Commun. Korean Math. Soc. **35** (2020), No. 1, pp. 83–116
https://doi.org/10.4134/CKMS.c190013
pISSN: 1225-1763 / eISSN: 2234-3024

ON MINIMAL PRODUCT-ONE SEQUENCES OF MAXIMAL LENGTH OVER DIHEDRAL AND DICYCLIC GROUPS

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Abstract. Let G be a finite group. By a sequence over G, we mean a finite unordered sequence of terms from G, where repetition is allowed, and we say that it is a product-one sequence if its terms can be ordered such that their product equals the identity element of G. The large Davenport constant $\mathsf{D}(G)$ is the maximal length of a minimal product-one sequence, that is, a product-one sequence which cannot be factored into two non-trivial product-one subsequences. We provide explicit characterizations of all minimal product-one sequences of length $\mathsf{D}(G)$ over dihedral and dicyclic groups. Based on these characterizations we study the unions of sets of lengths of the monoid of product-one sequences over these groups.

1. Introduction

Let G be a finite group. A sequence S over G means a finite sequence of terms from G which is unordered, repetition of terms allowed. We say that S is a product-one sequence if its terms can be ordered so that their product equals the identity element of the group. The *small Davenport constant* $\mathsf{d}(G)$ is the maximal integer ℓ such that there is a sequence of length ℓ which has no non-trivial product-one subsequence. The *large Davenport constant* $\mathsf{D}(G)$ is the maximal length of a minimal product-one sequence (this is a product-one sequence which cannot be factored into two non-trivial product-one subsequences). We have $1 + \mathsf{d}(G) \le \mathsf{D}(G)$ and equality holds if G is abelian. The study of the Davenport constant of finite abelian groups has been a central topic in zero-sum theory since the 1960s (see [13] for a survey). Both the direct problem, asking for the precise value of the Davenport constant in terms of the group invariants, as well as the associated inverse problem, asking for the structure of extremal sequences, have received wide attention in the literature. We refer to [4, 14, 15, 21–23, 29, 30] for progress with respect to the direct and

Received January 13, 2019; Revised April 5, 2019; Accepted May 2, 2019.
2010 *Mathematics Subject Classification.* 20D60, 20M13, 11B75, 11P70.
Key words and phrases. Product-one sequences, Davenport constant, dihedral groups, dicyclic groups, sets of lengths, unions of sets of lengths.
This work was supported by the Austrian Science Fund FWF, W1230 Doctoral Program Discrete Mathematics and Project No. P28864–N35.

to the inverse problem. Much of this research was stimulated by and applied to factorization theory and we refer to [16, 18] for more information on this interplay.

Applications to invariant theory (in particular, the relationship of the small and large Davenport constants with the Noether number, see [5–9, 26]) pushed forward the study of the Davenport constants for finite non-abelian groups. Geroldinger and Grynkiewicz ([17, 24]) studied the small and the large Davenport constant of non-abelian groups and derived their precise values for groups having a cyclic index 2 subgroup. Brochero Martínez and Ribas ([2, 3]) determined, among others, the structure of product-one free sequences of length $\mathsf{d}(G)$ over dihedral and dicyclic groups.

In this paper we establish a characterization of the structure of minimal product-one sequences of length $\mathsf{D}(G)$ over dihedral and dicyclic groups (Theorems 4.1, 4.2, and 4.3). It turns out that this problem is quite different from the study of product-one free sequence done by Brochero Martínez and Ribas. The minimal product-one sequences over G are the atoms (irreducible elements) of the monoid $\mathcal{B}(G)$ of all product-one sequences over G. Algebraic and arithmetic properties of $\mathcal{B}(G)$ were recently studied in [27, 28]. Based on our characterization results of minimal product-one sequences of length $\mathsf{D}(G)$ we give a description of all unions of sets of lengths of $\mathcal{B}(G)$ (Theorems 5.4 and 5.5).

We proceed as follows. In Section 2, we fix our notation and gather the required tools. In Section 3, we study the structure of minimal product-one sequences fulfilling certain requirements on their length and their support (Propositions 3.2 and 3.3). Based on these preparatory results, we establish an explicit characterization of all minimal product-one sequences having length $\mathsf{D}(G)$ for dihedral groups (Theorems 4.1 and 4.2) and for dicyclic groups (Theorem 4.3). Our results on unions of sets of lengths are given in Section 5.

2. Preliminaries

We denote by $\mathbb{N}$ the set of positive integers and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For each $k \in \mathbb{N}$, we also denote by $\mathbb{N}_{\geq k}$ the set of positive integers greater than or equal to k. For integers $a, b \in \mathbb{Z}$, $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$ is the discrete interval.

Groups. Let G be a multiplicatively written finite group with identity element 1_G. For an element $g \in G$, we denote by $\operatorname{ord}(g) \in \mathbb{N}$ the order of g, and for subsets $A, B \subset G$, we set

$$AB = \{ab \mid a \in A \text{ and } b \in B\} \quad \text{ and } \quad gA = \{ga \mid a \in A\}.$$

If $G_0 \subset G$ is a non-empty subset, then we denote by $\langle G_0 \rangle \subset G$ the subgroup generated by G_0, and by $\mathsf{H}(G_0) = \{g \in G \mid gG_0 = G_0\}$ the left *stabilizer* of G_0. Then $\mathsf{H}(G_0) \subset G$ is a subgroup, and G_0 is a union of right $\mathsf{H}(G_0)$-cosets. Of course, if G is abelian, then we do not need to differentiate between left and right stabilizers and simply speak of the stabilizer of G_0, and when G is written additively, we have that $\mathsf{H}(G_0) = \{g \in G \mid g + G_0 = G_0\}$. Furthermore, for every $n \in \mathbb{N}$ and for a subgroup $H \subset G$, we denote by

- $[G : H]$ the *index* of H in G,
- $\phi_H : G \to G/H$ the canonical epimorphism if $H \subset G$ is normal,
- C_n an (additively written) *cyclic group* of order n,
- D_{2n} a *dihedral group* of order $2n$, and by
- Q_{4n} a *dicyclic group* of order $4n$.

