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GROUP RINGS SATISFYING NIL CLEAN PROPERTY

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Abstract. In 2013, Diesl defined a nil clean ring as a ring of which all elements can be expressed as the sum of an idempotent and a nilpotent. Furthermore, in 2017, Y. Zhou, S. Sahinkaya, G. Tang studied nil clean group rings, finding both necessary condition and sufficient condition for a group ring to be a nil clean ring. We have proposed a necessary and sufficient condition for a group ring to be a uniquely nil clean ring. Additionally, we provided theorems for general nil clean group rings, and some examples of trivial-center groups of which group ring is not nil clean over any strongly nil clean rings.

1. Introduction

In 1977, in his paper [8], Nicholson defined the concept of a suitable ring by proving the existence of an idempotent that satisfies certain conditions of a ring. In this paper, he defined a clean ring as a ring such that there is an idempotent e for every element r of the ring so that $r - e$ becomes a unit. Following [9], he defined a strongly clean ring and suggested it as a class of a ring satisfying Fitting's lemma, an important concept of that time.

While following researches were on clean rings in order to solve module-related problems, Diesl defined a nil clean ring and a strongly nil clean ring in [3]. He defined a ring R to be nil clean if there is an idempotent e and a nilpotent b such that $r = e + b$ for each element r in ring R, and a ring R to be strongly nil clean if such idempotent e and nilpotent b can be selected to satisfy $eb = be$.

Diesl [3] dealt with important questions related to the properties of nil clean rings and strongly nil clean rings. When R is a nil clean ring, the element $2 \in R$ is always a (central) nilpotent, therefore is included in $J(R)$ (Jacobson radical of R). Several fundamental structure theorems of nil clean rings have also been studied.

Recently, there have been many investigations concerning the conditions of a group ring $R[G]$ to be nil clean. In 2015, McGovern [7] proved that a commutative group ring $R[G]$ is nil clean, if and only if R is nil clean and G

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is a torsion 2-group. In [10], for a general group ring without the commutative condition, Zhou proposed the necessary condition and sufficient condition for such a ring to be nil clean. Furthermore, he suggested the necessary and sufficient condition for such a ring to be strongly nil clean. However, it was not able to clearly explain the properties of a group ring which satisfies the necessary conditions for a group ring to be nil clean but does not satisfy the sufficient conditions for a group ring to be nil clean from the result, and it still remains as an open question.

Throughout the paper, all rings R considered to be associative and unital with additive identity 0 and multiplicative identity 1. For a ring R, the notations $N(R)$, $ID(R)$, and $J(R)$ will be used, respectively, for the set of nilpotent elements in R, set of idempotent elements in R, and the Jacobson radical of R, which is the intersection of every maximal right (left) ideal in R. $\mathbb{Z}_n$ denotes the cyclic group of order n or the ring of residue classes modulo n, according to the context. For a group G, 0_g and $Z(G)$ denotes the identity of the group and the center of the group, respectively. For other general ring theory terms, we referred to [5].

2. Some preliminaries on nil clean rings

In this section, we deal with the definition and some basic properties of nil clean rings, following [3].

2.1. Definitions of nil cleanness

The notions of a nil clean ring, a strongly nil clean ring, and a uniquely nil clean ring are defined as a ring which satisfies the following condition.

Definition ([3, Definition 3.1])**.** Let R be a ring. An element $r \in R$ is called nil clean if there is an idempotent $e \in R$ and a nilpotent $b \in R$ such that $r = e + b$. The element r is further called strongly nil clean if such an idempotent and nilpotent can be chosen to satisfy $eb = be$. A ring is called nil clean (respectively, strongly nil clean) if every one of its elements is nil clean (respectively, strongly nil clean).

Definition ([3, Definition 5.1])**.** Let R be a ring. An element $r \in R$ is called uniquely nil clean if there is a unique idempotent $e \in R$ such that $r - e$ is nilpotent. A ring is called uniquely nil clean if every one of its elements is uniquely nil clean.

From these definitions, the following inclusion relation can be deduced.

$$\text{Boolean} \Rightarrow \text{Uniquely nil clean} \Rightarrow \text{Strongly nil clean} \Rightarrow \text{Nil clean.}$$

By the following theorem, every uniquely nil clean ring is an abelian. Thus, every uniquely nil clean ring is strongly nil clean. The other implications can be deduced directly from the definitions.

Proposition 2.1 ([3]). *Let R be a ring. Then, the following conditions are equivalent.*

(1) *R is nil clean and abelian.*
(2) *R is uniquely nil clean.*

Proof. ($\Rightarrow$) Since R is abelian, any nil clean decomposition of any element of R is a strongly nil clean decomposition. By Corollary 3.8. of [3], such strongly nil clean decomposition is unique. Therefore, R is uniquely nil clean.

