\right)(x) = \left\langle f(t), D_x \left(\lambda_{\gamma,\sigma}^{(\beta)}(xt)\right)\right\rangle.$$

This remark can be easily checked by following elementary techniques of distributions. Therefore we omit the details.

3. Boehmians for the modified Bessel-type integral

In this section, we derive two mandatory spaces of Boehmians for our integral transform by defining an appropriate convolution product; the necessary ingredient for the construction of a Boehmian space. We usually utilize the ordinary definition of the Mellin-type convolution defined by

$$(\phi * \psi)(y) = \int_0^\infty \varphi\left(x^{-1}y\right)\psi(x)\, d\mu(x) \quad \left(d\mu(x) =: \frac{dx}{x}\right) \tag{10}$$

and make use of the familiar properties

$$\begin{aligned} &\text{(i) commutativity: } \varphi * \psi = \psi * \varphi, \\ &\text{(ii) associativity: } \varphi * (\psi_1 * \psi_2) = (\varphi * \psi_1) * \psi_2. \end{aligned} \tag{11}$$

The convolution product which in this case coincides with (10) can be identified as follows

$$(\varphi \odot \psi)(y) = \int_0^\infty \varphi(xy)\psi(x)\, dx, \tag{12}$$

provided the integral exists.

We also request that the F_c^d elements be supplied with an absolute integrability condition to yield the space $F_c^{d,a}$. The subspace of $F_c^{d,a}$ consisting of smooth functions of compact supports on I is denoted by D_I. By exchanging $\star$ by $\odot$ and $\bullet$ by $*$, we commence the proofs of the necessary axioms for constructing the Boehmian space $\beta(\odot, *) := \beta\left(\left(F_c^{d,a}, \odot\right), (D_I, *)\right)$ with some mandatory theorems.

Theorem 4. *Let $\varphi \in F_c^{d,a}$ and $\psi \in D_I$. Then $\varphi \odot \psi \in F_c^{d,a}$.*

Proof. Let $\varphi \in F_c^{d,a}$ and $\psi \in D_I$ be given. Then, we, by Eq. (12), write

$$\left|\lambda_{c,d}(\log t)(tD_t)^k \sqrt{t}(\varphi \odot \psi)(t)\right| \leq \int_a^b |\psi(x)| \left|\lambda_{c,d}(\log t)\left(tD_t^k\right)\sqrt{t}(tx)\right| dx$$

$$\leq A\bar{\delta}_k(\phi)\int_a^b |\psi(x)|\, dx, \tag{13}$$

where $[a, b]$ is a closed interval containing the support of $\psi(A > 0)$. Since $\int_a^b |\psi(x)|\, dx < M$ for some positive constant M, we, by allowing the supremum to range over all $t \in (0, \infty)$, have $\bar{\delta}_k(\varphi \odot \psi) \leq M\bar{\delta}_k(\varphi) A < \infty$ for every choice of $k \in N_0$.

The proof of the theorem is finished. □

Theorem 5. *If $\varphi \in F_c^{d,a}$ and $\psi_1, \psi_2 \in D_I$, then we have*

$$\varphi \odot (\psi_1 * \psi_2) = (\varphi \odot \psi_1) \odot \psi_2 \in F_c^{d,a}. \tag{14}$$

Proof. As the right hand side of Eq. (14) lies in $F_c^{d,a}$, it will be sufficient for (14) to be verified in terms of equality for every choice of ψ_1, ψ_2 and φ. Now, Eq. (12) reads

$$(\varphi \odot (\psi_1 * \psi_2))(y) = \int_0^\infty \varphi(yx)(\psi_1 * \psi_2)(x)\, dx.$$

Hence, Eq. (10) and Fubini's theorem imply

$$(\varphi \odot (\psi_1 * \psi_2))(y) = \int_0^\infty \psi_2(t) \int_0^\infty \varphi(yx)\, \psi_1\left(t^{-1}x\right) d\mu(t)\, dx.$$

By changing the variables, $tx^{-1} = z$, it follows

$$(\varphi \odot (\psi_1 * \psi_2))(y) = \int_0^\infty \psi_2(t) \int_0^\infty \varphi(ytz)\, \psi_1(z)\, dtdz.$$