Sequences over groups. Let G be a finite group with identity element 1_G and $G_0 \subset G$ a subset. The elements of the free abelian monoid $\mathcal{F}(G_0)$ will be called *sequences* over G_0. This terminology goes back to Combinatorial Number Theory. Indeed, a sequence over G_0 can be viewed as a finite unordered sequence of terms from G_0, where the repetition of elements is allowed. We briefly discuss our notation which follows the monograph [25, Chapter 10.1]. In order to avoid confusion between multiplication in G and multiplication in $\mathcal{F}(G_0)$, we denote multiplication in $\mathcal{F}(G_0)$ by the boldsymbol $\boldsymbol{\cdot}$ and we use brackets for all exponentiation in $\mathcal{F}(G_0)$. In particular, a sequence $S \in \mathcal{F}(G_0)$ has the form

$$S = g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_\ell = \prod^\bullet_{i\in[1,\ell]} g_i \in \mathcal{F}(G_0), \tag{2.1}$$

where $g_1, \ldots, g_\ell \in G_0$ are the terms of S. For $g \in G_0$,

- $\mathsf{v}_g(S) = |\{i \in [1,\ell] \mid g_i = g\}|$ denotes the *multiplicity* of g in S,
- $\mathrm{supp}(S) = \{g \in G_0 \mid \mathsf{v}_g(S) > 0\}$ denotes the *support* of S, and
- $\mathsf{h}(S) = \max\{\mathsf{v}_g(S) \mid g \in G_0\}$ denotes the *maximal multiplicity* of S.

A *subsequence* T of S is a divisor of S in $\mathcal{F}(G_0)$ and we write $T \mid S$. For a subset $H \subset G_0$, we denote by S_H the subsequence of S consisting of all terms from H. Furthermore, $T \mid S$ if and only if $\mathsf{v}_g(T) \le \mathsf{v}_g(S)$ for all $g \in G_0$, and in such case, $S \boldsymbol{\cdot} T^{[-1]}$ denotes the subsequence of S obtained by removing the terms of T from S so that $\mathsf{v}_g\big(S \boldsymbol{\cdot} T^{[-1]}\big) = \mathsf{v}_g(S) - \mathsf{v}_g(T)$ for all $g \in G_0$. On the other hand, we set $S^{-1} = g_1^{-1} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_\ell^{-1}$ to be the sequence obtained by taking elementwise inverse from S.

Moreover, if $S_1, S_2 \in \mathcal{F}(G_0)$ and $g_1, g_2 \in G_0$, then $S_1 \boldsymbol{\cdot} S_2 \in \mathcal{F}(G_0)$ has length $|S_1| + |S_2|$, $S_1 \boldsymbol{\cdot} g_1 \in \mathcal{F}(G_0)$ has length $|S_1| + 1$, $g_1 g_2 \in G$ is an element of G, but $g_1 \boldsymbol{\cdot} g_2 \in \mathcal{F}(G_0)$ is a sequence of length 2. If $g \in G_0$, $T \in \mathcal{F}(G_0)$, and $k \in \mathbb{N}_0$, then

$$g^{[k]} = \underbrace{g \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g}_{k} \in \mathcal{F}(G_0) \quad \text{and} \quad T^{[k]} = \underbrace{T \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T}_{k} \in \mathcal{F}(G_0)\,.$$

Let $S \in \mathcal{F}(G_0)$ be a sequence as in (2.1). When G is written multiplicatively, we denote by

$$\pi(S) = \{g_{\tau(1)} \cdots g_{\tau(\ell)} \in G \mid \tau \text{ a permutation of } [1,\ell]\} \subset G$$

the *set of products* of S, and it can easily be seen that $\pi(S)$ is contained in a G'-coset, where G' is the commutator subgroup of G. Note that $|S| = 0$ if and only if $S = 1_{\mathcal{F}(G)}$, and in that case we use the convention that $\pi(S) = \{1_G\}$. When G is written additively with commutative operation, we likewise define

$$\sigma(S) \,=\, g_1 + \cdots + g_\ell \,\in\, G$$

to be the *sum* of S. More generally, for any $n \in \mathbb{N}_0$, the *n-sums* and *n-products* of S are respectively denoted by

$$\Sigma_n(S) \,=\, \{\sigma(T) \bigm| T \mid S \text{ and } |T| = n\} \,\subset\, G \quad\text{and}\quad \Pi_n(S) \,=\, \bigcup_{\substack{T \mid S \\ |T|=n}} \pi(T) \,\subset\, G\,,$$

and the *subsequence sums* and *subsequence products* of S are respectively denoted by

$$\Sigma(S) \,=\, \bigcup_{n\geq 1} \Sigma_n(S) \,\subset\, G \qquad\text{and}\qquad \Pi(S) \,=\, \bigcup_{n\geq 1} \Pi_n(S) \,\subset\, G\,.$$

The sequence S is called

- a *product-one sequence* if $1_G \in \pi(S)$,
- *product-one free* if $1_G \notin \Pi(S)$,
- *square-free* if $\mathsf{h}(S) \leq 1$.

If $S = g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_\ell \in \mathcal{B}(G)$ is a product-one sequence with $1_G = g_1 \cdots g_\ell$, then $1_G = g_i \cdots g_\ell g_1 \cdots g_{i-1}$ for every $i \in [1, \ell]$. Every map of groups $\theta : G \to H$ extends to a monoid homomorphism $\theta : \mathcal{F}(G) \to \mathcal{F}(H)$, where $\theta(S) = \theta(g_1) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} \theta(g_\ell)$. If θ is a group homomorphism, then $\theta(S)$ is a product-one sequence if and only if $\pi(S) \cap \ker(\theta) \neq \emptyset$. We denote by

$$\mathcal{B}(G_0) \,=\, \big\{S \in \mathcal{F}(G_0) \mid 1_G \in \pi(S)\big\}$$

the set of all product-one sequences over G_0, and clearly $\mathcal{B}(G_0) \subset \mathcal{F}(G_0)$ is a submonoid. We denote by $\mathcal{A}(G_0)$ the set of irreducible elements of $\mathcal{B}(G_0)$ which, in other words, is the set of minimal product-one sequences over G_0. Moreover,

$$\mathsf{D}(G_0) \,=\, \sup\big\{|S| \mid S \in \mathcal{A}(G_0)\big\} \,\in\, \mathbb{N} \cup \{\infty\}$$

is the *large Davenport constant* of G_0, and

$$\mathsf{d}(G_0) \,=\, \sup\big\{|S| \mid S \in \mathcal{F}(G_0) \text{ is product-one free}\big\} \,\in\, \mathbb{N}_0 \cup \{\infty\}$$

is the *small Davenport constant* of G_0. It is well known that $\mathsf{d}(G) + 1 \leq \mathsf{D}(G) \leq |G|$, with equality in the first bound when G is abelian, and equality in the second bound when G is cyclic ([17, Lemma 2.4]). Moreover, Geroldinger and Grynkiewicz provide the precise value of the Davenport constants for non-cyclic groups having a cyclic index 2 subgroups (see [17, 24]), whence we have that, for every $n \in \mathbb{N}_{\geq 2}$,

$$\mathsf{D}(Q_{4n}) \,=\, 3n \qquad\text{and}\qquad \mathsf{D}(D_{2n}) \,=\, \begin{cases} 2n & \text{if } n \geq 3 \text{ is odd,} \\ \frac{3n}{2} & \text{if } n \geq 4 \text{ is even.} \end{cases}$$