($\Leftarrow$) ([3, Lemma 5.5]) Let $e \in ID(R)$, and r be any element of R. Then, $e + er(1 - e) \in R$ has two nil clean decompositions, namely, $e + er(1 - e) = (e+er(1-e))+0 = (e)+(er(1-e))$. Since R is uniquely nil clean, $er(1-e) = 0$, which implies $ere = er$. Similarly, we obtain $ere = re$. Thus, $er = re$, R is abelian. $\square$

3. Uniquely nil clean group rings

3.1. Characterization of uniquely nil clean rings

Throughout the paper, we denote the group ring of a group G over a ring R by $R[G]$, which is the set of formal sum $\sum_{i\in I} r_i \cdot g_i$, where r_i and g_i is an element of ring R and group G, respectively. The trace map ω is defined as a mapping from the group ring $R[G]$ to ring R as the following.

$$\omega : \sum_{g\in G} r_g \cdot g \mapsto \sum_{g\in G} r_g.$$

The trace map is a ring homomorphism, and its kernel $ker(\omega)$ is called the augmentation ideal of the group ring. We denote the augmentation ideal of group ring $R[G]$ as $\triangle(R[G])$.

For other group-theoretic terms, we denote the hypercenter of a group G, which is the limit of the (transfinite) upper central series of G, as $H(G)$. We say a group G is locally finite, if every finitely generated subgroup of G is finite. Finally, for a prime number p, we say a group G is a p-group if every element of G has the order which is a power of p.

Theorem 3.1 ([2, Corollary of Theorem 9]). *The augmentation ideal $\triangle(R[G])$ of a group ring $R[G]$ is a nilpotent ideal if and only if*

(1) *G is a locally finite p-group, and*
(2) *$p \in R$ is a nilpotent.*

This theorem is one of the basic results of the finite group ring theory. With this theorem, it is possible to explicitly describe the set of nilpotent elements in a group ring.

Proposition 3.2. *For any group ring $R[G]$ of a locally finite 2-group G, if the element $2 \in R$ is a nilpotent, then the equality $N(R[G]) = \omega^{-1}(N(R))$ holds, where ω is the trace map defined above.*

Proof. $N(R[G]) \subseteq \omega^{-1}(N(R))$: Let $a \in N(R[G])$. Then, $a^N = 0_{R[G]}$ for some $N \in \mathbb{N}$. It follows that $\omega(a)^N = \omega(a^N) = \omega(0_{R[G]}) = 0_R$, $\omega(a) \in N(R)$, $a \in \omega^{-1}(N(R))$.

$\omega^{-1}(N(R)) \subseteq N(R[G])$: Let $a \in \omega^{-1}(N(R))$. Then, for some $N \in \mathbb{N}$, $a^N \in \triangle(R[G])$. Thus $a^N \in R[G]$ is nilpotent, since $\triangle(R[G])$ is a nil ideal by Theorem 3.1. Therefore, $a \in N(R[G])$. □

Proposition 3.3 ([3, Proposition 3.8])**.** *If an element of a ring is strongly nil clean, then it has precisely one strongly nil clean decomposition.*

In 2017, Y. Zhou, T. Kosan, and Z. Wang suggested the necessary and sufficient condition for a group ring to be strongly nil clean in [4].

Theorem 3.4 ([4, Theorem 4.7])**.** *Let G be a locally finite group and R be a ring. Then, the following conditions are equivalent.*

(1) *Group ring $R[G]$ is strongly nil clean.*
(2) *R is strongly nil clean and G is a 2-group.*

With these theorem and proposition, now it is possible to prove our main theorem, dealing with a necessary and sufficient condition of a group ring to be uniquely nil clean.

Theorem 3.5. *Let G be a locally finite group and R be a ring. Then, the following conditions are equivalent.*

(1) *A group ring $R[G]$ is uniquely nil clean.*
(2) *R is uniquely nil clean and G is a 2-group.*

Proof. ($\Rightarrow$) Let $R[G]$ be a uniquely nil clean group ring. From the first isomorphism theorem, we have $R[G]/\triangle(R[G]) \cong R$. Since $R[G]$ is uniquely nil clean, R is uniquely nil clean by [3], Corollary 5.10. Also, by Theorem 3.4, G is a 2-group.