Once again, making use of Eq. (12) completes the proof of the theorem. □

Theorem 6. *Let $\varphi, (\varphi_n) \in F_c^{d,a}$, $\psi, \psi_1, \psi_2 \in D_I$ and $\alpha \in C$. Then, we have*

(i) $\bar{\delta}_k(\varphi_n - \varphi) \to 0$ *as* $n \to \infty \Rightarrow \bar{\delta}_k(\varphi_n \odot \psi - \varphi \odot \psi) \to 0$ *as* $n \to \infty$ *in* $F_c^{d,a}$,

(ii) $\varphi \odot (\psi_1 + \psi_2) = \varphi \odot \psi_1 + \varphi \odot \psi_2$,

(iii) $\alpha(\varphi \odot \psi_1) = \alpha(\varphi \odot \psi_1)$.

Proof of this theorem follows from applying simple integral calculus.

Now we define a set of delta sequences as follows.

Definition 7. By $(\Delta, *)$ we denote the set of sequences (δ_n) satisfying the following properties: $(\delta_n) \in D_I\ (\forall n \in N)$, $\int_0^\infty \delta_n(x)\, dx = 1$, $|\delta_n| < M\ (\forall n \in N)$ and $\mathrm{supp}\delta_n(x) \subseteq (a_n, b_n)$ $(a_n, b_n \to 0$ as $n \to \infty)$.

Theorem 8. $(\delta_n * \epsilon_n) \in (\Delta, *)$ *for every* (δ_n),$(\epsilon_n) \in (\Delta, *)$.

Proof of this theorem follows by the commutativity of $*$ and Definition 7. Details are therefore deleted.

We prove the following relation.

Theorem 9. *Let $(\delta_n) \in (\Delta, *)$ and $\varphi \in F_c^{d,a}$. Then, $\bar{\delta}_k(\varphi \odot \delta_n - \varphi) \to 0$ in $F_c^{d,a}$ as $n \to \infty$.*

Proof. Let $(\delta_n) \in (\Delta, *)$ and $\varphi \in F_c^{d,a}$ be given, then by Definition 7 we have

$$\begin{aligned} &\left|\lambda_{c,d}(\log t)(tD_t)^k \sqrt{t}(\varphi \odot \delta_n - \varphi)(t)\right| \\ \leq &\left|\lambda_{c,d}(\log t)(tD_t)^k \sqrt{t}\right| \int_0^\infty (|\varphi(xt) - \varphi(t)|) |\delta_n(x)| \, dx. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\left|\lambda_{c,d}(\log t)(tD_t)^k \sqrt{t}(\varphi \odot \delta_n - \varphi)(t)\right| \\ \leq &\int_{a_n}^{b_n} |\delta_n| \left|\lambda_{c,d}(\log t)(tD_t)^k \sqrt{t}(\varphi_x(t))\right| dx, \end{aligned} \tag{15}$$

where $\varphi_x(t) = \varphi(tx) - \varphi(t) \in F_c^{d,a}$. Hence, by using Definition 7 we get

$$\left|\lambda_{c,d}(\log t)(tD_t)^k \sqrt{t}(\varphi \odot \delta_n - \varphi)(t)\right| \leq M(a_n, b_n)\, \bar{\delta}_k(\varphi).$$

Hence, this yields

$$\bar{\delta}_k(\varphi \odot \delta_n - \varphi) \to 0$$

as $n \to \infty$.

Hence the theorem is established. $\square$

Finally, we demand the following theorem to be established.

Theorem 10. *Let $\beta > 0, \mathrm{Re}(\gamma) > \frac{1}{\beta} - 1$ and $\sigma \in R$. Then the followings hold:*

(i) $l_{\gamma,\sigma}^{(\beta)}\varphi \in F_c^{d,a}$ *for every* $\varphi \in F_c^{d,a}$,

(ii) $l_{\gamma,\sigma}^{(\beta)}(\varphi_2 * \varphi_1)(x) = \left(\left(l_{\gamma,\sigma}^{(\beta)}\varphi_2\right) \odot \varphi_1\right)(x)$ $(x > 0)$,

for every $\varphi_2 \in F_c^{d,a}$ *and* $\varphi_1 \in D_I$.