Ordered sequences over groups. These are an important tool used to study (unordered) sequences over non-abelian groups. Indeed, it is quite useful to have related notation for sequences in which the order of terms matters. Thus, for a subset $G_0 \subset G$, we denote by $\mathcal{F}^*(G_0) = \big(\mathcal{F}^*(G_0), \boldsymbol{\cdot}\big)$ the free (non-abelian) monoid with basis G_0, whose elements will be called the *ordered sequences* over G_0.

Taking an ordered sequence in $\mathcal{F}^*(G_0)$ and considering all possible permutations of its terms gives rise to a natural equivalence class in $\mathcal{F}^*(G_0)$, yielding a natural map

$$[\boldsymbol{\cdot}] : \mathcal{F}^*(G_0) \quad \to \quad \mathcal{F}(G_0)$$

given by abelianizing the sequence product in $\mathcal{F}^*(G_0)$. For any sequence $S \in \mathcal{F}(G_0)$, we say that an ordered sequence $S^* \in \mathcal{F}^*(G_0)$ with $[S^*] = S$ is an *ordering* of the sequence $S \in \mathcal{F}(G_0)$.

All notation and conventions for sequences extend naturally to ordered sequences. We sometimes associate an (unordered) sequence S with a fixed (ordered) sequence having the same terms, also denoted by S. While somewhat informal, this does not give rise to confusion, and will improve the readability of some of the arguments.

For an ordered sequence $S = g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_\ell \in \mathcal{F}^*(G)$, we denote by $\pi^* : \mathcal{F}^*(G) \to G$ the unique homomorphism that maps an ordered sequence onto its product in G, so

$$\pi^*(S) = g_1 \cdots g_\ell \in G\,.$$

If G is a multiplicatively written abelian group, then for every sequence $S \in \mathcal{F}(G)$, we always use $\pi^*(S) \in G$ to be the unique product, and $\Pi(S) = \bigcup \big\{\pi^*(T) \,|\, T \text{ divides } S \text{ and } |T| \ge 1\big\} \subset G$.

For the proof of our main results, the structure of product-one free sequences over cyclic groups plays a crucial role. Thus we gather some necessary lemmas regarding sequences over cyclic groups. Let G be an additively written finite cyclic group. A sequence $S \in \mathcal{F}(G)$ is called *smooth* (more precisely, *g-smooth*) if $S = (n_1 g) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} (n_\ell g)$, where $|S| = \ell \in \mathbb{N}$, $g \in G$, $1 = n_1 \le \cdots \le n_\ell$, $m = n_1 + \cdots + n_\ell < \operatorname{ord}(g)$, and $\Sigma(S) = \{g, 2g, \ldots, mg\}$.

Lemma 2.1 ([16, Lemma 5.1.4])**.** *Let G be an additively written cyclic group of order $|G| = n \ge 3$, $g \in G$, and $k, l, n_1, \ldots, n_l \in \mathbb{N}$ such that $l \ge \frac{k}{2}$ and $m = n_1 + \cdots + n_l < k \le \operatorname{ord}(g)$. If $1 \le n_1 \le \cdots \le n_l$ and $S = (n_1 g) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} (n_l g)$, then $\sum(S) = \big\{g, 2g, \ldots, mg\big\}$, and S is g-smooth.*

Lemma 2.2. *Let G be an additively written cyclic group of order $|G| = n \ge 3$ and $S \in \mathcal{F}(G)$ a product-one free sequence of length $|S| \ge \frac{n+1}{2}$. Then S is g-smooth for some $g \in G$ with $\operatorname{ord}(g) = n$, and for every $h \in \sum(S)$, there exists a subsequence $T \,|\, S$ such that $\sigma(T) = h$ and $g \,|\, T$. In particular,*

1. *if $|S| = n - 1$, then $S = g^{[n-1]}$.*
2. *if $|S| = n - 2$, then $S = (2g) \boldsymbol{\cdot} g^{[n-3]}$ or $S = g^{[n-2]}$.*

3. *if $n \ge 4$, then, for every subsequence $W \mid S$ with $|W| \ge \frac{n}{2} - 1$, we obtain that $g \mid W$.*

Proof. The first statement, that S is g-smooth for some $g \in G$ with $\mathrm{ord}(g) = n$, was found independently by Savchev–Chen and by Yuan, and we cite it in the formulation of [16, Theorem 5.1.8.1].

Suppose now that $S = (n_1 g) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} (n_\ell g)$ with $1 = n_1 \le \cdots \le n_\ell$. Then $n_2 + \cdots + n_\ell < n-1$ and $\ell - 1 \ge \frac{n-1}{2}$. Applying Lemma 2.1 (with $k = n-1$), we obtain that $S \boldsymbol{\cdot} g^{[-1]}$ is still g-smooth. Let $h \in \sum(S) = \big\{g, 2g, \ldots, (n_1 + \cdots + n_\ell)g\big\}$. If $h = g$, then we take $T = g$. If $h \ne g$, then since $S \boldsymbol{\cdot} g^{[-1]}$ is g-smooth, it follows that $h + (-g) \in \sum\big(S \boldsymbol{\cdot} g^{[-1]}\big)$, and hence there exists $W \mid S \boldsymbol{\cdot} g^{[-1]}$ such that $\sigma(W) = h + (-g)$. Thus $W \boldsymbol{\cdot} g$ is a subsequence of S with $\sigma(W \boldsymbol{\cdot} g) = h$.

1. and 2. This follows immediately from the main statement.