($\Leftarrow$) Let R be a uniquely nil clean ring, G be a (locally finite) 2-group. Then, $R[G]$ is a (strongly) nil clean ring. We claim that $ID(R[G]) = \{e \cdot 0_g \mid e \in ID(R)\}$. If this assertion is true, the group ring $R[G]$ is abelian, and the theorem follows. The verification of $\{e \cdot 0_g \mid e \in ID(R)\} \subseteq ID(R[G])$ is straightforward. Let $a \in ID(R[G])$. Since R is uniquely nil clean, $\omega(a) = e + b$ for some $e \in ID(R)$, $b \in N(R)$. Then, a has two strongly nil clean decompositions, namely, $a = (a) + 0_{R[G]} = (e \cdot 0_g) + (a - e \cdot 0_g)$. (Since $w(a - e \cdot 0_g) = b \in N(R)$, $(a - e \cdot 0_g) \in R[G]$ is a nilpotent by Theorem 3.2. $(e \cdot 0_g)(a - e \cdot 0_g) = (a - e \cdot 0_g)(e \cdot 0_g)$ follows from the fact that e is a central idempotent.) By Proposition 3.3, we obtain $a = e \cdot 0_g \in \{e \cdot 0_g \mid e \in ID(R)\}$. Thus, $ID(R[G]) = \{e \cdot 0_g \mid e \in ID(R)\}$. This completes the proof. □

4. General nil clean group rings

In [10], a necessary condition and a sufficient condition of a group ring to be a nil clean ring were suggested. However, the nil-cleanness of a group ring

$R[G]$ such that G is not a 2-group but its center $Z(G)$ is a 2-group is still remaining as an open question. In this section, we propose a theorem which corresponds to [3], Proposition 3.15 to deal with this problem. Furthermore, we suggest some examples of a group ring over a trivial-center group which is not nil clean.

4.1. The center of group and nil cleanness

The following result dealing with the center of a group and the nil cleanness holds for general group rings.

Theorem 4.1. *Let G be a group of which center $Z(G)$ is a locally finite 2-group. Then, the following conditions are equivalent.*

(1) *A group ring $R[G]$ is nil clean.*
(2) *A group ring $R[G/Z(G)]$ is nil clean.*

Proof. Let $\psi : G \to G/Z(G)$ be the canonical homomorphism. Then, consider the following homomorphism $\phi : R[G] \to R[G/Z(G)]$:

$$\phi : \sum_{g\in G} r_g \cdot g \mapsto \sum_{g\in G} r_g \cdot \psi(g).$$

Then, ϕ is onto, hence it is enough to show that $ker(\phi)$ is a nil ideal, by [3], Proposition 3.15 and the first isomorphism theorem. From the definition of the mapping, it follows that

$$ker(\phi) = \{\sum_{g\in G} r_g \cdot g | \sum_{g\in \overline{g}} r_g = 0 \quad \forall \overline{g} = Z(G)g\}.$$

Let $G = \cup_{i\in I}\overline{g_i}$, where $\overline{g_i}$ is a coset of $Z(G)$ containing g_i. Then, we can express each element in $ker(\phi)$ as

$$\begin{aligned}\sum_{g\in G} r_g \cdot g &= \sum_{i\in I}\left(\sum_{z\in Z(G)} r_{zg_i} z\right) g_i, \text{ where } \sum_{z\in Z(G)} r_{zg_i} = 0 \ \forall i\in I \\ &= \sum_{i\in I} a_i \cdot g_i, \text{ where } a_i \in \triangle = \triangle(R[Z(G)]) \ \forall i \in I.\end{aligned}$$

Since the ring R is nil clean, the element $2 \in R$ is a nilpotent. Thus, from the fact that the group $Z(G)$ is a locally finite 2-group, $\triangle$ is a nilpotent ideal, by Theorem 3.1. Therefore, there is N such that $\triangle^N = 0$. From

$$\begin{aligned}(\sum_{g\in G} r_g \cdot g)^N &= (\sum_{i\in I} a_i \cdot g_i)^N \\ &\in \sum_{i_1,i_2,\ldots,i_N\in I} \triangle^N \cdot g_{i_1}g_{i_2}\cdots g_{i_N} = 0\end{aligned}$$

each element in $ker(\phi)$ is nilpotent, so it is a nil ideal. This completes the proof. $\square$

If a group G has nontrivial center, we can determine whether group ring $R[G]$ is nil clean or not by investigating $R[G/Z(G)]$. If $Z(G)$ is not a 2-group, $R[G]$ is not nil clean, and if $Z(G)$ is a nontrivial 2-group, we can reduce the size of the group ring by using the theorem above. Thus, the nil cleanness of a group ring $R[G]$ of finite group G is equivalent to the nil cleanness of $R[G/H(G)]$, if $H(G)$ is a 2-group. Therefore, it suffices to characterize the nil cleanness of group rings over a trivial-center group.

4.2. An example of trivial-center case

In the following proposition, we propose an example of a trivial-center group of which a group ring over a strongly nil clean ring is never a nil clean ring.