Proof. Proof of (i). By Eq. (4) we write

$$\begin{aligned} &\left|\lambda_{c,d}(\log t)(tD_t)^k \sqrt{t}\left(l_{\gamma,\sigma}^{(\beta)}\varphi\right)(t)\right| \\ \leq &\int_0^\infty |\varphi(x)| \left|\lambda_{c,d}(\log t)(tD_t)^k \sqrt{t} H_{1,2}^{2,0}\left(xt \left| \begin{matrix} \left(1 - \frac{(\sigma+1)}{\beta}\right), \frac{1}{\beta} \\ (0,1), \left(-\gamma - \frac{\sigma}{\beta}, \frac{1}{\beta}\right) \end{matrix} \right.\right)\right| dx. \end{aligned}$$

Hence,

$$\left|\lambda_{c,d}(\log t)(tD_t)^k \sqrt{t}\left(l_{\gamma,\sigma}^{(\beta)}\varphi\right)(t)\right| \leq M \int_0^\infty |\varphi(x)| \, dx.$$

By the integrability of φ and by allowing the supremum to range over all $0 < t < \infty$ we get

$$\bar{\delta}_k\left(l_{\gamma,\sigma}^{(\beta)}\varphi\right) \leq MA, \quad (A > 0, A \in R).$$

Proof of (ii). Let the hypothesis be satisfied, then, making use of the defined convolution products and the Fubini's theorem we write

$$l_{\gamma,\sigma}^{(\beta)}\left(\varphi_2 * \varphi_1\right)(x)=\int_0^{\infty} \lambda_{\gamma,\sigma}^{(\beta)}(xt) \int_0^{\infty} \varphi_2\left(y^{-1} t\right) \varphi_1(t)\, d\mu(y)\, dt.$$

Setting variables as $y^{-1}t = z$ yields

$$l_{\gamma,\sigma}^{(\beta)}\left(\varphi_2 * \varphi_1\right)(x)=\int_0^{\infty} \varphi_1(y) \int_0^{\infty} \lambda_{\gamma,\sigma}^{(\beta)}(xyz)\, \varphi_2(z)\, dz dy.$$

This completes the second part of the theorem.

Hence the theorem is completely proved. □

By proving the above axioms the Boehmian space $\beta(\odot,*)$ is established. Similarly, we generate the similar space $\beta(*,*) := \beta\left(\left(F_c^{d,a},*\right),\left(D_I,*\right)\right)$ with the set $(\Delta,*)$, where sum, multiplication by a scalar, $\odot$ and differentiation in $\beta(\odot,*)$ are respectively defined as

$$\begin{aligned}
\frac{f_n}{\delta_n}+\frac{g_n}{\psi_n} &= \frac{f_n \odot \psi_n + g_n \odot \delta_n}{\delta_n * \psi_n}, \alpha \frac{f_n}{\delta_n} = \frac{\alpha f_n}{\delta_n},\\
\frac{f_n}{\delta_n} \odot \frac{g_n}{\psi_n} &= \frac{f_n \odot g_n}{\delta_n \odot \psi_n}, \mathcal{D}^{\alpha} \frac{f_n}{(\delta_n)} = \frac{\mathcal{D}^{\alpha} f_n}{\delta_n},
\end{aligned}$$

where $\alpha \in C$. Whereas, sum, multiplication by a scalar, $*$ and differentiation in $\beta(*,*)$ can respectively be defined as

$$\begin{aligned}
\frac{f_n}{\delta_n}+\frac{g_n}{\psi_n} &= \frac{f_n * \psi_n + g_n * \delta_n}{\delta_n * \psi_n}, \alpha \frac{f_n}{\delta_n} = \frac{\alpha f_n}{\delta_n},\\
\frac{f_n}{\delta_n} * \frac{g_n}{\psi_n} &= \frac{f_n * g_n}{\delta_n * \psi_n}, \mathcal{D}^{\alpha} \frac{f_n}{(\delta_n)} = \frac{\mathcal{D}^{\alpha} f_n}{\delta_n},
\end{aligned}$$

where $\alpha \in C$.