3. Let $n \ge 4$, and $W \mid S$ be a subsequence with $|W| \ge \frac{n}{2} - 1$. Then there exists a subset $I \subset [1, \ell]$ with $|I| \ge \frac{n}{2} - 1$ such that $W = \prod^\bullet_{i \in I}(n_i g)$. Assume to the contrary that $n_i \ge 2$ for all $i \in I$. Then

$$n-1 \ \ge\ \sum_{j=1}^{\ell} n_j = \sum_{i \in I} n_i + \sum_{j \in [1,\ell] \setminus I} n_j \ \ge\ 2|W| + \big(|S| - |W|\big) \ =\ |S| + |W| \ \ge\ n - \frac{1}{2},$$

a contradiction. □

3. On special sequences

In this section, we study the structure of minimal product-one sequences under certain additional conditions (Propositions 3.2 and 3.3). These results will be used substantially in the proofs of our main results in next section. We need the Theorem of DeVos–Goddyn–Mohar (see Theorem 13.1 of [25] and the proceeding special cases).

Lemma 3.1. *Let G be a finite abelian group, $S \in \mathcal{F}(G)$ a sequence, $n \in [1, |S|]$, and $H = \mathsf{H}\big(\sum_n(S)\big)$. Then*

$$|\Sigma_n(S)| \ \ge\ \left(\sum_{g \in G/H} \min\{n, \mathsf{v}_g\big(\phi_H(S)\big)\} - n + 1 \right) |H| \, .$$

Let G be an additively (resp. multiplicatively) written finite abelian group. Then $2G = \{2g \mid g \in G\}$ $\big($resp. $G^2 = \{g^2 \mid g \in G\}\big)$. Likewise, given a sequence $S = g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_\ell \in \mathcal{F}(G)$, we set

$$(3.1) \qquad 2S \ =\ 2g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} 2g_\ell \ \in\ \mathcal{F}(2G) \quad \big(\text{resp. } S^2 \ =\ g_1^2 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_\ell^2 \ \in\ \mathcal{F}\big(G^2\big)\big) \, .$$

The Erdős-Ginzburg-Ziv constant $\mathsf{s}(G)$ is the smallest integer $\ell \in \mathbb{N}$ such that every sequence $S \in \mathcal{F}(G)$ of length $|S| \ge \ell$ has a subsequence $T \in \mathcal{B}(G)$ of length $|T| = \exp(G)$. If $G = C_{n_1} \oplus C_{n_2}$ with $1 \le n_1 \mid n_2$, then $\mathsf{s}(G) =$

$2n_1+2n_2-3$ ([18, Theorem 5.8.3]). Results on groups of higher rank can be found in [10].

Proposition 3.2. *Let $G = \langle \alpha, \tau \mid \alpha^n = \tau^2 = 1_G$ and $\tau\alpha = \alpha^{-1}\tau\rangle$ be a dihedral group, where $n \in \mathbb{N}_{\ge 4}$ is even. Let $S \in \mathcal{F}(G)$ be a minimal product-one sequence such that $|S| \ge n$ and $\operatorname{supp}(S) \subset G \setminus \langle\alpha\rangle$. Then S is a sequence of length $|S| = n$ having the following form:*

(a) *If $n = 4$, then*

$$S \,=\, \tau \boldsymbol{\cdot} \alpha\tau \boldsymbol{\cdot} \alpha^2\tau \boldsymbol{\cdot} \alpha^3\tau \qquad or \qquad S \,=\, (\alpha^x\tau)^{[2]} \boldsymbol{\cdot} \alpha^y\tau \boldsymbol{\cdot} \alpha^{y+2}\tau\,,$$

where $x, y \in [0,3]$ with $x \equiv y+1 \pmod 2$.

(b) *If $n \ge 6$, then*

$$S \,=\, (\alpha^x\tau)^{[v]} \boldsymbol{\cdot} (\alpha^{\frac{n}{2}+x}\tau)^{[\frac{n}{2}-v]} \boldsymbol{\cdot} (\alpha^y\tau)^{[w]} \boldsymbol{\cdot} (\alpha^{\frac{n}{2}+y}\tau)^{[\frac{n}{2}-w]}\,,$$

where $x, y \in [0, n-1]$ such that $2x \not\equiv 2y \pmod n$ and $\gcd(x-y, \frac{n}{2}) = 1$, and $v, w \in [0, \frac{n}{2}]$ such that $x - y \equiv v - w \pmod 2$.

In particular, there are no minimal product-one sequences S over G such that $S = S_1 \boldsymbol{\cdot} S_2$ for some $S_1 \in \mathcal{F}\big(\langle\alpha\rangle\big)$ and $S_2 \in \mathcal{F}\big(G \setminus \langle\alpha\rangle\big)$ of length $|S_2| \ge n+2$.

Proof. For every $x \in \mathbb{Z}$, we set $\overline{x} = x + n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z}$. Let $S = \prod^\bullet_{i\in[1,|S|]} \alpha^{x_i}\tau \in \mathcal{A}(G)$ be of length $|S| \ge n$ with $\alpha^{x_1}\tau \cdots \alpha^{x_{|S|}}\tau = 1_G$, where $x_1, \ldots, x_{|S|} \in [0, n-1]$. Since $S \in \mathcal{A}(G)$, it follows that $|S|$ is even, and after renumbering if necessary, we set

$$W \,=\, \overline{x_1} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} \overline{x_{|S|}} \,=\, W_1 \boldsymbol{\cdot} W_2 \,\in\, \mathcal{F}(\mathbb{Z}/n\mathbb{Z})\,,$$

where $W_1 = \prod^\bullet_{i\in[1,|S|/2]} \overline{x_{2i-1}}$ and $W_2 = \prod^\bullet_{i\in[1,|S|/2]} \overline{x_{2i}}$. Thus we have $\sigma(W_1) = \sigma(W_2)$. If we shift the sequence W by $\overline{y}$ for some $y \in \mathbb{Z}$, then the corresponding sequence $S' = \prod^\bullet_{i\in[1,|S|]} \alpha^{x_i+y}\tau$ is still a minimal product-one sequence. If S' has the asserted structure, then the same is true for S whence we may shift the sequence W whenever this is convenient. For every subsequence $U = \overline{y_1} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} \overline{y_v}$ of W, we denote by $\psi(U) = \alpha^{y_1}\tau \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} \alpha^{y_v}\tau$ the corresponding subsequence of S.