Proposition 4.2. *The group ring $\mathbb{Z}_2[D_{2n+1}]$ is not nil clean for $n > 1$.*

Proof. By [1], we have $\mathbb{Z}_2[D_{2n+1}] \cong \mathbb{Z}_2[\mathbb{Z}_2] \oplus M_2(\Delta)$, where

$$\Delta = \left\{ \sum_{i=1}^{n} r_i(g^i + g^{-i}) \;\middle|\; r_i \in \mathbb{Z}_2 \right\}, g^n = 1$$

is a ring. For $a = g + g^{-1} \in \Delta$, $a - a^2 = g^2 + g^1 + g^{-1} + g^{-2}$. In order to show that Δ is not nil clean, it is sufficient to show that $a - a^2 \in \Delta$ is not a nilpotent by [4], Theorem 2.1. If $a - a^2$ is a nilpotent, $(a - a^2)^{2^N} = 0$ for some $N \in \mathbb{N}$. However, $0 = (a-a^2)^{2^N} = (g^2+g^1+g^{-1}+g^{-2})^{2^N} = g^{2^{N+1}}+g^{2^N}+g^{-2^N}+g^{-2^{N+1}}$ plies $(2n + 1) \mid 2^{N+2}$ or $(2n + 1) \mid 3 \times 2^N$, which is not possible since n is an odd number greater than 3. Hence, we can conclude that Δ is not strongly nil clean.

Therefore, Δ and $M_2(\Delta)$ is not nil clean by [4], Corollary 6.2. Hence, $\mathbb{Z}_2[D_{2n+1}] \cong \mathbb{Z}_2[\mathbb{Z}_2] \oplus M_2(\Delta)$ is not nil clean. This completes the proof. □

For even n, by using the fact $D_{2n}/Z(D_{2n}) \cong D_n$ and $\mathbb{Z}_2[D_3]$ is nil clean, we can conclude that $\mathbb{Z}_2[D_n]$ is nil clean if and only if $n = 2^k$ or $n = 3 \times 2^k$ for some nonnegative integer k. Also, this proposition can be utilized to show that any group ring of D_{2n+1} over a finite strongly nil clean ring is not a nil clean ring. If $R[D_{2n+1}]$ is nil clean, $(R/J(R))[D_{2n+1}]$ should be nil clean as well, where $R/J(R)$ is a Boolean ring ([4, Theorem 2.7]). By Proposition 4.2, $(R/J(R))[D_{2n+1}]$, which has $\mathbb{Z}_2[D_{2n+1}]$ as a direct summand, is not a nil clean ring.

5. Left-open problems

In this research, we found out that a group ring $R[G]$ is uniquely nil clean if and only if R is uniquely nil clean and G is (locally finite) 2-group, developing the result of [10]. Also, we proposed a basic structure theorem of nil clean group rings, and calculated an example of a group ring of a trivial-center group. We leave some open questions for readers.

Conjecture 5.1. *Does there exist any ring R such that R is a uniquely nil clean ring while the group ring $R[S_3]$ is not nil clean?*

This question is related to Question 3 in [3], which is, 'Is $M_n(R)$, which is the full matrix ring over R, nil clean if R is a nil clean ring?'. If the question we suggested is proven to be wrong, by Proposition 2.9 of [10] and Theorem 3.1 of [6], the full matrix ring $M_n(R)$ is nil clean if R is nil clean. Meanwhile, in [4], it is shown that the full matrix ring of a strongly nil clean ring with its index bounded is nil clean. So, to find the ring satisfying the condition, we should consider a ring which is not bounded index, or nil clean but not strongly nil clean.

Conjecture 5.2. *What is the condition of a group G that the group ring $R[G]$ becomes a nil clean ring where R is a strongly nil clean ring?*

This question is about a generalization of [7], Theorem 2.6. As we showed in Proposition 4.2, the fact that $\mathbb{Z}_2[G]$ is not a nil clean ring implies that a group ring of G over any finite strongly nil clean ring is not nil clean. Thus, it should be preceded to complete investigating groups of which the group ring over $\mathbb{Z}_2$ is nil clean. As a partial solution for this question to this problem, we propose the following theorem.

Theorem 5.3. *If G is a nontrivial group of odd order, then a group ring $R[G]$ is never nil clean for any ring R.*

Proof. By the Feit-Thompson odd order theorem, the group G is solvable, so that it has a normal subgroup H such that G/H is abelian. By Lagrange's theorem, we know that G/H has odd order, so $R[G/H]$ is not nil clean. Therefore, $R[G]$ is not nil clean, as well. This completes the proof. □

From this theorem, we can find a lot of groups of which group rings are never nil clean. As an example, for a group G which possesses a normal 2-Sylow subgroup, $R[G]$ is not a nil clean ring.

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