4. $l_{\gamma,\sigma}^{(\beta)}$ of a Boehmian

In view of above detailed analysis, the $l_{\gamma,\sigma}^{(\beta)}$ transform of a Boehmian $\frac{\gamma_n}{\delta_n}$ in $\beta(*,*)$ is the Boehmian given as

$$L_{\gamma,\sigma}^{(\beta)}\left(\frac{\gamma_n}{\delta_n}\right)=\frac{l_{\gamma,\sigma}^{(\beta)} \gamma_n}{\delta_n}. \tag{16}$$

The right-hand side of Eq. (16) is, indeed, well-defined by Theorem 10.

Linearity of $L_{\gamma,\sigma}^{(\beta)}$ can be read as follows: Let $\frac{\theta_n}{\epsilon_n}, \frac{\gamma_n}{\delta_n} \in \beta(*,*)$ $(\forall n \in N)$. Then, by Theorem 10, we have

$$L_{\gamma,\sigma}^{(\beta)}\left(\frac{\theta_n}{\epsilon_n}+\frac{\gamma_n}{\delta_n}\right)=L_{\gamma,\sigma}^{(\beta)}\left(\frac{\theta_n * \delta_n+\gamma_n * \epsilon_n}{\epsilon_n * \delta_n}\right)=\frac{l_{\gamma,\sigma}^{(\beta)} \theta_n \odot \delta_n + l_{\gamma,\sigma}^{(\beta)} \gamma_n \odot \epsilon_n}{\epsilon_n * \delta_n}.$$

This can be read as

$$L_{\gamma,\sigma}^{(\beta)}\left(\frac{\theta_n}{\epsilon_n}+\frac{\gamma_n}{\delta_n}\right)=L_{\gamma,\sigma}^{(\beta)} \frac{\theta_n}{\epsilon_n}+L_{\gamma,\sigma}^{(\beta)} \frac{\gamma_n}{\delta_n} \ (\forall n \in N).$$

Let $\Omega \in \mathbb{C}$, then of course $\Omega L_{\gamma,\sigma}^{(\beta)} \frac{\theta_n}{\epsilon_n} = \Omega \frac{l_{\gamma,\sigma}^{(\beta)} \theta_n}{\epsilon_n} = \frac{l_{\gamma,\sigma}^{(\beta)} (\Omega\theta_n)}{\epsilon_n}$. Hence, we are lead to write

$$\Omega L_{\gamma,\sigma}^{(\beta)} \frac{\theta_n}{\epsilon_n} = L_{\gamma,\sigma}^{(\beta)} \left(\Omega \frac{\theta_n}{\epsilon_n} \right) \quad (\forall n \in N) .$$

The proof is finished.

Theorem 11. *The operator $L_{\gamma,\sigma}^{(\beta)}$ is a one-one mapping from $\beta(*,*)$ onto $\beta(\odot,*)$.*

Proof. Assume $L_{\gamma,\sigma}^{(\beta)} \frac{\gamma_n}{\delta_n} = L_{\gamma,\sigma}^{(\beta)} \frac{\theta_n}{\epsilon_n}$. Then $\frac{l_{\gamma,\sigma}^{(\beta)} \gamma_n}{\delta_n} = \frac{l_{\gamma,\sigma}^{(\beta)} \theta_n}{\epsilon_n} (\forall n \in N)$. On using the concept of quotients in $\beta(\odot,*)$ and the convolution theorem we write

$$l_{\gamma,\sigma}^{(\beta)} \gamma_n \odot \epsilon_m = l_{\gamma,\sigma}^{(\beta)} \theta_m \odot \delta_n \ \ (\forall m,n \in N) .$$

Theorem 10 implies $l_{\gamma,\sigma}^{(\beta)} (\gamma_n * \epsilon_m) = l_{\gamma,\sigma}^{(\beta)} (\theta_m * \delta_n) \ (\forall m,n \in N)$. Hence $\gamma_n * \epsilon_m = \theta_m * \delta_n$. Therefore $\frac{\gamma_n}{\delta_n} = \frac{\theta_n}{\epsilon_n} \ (\forall n \in N)$. Surjectivity can be checked by similar techniques.