A1. *Let $U = U_1 \boldsymbol{\cdot} U_2$ be a subsequence of W such that $|U_1| = |U_2|$ and $\sigma(U_1) = \sigma(U_2)$. Then $\psi(U)$ is a product-one sequence.*

Proof of **A1**. Suppose that $U_1 = \overline{y_1} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} \overline{y_{|U_1|}}$ and $U_2 = \overline{z_1} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} \overline{z_{|U_1|}}$. Since $\sigma(U_1) = \sigma(U_2)$, it follows that

$$\alpha^{y_1}\tau\alpha^{z_1}\tau \cdots \alpha^{y_{|U_1|}}\tau\alpha^{z_{|U_1|}}\tau \,=\, \alpha^{(y_1+\cdots+y_{|U_1|})-(z_1+\cdots+z_{|U_1|})} \,=\, 1_G\,,$$

whence $\psi(U)$ is a product-one sequence. $\square$

If $\operatorname{supp}(W_1) \cap \operatorname{supp}(W_2) \neq \emptyset$, say $\overline{x_1} = \overline{x_2}$, then since $\sigma(W_1) = \sigma(W_2)$, it follows by **A1** that $\psi(\overline{x_1} \boldsymbol{\cdot} \overline{x_2})$ and $\psi\big(W \boldsymbol{\cdot} (\overline{x_1} \boldsymbol{\cdot} \overline{x_2})^{[-1]}\big)$ are both product-one sequences, a contradiction. Therefore $\operatorname{supp}(W_1) \cap \operatorname{supp}(W_2) = \emptyset$.

CASE 1. $\mathsf{h}(W) = 1$.

Since $|W| \ge n = |\mathbb{Z}/n\mathbb{Z}|$, it follows that $|W| = n$, and hence $\mathrm{supp}(W) = \mathbb{Z}/n\mathbb{Z}$. Since $\sigma(W_1) = \sigma(W_2)$, it follows that

$$2(x_1 + x_3 + \cdots + x_{|S|-1}) \equiv \frac{n(n-1)}{2} \pmod n\,, \text{ whence } 2 \,\Big|\, \frac{n}{2}(n-1)\,.$$

Since n is even, we have $\gcd(2, n-1) = 1$, which implies that $\frac{n}{2}$ is even. Note that, for any distinct two elements $x_{i_1}, x_{i_3} \in [1, \frac{n}{2}]$ with $x_{i_2} = x_{i_1} + \frac{n}{2}$ and $x_{i_4} = x_{i_3} + \frac{n}{2}$, the sequence $\prod^\bullet_{k\in[1,4]} \alpha^{x_{i_k}}\tau$ is a product-one sequence. Since $\mathrm{supp}(W) = \mathbb{Z}/n\mathbb{Z}$, we have that S is a product of $\frac{n}{4}$ product-one sequences of length 4. Since $S \in \mathcal{A}(G)$, we must have that $n = 4$ and W is a sequence over $\mathbb{Z}/4\mathbb{Z}$ with $\mathsf{h}(W) = 1$, whence $\psi(W)$ is the desired sequence for (a).

CASE 2. $\mathsf{h}(W) \ge 2$.

Then there exists $i \in [1, |W|]$, say $i = 1$, such that $\mathsf{v}_{\overline{x_1}}(W) \ge 2$. In view of $\mathrm{supp}(W_1) \cap \mathrm{supp}(W_2) = \emptyset$, we may assume without loss of generality that $\overline{x_1} = \overline{x_3}$. Let

$$W' = \big(W_1 \boldsymbol{\cdot} (\overline{x_1} \boldsymbol{\cdot} \overline{x_3})^{[-1]}\big) \boldsymbol{\cdot} W_2 \quad \text{ and } \quad \ell = \frac{|W'|}{2} = \frac{|W|}{2} - 1\,.$$

If $\sum_\ell(2W') = 2(\mathbb{Z}/n\mathbb{Z})$, it follows by $\sigma(W') = 2\sigma(W_2) - 2\overline{x_1} \in 2(\mathbb{Z}/n\mathbb{Z})$ that there exists a subsequence $T \mid W'$ of length $|T| = \ell$ such that $2\sigma(T) = \sigma(W')$. Hence we infer that $\sigma(T) = \sigma(W' \boldsymbol{\cdot} T^{[-1]})$ and $|T| = |W' \boldsymbol{\cdot} T^{[-1]}|$. Thus **A1** implies that $\psi(\overline{x_1} \boldsymbol{\cdot} \overline{x_3})$ and $\psi(W')$ are both product-one sequences, a contradiction. Therefore $\sum_\ell(2W') \subsetneq 2(\mathbb{Z}/n\mathbb{Z})$.

Let $H = \mathsf{H}\big(\sum_\ell(2W')\big)$. By Lemma 3.1, we obtain that

$$|\Sigma_\ell(2W')| \ \ge \ \left(\sum_{g\in(2(\mathbb{Z}/n\mathbb{Z}))/H} \min\{\ell, \mathsf{v}_g\big(\phi_H(2W')\big)\} - \ell + 1\right)|H|\,.$$

If $\mathsf{h}\big(\phi_H(2W')\big) \le \ell$, then

$$|\Sigma_\ell(2W')| \ \ge \ \big(|2W'| - \ell + 1\big)|H| \ \ge \ \frac{n}{2} = |2(\mathbb{Z}/n\mathbb{Z})|\,,$$

a contradiction. If there exist distinct $g_1, g_2 \in (2(\mathbb{Z}/n\mathbb{Z}))/H$ such that $\ell < \mathsf{v}_{g_k}\big(\phi_H(2W')\big)$ for all $k \in [1,2]$, then

$$|\Sigma_\ell(2W')| \ \ge \ (2\ell - \ell + 1)|H| \ \ge \ \frac{n}{2} = |2(\mathbb{Z}/n\mathbb{Z})|\,,$$

a contradiction. Thus there exists only one element, say $g \in (2(\mathbb{Z}/n\mathbb{Z}))/H$, such that $\mathsf{v}_g\big(\phi_H(2W')\big) > \ell$, which implies that

$$\mathsf{v}_g\big(\phi_H(2W')\big) \ge |2W'| + 1 - \frac{|\Sigma_\ell(2W')|}{|H|} \ge |W'| + 2 - \frac{n}{2|H|}\,.$$