This establishes the theorem. □

Definition 12. If $\frac{\hat{\gamma}_n}{\delta_n} \in \beta(*,*)$, then the inverse $L_{\gamma,\sigma}^{(\beta)}$ transform of $\frac{\hat{\gamma}_n}{\delta_n} \in \beta(\odot,*)$ can be given as

$$I_{\gamma,\sigma}^{(\beta)} \frac{\hat{\gamma}_n}{\delta_n} = \frac{i_{\gamma,\sigma}^{(\beta)} \hat{\gamma}_n}{\delta_n}, \tag{17}$$

where $i_{\gamma,\sigma}^{(\beta)}$ is the inverse transform of $l_{\gamma,\sigma}^{(\beta)}$.

Theorem 13. *The mapping $I_{\gamma,\sigma}^{(\beta)}$ is well-defined.*

Proof. The assumption $\frac{\hat{\gamma}_n}{\delta_n} = \frac{\hat{\theta}_n}{\epsilon_n}$ leads to $\hat{\gamma}_n * \epsilon_m = \hat{\theta}_m * \delta_n$ in $\beta(\odot,*)$, $m,n \in N$. Hence, applying the inverse $l_{\gamma,\sigma}^{(\beta)}$ transform then investing Theorem 10 yields $i_{\gamma,\sigma}^{(\beta)} \hat{\gamma}_n \odot \epsilon_m = i_{\gamma,\sigma}^{(\beta)} \hat{\theta}_m \odot \delta_n$ in $\beta(*,*)$. This reveals

$$\frac{i_{\gamma,\sigma}^{(\beta)} \hat{\gamma}_n}{\delta_n} = \frac{i_{\gamma,\sigma}^{(\beta)} \hat{\theta}_n}{\epsilon_n} \Rightarrow I_{\gamma,\sigma}^{(\beta)} \frac{\hat{\gamma}_n}{\delta_n} = I_{\gamma,\sigma}^{(\beta)} \frac{\hat{\theta}_n}{\epsilon_n} \, (\forall n \in N) .$$

This finishes the proof. □

Linearity of $I_{\gamma,\sigma}^{(\beta)}$ can be checked easily. Continuity of $I_{\gamma,\sigma}^{(\beta)}$ and $L_{\gamma,\sigma}^{(\beta)}$, with respect to δ and Δ convergence, can be checked by techniques similar to that in [2, 7].

Theorem 14. *The operator $L_{\gamma,\sigma}^{(\beta)}$ is sequentially continuous from $\beta(*,*)$ onto $\beta(\odot,*)$, i.e., if $\Delta - \lim_{n\to\infty} \beta_n = \beta$ in $\beta(*,*)$, then $\Delta - \lim_{n\to\infty} L_{\gamma,\sigma}^{(\beta)} \beta_n = L_{\gamma,\sigma}^{(\beta)} \beta$ in $\beta(\odot,*)$.*

Proof. Let $\Delta - \lim_{n\to\infty} \beta_n = \beta$ in $\beta(*,*)$, then there is $\{\delta_n\} \in \Delta$ such that

$$\Delta - \lim_{n\to\infty} (\beta_n - \beta) * \delta_n = 0 \text{ in } F_{\rho,\mu}.$$

Continuity of $l_{\gamma,\sigma}^{(\beta)}\varphi$ integral operator yields

$$\Delta - \lim_{n\to\infty} L_{\gamma,\sigma}^{(\beta)}\left((\beta_n - \beta) * \delta_n\right) = \Delta - \lim_{n\to\infty}\left(\left(L_{\gamma,\sigma}^{(\beta)}\beta_n - L_{\gamma,\sigma}^{(\beta)}\beta\right)\odot \delta_n\right) = 0.$$

Thus we have $\Delta - \lim_{n\to\infty} L_{\gamma,\sigma}^{(\beta)}\beta_n = L_{\gamma,\sigma}^{(\beta)}\beta$ in $\beta\left(\odot, *\right)$.

This finishes the proof of the theorem. □

5. Conclusion

In this article, the modified integral operator with the specific kernel function has been extended to certain generalized Zemanian spaces of Boehmians. By aid of well-defined Dirac delta sequences and an appropriately established pair of convolution products, the modified Bessel integral has also been derived. It is well-defined, one-to-one and onto mapping from one space of Boehmians onto another. Further, this integral with the Boehmian character satisfies the desired properties that the classical integral does have.

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