A2. *If H is trivial, then $|W| = n$ and $2W_2 = (2\overline{x_2})^{[\frac{n}{2}]}$ with $\mathsf{v}_{2\overline{x_2}}(2W_1) = 0$.*

Proof of **A2**. Suppose that H is trivial. Then there exists $g \in 2(\mathbb{Z}/n\mathbb{Z})$ such that $\mathsf{v}_g(2W') \ge |W'| + 2 - \frac{n}{2} \ge \ell + 1$, and then we set $g = 2\overline{y}$ for some $y \in \mathbb{Z}$. If $\max\big\{\mathsf{v}_{2\overline{y}}(2W_1), \mathsf{v}_{2\overline{y}}(2W_2)\big\} \le 1$, then $\ell + 1 \le \mathsf{v}_{2\overline{y}}(2W') \le \mathsf{v}_{2\overline{y}}(2W_1) + \mathsf{v}_{2\overline{y}}(2W_2) \le 2$, and thus $\ell \le 1$. Since $\ell \ge 1$, we obtain that $\ell = 1$, and it follows by $\ell = \frac{|W|}{2} - 1$ that $|W| = n = 4$ and $|W_1| = |W_2| = 2$. Since $\max\big\{\mathsf{v}_{2\overline{y}}(2W_1), \mathsf{v}_{2\overline{y}}(2W_2)\big\} \le 1$, we obtain that $2\overline{x_1} \ne 2\overline{y}$, and hence $2 = \ell + 1 \le \mathsf{v}_{2\overline{y}}(2W') = \mathsf{v}_{2\overline{y}}(2W_2) \le 1$, a contradiction. Thus we must have that $\max\big\{\mathsf{v}_{2\overline{y}}(2W_1), \mathsf{v}_{2\overline{y}}(2W_2)\big\} \ge 2$, and assert that $\min\big\{\mathsf{v}_{2\overline{y}}(2W_1), \mathsf{v}_{2\overline{y}}(2W_2)\big\} = 0$. Assume to the contrary that $\min\big\{\mathsf{v}_{2\overline{y}}(2W_1), \mathsf{v}_{2\overline{y}}(2W_2)\big\} \ge 1$. Then we may suppose by shifting if necessary that $2y \equiv 0 \pmod n$, and by symmetry that $\mathsf{v}_{2\overline{y}}(2W_1) \le \mathsf{v}_{2\overline{y}}(2W_2)$. Since $\mathrm{supp}(W_1) \cap \mathrm{supp}(W_2) = \emptyset$, we can assume that $\mathsf{v}_{\overline{y}}(W_1) = 0$ and $\mathsf{v}_{\overline{y}}(W_2) \ge 2$, and it follows that

$$\sigma\bigg(W_1 \cdot (\overline{y} + \overline{\frac{n}{2}})^{[-1]}\bigg) = \sigma\bigg(W_2 \cdot (\overline{y} + \overline{\frac{n}{2}}) \cdot (\overline{y} \cdot \overline{y})^{[-1]}\bigg).$$

Thus **A1** ensures that $\psi(\overline{y} \cdot \overline{y})$ and $\psi(W \cdot (\overline{y} \cdot \overline{y})^{[-1]})$ are both product-one sequences, a contradiction. Hence $\min\big\{\mathsf{v}_{2\overline{y}}(2W_1), \mathsf{v}_{2\overline{y}}(2W_2)\big\} = 0$, and it follows that

$$\ell + 1 \le \mathsf{v}_{2\overline{y}}(2W') = \max\big\{\mathsf{v}_{2\overline{y}}(2(W_1 \cdot (\overline{x_1} \cdot \overline{x_3})^{[-1]})), \mathsf{v}_{2\overline{y}}(2W_2)\big\} \le \ell + 1\,.$$

Thus $\mathsf{v}_{2\overline{y}}(2W') = \mathsf{v}_{2\overline{y}}(2W_2) = |W_2| = \ell + 1$. If $|W| \ge n + 2$, then $\ell \ge \frac{n}{2}$, and thus $\mathsf{v}_{2\overline{y}}(2W') \ge |W'| + 2 - \frac{n}{2} \ge \ell + 2$, a contradiction. Therefore $|W| = n$ and $2W_2 = (2\overline{y})^{[\frac{n}{2}]} = (2\overline{x_2})^{[\frac{n}{2}]}$ with $\mathsf{v}_{2\overline{x_2}}(2W_1) = 0$. □

From now on, we assume that $(\overline{x_1}, \overline{x_3})$ is chosen to make $|H|$ maximal.

SUBCASE 2.1. H is non-trivial.

If $n = 4$, then $H \subset 2(\mathbb{Z}/4\mathbb{Z}) \cong C_2$ implies that $H = 2(\mathbb{Z}/4\mathbb{Z})$, whence $\sum_\ell(2W') = 2(\mathbb{Z}/4\mathbb{Z})$, a contradiction. Thus we can assume that $n \ge 6$.

Suppose that $[2(\mathbb{Z}/n\mathbb{Z}) : H] \ge 3$. Then $|H| \le \frac{n}{6}$, and since $\ell \ge \frac{n}{2} - 1$, we have

$$\mathsf{v}_g\big(\phi_H(2W')\big) \ge \ell + 1 + \frac{n}{2} - \frac{n}{2|H|} \ge \ell + 1 + 3|H| - 3\,.$$

Then it follows that $\min\big\{\mathsf{v}_g\big(\phi_H(2W_1)\big), \mathsf{v}_g\big(\phi_H(2W_2)\big)\big\} \ge 3|H| - 3$, for otherwise, we obtain that

$$\mathsf{v}_g\big(\phi_H(2W')\big) \le \mathsf{v}_g\big(\phi_H(2W_1)\big) + \mathsf{v}_g\big(\phi_H(2W_2)\big) \le (\ell + 1) + (3|H| - 4)\,,$$

a contradiction. Moreover, we obtain that $\max\big\{\mathsf{v}_g\big(\phi_H(2W_1)\big), \mathsf{v}_g\big(\phi_H(2W_2)\big)\big\} \ge 3|H| - 1$, for otherwise $3|H| - 2 \le \frac{n}{2} - 2 \le \ell - 1$ implies that

$$\mathsf{v}_g\big(\phi_H(2W')\big) \le \mathsf{v}_g\big(\phi_H(2W_1)\big) + \mathsf{v}_g\big(\phi_H(2W_2)\big) \le (\ell - 1) + (3|H| - 2)\,,$$

a contradiction. Then it suffices to show the case when $\mathsf{v}_g\big(\phi_H(2W_1)\big) \le \mathsf{v}_g\big(\phi_H(2W_2)\big)$. Indeed the other case when $\mathsf{v}_g\big(\phi_H(2W_1)\big) \ge \mathsf{v}_g\big(\phi_H(2W_2)\big)$ follows by an identical argument. Since $g \in (2(\mathbb{Z}/n\mathbb{Z}))/H$, by shifting if necessary, we can assume that $g = H$, whence $|(2W_1)_H| \ge 3|H| - 3$ and $|(2W_2)_H| \ge$

$3|H|-1$. Since H is a non-trivial cyclic group, it follows by $\mathsf{s}(H)=2|H|-1$ that there exist $U_1 \mid W_1$ and $U_2 \mid W_2$ such that $2U_1$ and $2U_2$ are zero-sum sequences over H of length $|U_1| = |U_2| = |H|$. Since $\left|\big(2(W_2 \boldsymbol{\cdot} U_2^{[-1]})\big)_H\right| \ge 2|H|-1$, there also exists $U_3 \mid W_2 \boldsymbol{\cdot} U_2^{[-1]}$ such that $2U_3$ is a zero-sum sequence over H of length $|U_3| = |H|$. Since $\sigma(U_k) \in \{\overline{0}, \overline{\frac{n}{2}}\}$ for all $k \in [1,3]$, there exist distinct $i,j \in [1,3]$ such that $\sigma(U_i) = \sigma(U_j)$. If $\sigma(U_1) = \sigma(U_j)$ for some $j \in [2,3]$, then $\sigma(W_1 \boldsymbol{\cdot} U_1^{[-1]}) = \sigma(W_2 \boldsymbol{\cdot} U_j^{[-1]})$, and thus **A1** implies that $\psi(U_1 \boldsymbol{\cdot} U_j)$ and $\psi\big(W \boldsymbol{\cdot} (U_1 \boldsymbol{\cdot} U_j)^{[-1]}\big)$ are both product-one sequences, a contradiction. If $\sigma(U_2) = \sigma(U_3)$, then $\sigma\big(W_1 \boldsymbol{\cdot} U_1^{[-1]}\big) = \sigma\big(W_2 \boldsymbol{\cdot} U_1 \boldsymbol{\cdot} (U_2 \boldsymbol{\cdot} U_3)^{[-1]}\big)$ and $|W_1 \boldsymbol{\cdot} U_1^{[-1]}| = \frac{|W|}{2} - |H| = |W_2 \boldsymbol{\cdot} U_1 \boldsymbol{\cdot} (U_2 \boldsymbol{\cdot} U_3)^{[-1]}|$. Thus **A1** ensures that $\psi(U_2 \boldsymbol{\cdot} U_3)$ and $\psi\big(W \boldsymbol{\cdot} (U_2 \boldsymbol{\cdot} U_3)^{[-1]}\big)$ are both product-one sequences, a contradiction.

Hence $[2(\mathbb{Z}/n\mathbb{Z}) : H] = 2$, and we obtain that $\mathsf{v}_g\big(\phi_H(2W')\big) \ge |W'|$. Then we may assume by shifting if necessary that $\operatorname{supp}(2W') \subset H$, and hence $\operatorname{supp}(W') \subset 2(\mathbb{Z}/n\mathbb{Z})$. Since $\operatorname{supp}(W_1) \cap \operatorname{supp}(W_2) = \emptyset$ and $|W_2| \ge \frac{n}{2}$, we infer in view of $\operatorname{supp}(W_2) \subset 2(\mathbb{Z}/n\mathbb{Z})$ that there exists $\overline{y} \in \operatorname{supp}(W_2)$ with $\mathsf{v}_{\overline{y}}(W_2) \ge 2$. By swapping the role between $(\overline{x_1}, \overline{x_3})$ and $(\overline{y}, \overline{y})$, we have that $|K| = |\mathsf{H}\big(\sum_\ell(2W'')\big)| \le |H|$ by the choice of $(\overline{x_1}, \overline{x_3})$, where $W'' = W_1 \boldsymbol{\cdot} \big(W_2 \boldsymbol{\cdot} (\overline{y} \boldsymbol{\cdot} \overline{y})^{[-1]}\big)$. Then we assert that $2\overline{x_1} \in H$. If K is trivial, then **A2** ensures that $2W_1 = (2\overline{x_1})^{[\frac{n}{2}]}$, and it follows by $n \ge 6$ that $2\overline{x_1} \in H$. If K is non-trivial, then we must have $|K| = |H|$, for otherwise $[2(\mathbb{Z}/n\mathbb{Z}) : K] \ge 3$, and then the argument from the beginning of **SUBCASE 2.1** leads to a contradiction. As two subgroups of a finite cyclic group having the same order are equal, we obtain that $K = H$, and since W' and W'' share at least one term in common ($n \ge 6$), it follows that the K-coset containing $\operatorname{supp}(2W'')$ must be H, whence $2\overline{x_1} \in H$. Thus, in all cases, we obtain that

$$\sigma(W') \;=\; 2\sigma(W_2) - 2\overline{x_1} \;\in\; H \;=\; \Sigma_\ell(2W')\,,$$

where the final equality follows from the fact that H is the stabilizer of $\sum_\ell(2W')$. Hence there exists $T \mid W'$ of length $|T| = \ell$ such that $2\sigma(T) = \sigma(W')$, and thus we infer that $\sigma(T) = \sigma(W' \boldsymbol{\cdot} T^{[-1]})$ and $|T| = |W' \boldsymbol{\cdot} T^{[-1]}|$. Therefore **A1** ensures that $\psi(\overline{x_1} \boldsymbol{\cdot} \overline{x_3})$ and $\psi(W')$ are both product-one sequences, a contradiction.

SUBCASE 2.2. H is trivial.

By **A2**, we have $2W_2 = (2\overline{x_2})^{[\frac{n}{2}]}$. If $\mathsf{h}(W_2) \ge 2$, then we may assume that $\overline{x_2} = \overline{x_4}$. By swapping the role between $(\overline{x_1}, \overline{x_3})$ and $(\overline{x_2}, \overline{x_4})$, it follows by the choice of $(\overline{x_1}, \overline{x_3})$ that $\mathsf{H}\big(\sum_\ell(2W'')\big)$ is also trivial, where $W'' = W_1 \boldsymbol{\cdot} \big(W_2 \boldsymbol{\cdot} (\overline{x_2} \boldsymbol{\cdot} \overline{x_4})^{[-1]}\big)$. Again by **A2**, we obtain that $2W_1 = (2\overline{x_1})^{[\frac{n}{2}]}$ with $2\overline{x_1} \ne 2\overline{x_2}$.

If $n = 4$, then we may assume in view of $\mathsf{h}(W) \ge 2$ that

$$W \;=\; W_1 \boldsymbol{\cdot} W_2 \;=\; \overline{x_1}^{[2]} \boldsymbol{\cdot} \Big(\overline{x_2} \boldsymbol{\cdot} (\overline{x_2} + \overline{2})\Big)\,,$$

where $\overline{x_1}, \overline{x_2} \in \mathbb{Z}/4\mathbb{Z}$ with $2\overline{x_1} \neq 2\overline{x_2}$ (by **A2**); Indeed, the other possibility is that $W = W_1 \boldsymbol{\cdot} W_2 = \overline{x_1}^{[2]} \boldsymbol{\cdot} \overline{x_2}^{[2]}$, which implies that $\psi(\overline{x_1} \boldsymbol{\cdot} \overline{x_1})$ and $\psi(\overline{x_2} \boldsymbol{\cdot} \overline{x_2})$ are both product-one sequences, a contradiction. Since $\sigma(W_1) = \sigma(W_2)$, it follows that $x_1 \equiv x_2 + 1 \pmod 2$. Thus $\psi(W)$ is the desired sequence for (a).

If $n \geq 6$, then it follows by $2W_2 = (2\overline{x_2})^{[\frac{n}{2}]}$ and the Pigeonhole Principle that $\mathsf{h}(W_2) \geq 2$. Thus we obtain that $2W = (2\overline{x_1})^{[\frac{n}{2}]} \boldsymbol{\cdot} (2\overline{x_2})^{[\frac{n}{2}]}$, whence

$$W \;=\; W_1 \boldsymbol{\cdot} W_2 \;=\; \left(\overline{x_1}^{[v]} \boldsymbol{\cdot} \big(\overline{x_1} + \frac{\overline{n}}{2}\big)^{[\frac{n}{2}-v]}\right) \boldsymbol{\cdot} \left(\overline{x_2}^{[w]} \boldsymbol{\cdot} \big(\overline{x_2} + \frac{\overline{n}}{2}\big)^{[\frac{n}{2}-w]}\right),$$

where $\overline{x_1}, \overline{x_2} \in \mathbb{Z}/n\mathbb{Z}$ with $2\overline{x_1} \neq 2\overline{x_2}$ (by **A2**), and $v, w \in [0, \frac{n}{2}]$. Since $\sigma(W_1) = \sigma(W_2)$, it follows that $x_1 - x_2 \equiv v - w \pmod 2$. All that remains is to show that $\gcd(x_1 - x_2, \frac{n}{2}) = 1$. Assume to the contrary that $\gcd\big(x_1 - x_2, \frac{n}{2}\big) = d \geq 2$. Then we set $n' = \frac{n}{2d}$, and since $2W' = (2\overline{x_1})^{[\ell-1]} \boldsymbol{\cdot} (2\overline{x_2})^{[\ell+1]}$, it follows by $n'(2x_1 - 2x_2) \equiv 0 \pmod n$ that

$$\Sigma_\ell(2W') \;=\; \big\{k(2\overline{x_1} - 2\overline{x_2}) - 2\overline{x_2} \,|\, k \in [0, n'-1]\big\}\,.$$

Thus we obtain that $2\overline{x_1} - 2\overline{x_2} \in \mathsf{H}\big(\sum_\ell(2W')\big) = H$, and since H is trivial, it follows that $2\overline{x_1} = 2\overline{x_2}$, a contradiction. Therefore $\gcd(x_1 - x_2, \frac{n}{2}) = 1$.

To prove the "In particular" statement, we assume to the contrary that there exists a minimal product-one sequence S such that $S = S_1 \boldsymbol{\cdot} S_2$, where $S_1 \in \mathcal{F}\big(\langle\alpha\rangle\big)$ and $S_2 \in \mathcal{F}\big(G \setminus \langle\alpha\rangle\big)$ with $|S_2| \geq n+2$. Then we suppose that $S_2 = \prod^\bullet_{i\in[1,|S_2|]} \alpha^{x_i}\tau$ and $S_1 = T_1 \boldsymbol{\cdot} T_2$ such that $\pi^*(T_1)(\alpha^{x_1}\tau)\pi^*(T_2)(\alpha^{x_2}\tau \cdots \alpha^{x_{|S_2|}}\tau) = 1_G$. Since $S \in \mathcal{A}(G)$, it follows that

$$S'' \;=\; \big(\pi^*(T_1)\alpha^{x_1}\tau\big) \boldsymbol{\cdot} \big(\pi^*(T_2)\alpha^{x_2}\tau\big) \boldsymbol{\cdot} \left(\prod\nolimits^\bullet_{i\in[3,|S_2|]} \alpha^{x_i}\tau\right) \;\in\; \mathcal{A}\big(G \setminus \langle\alpha\rangle\big)$$

of length $|S''| = |S_2| \geq n+2$, but this is impossible by the main statement. $\square$

Proposition 3.3. *Let $G = \langle \alpha, \tau \,|\, \alpha^{2n} = 1_G,\ \tau^2 = \alpha^n,\ \text{ and }\ \tau\alpha = \alpha^{-1}\tau\rangle$ be a dicyclic group, where $n \geq 2$. Let $S \in \mathcal{F}(G)$ be a minimal product-one sequence such that $|S| \geq 2n+2$ and $\operatorname{supp}(S) \subset G \setminus \langle\alpha\rangle$. Then S is a sequence of length $|S| = 2n+2$ having the form*

$$S \;=\; (\alpha^x\tau)^{[n+2]} \,\boldsymbol{\cdot}\, S_0\,,$$

where $x \in [0, 2n-1]$, and S_0 is a sequence of length $|S_0| = n$ having one of the following two forms:

(a) *$S_0 = (\alpha^y\tau)^{[2]} \boldsymbol{\cdot} \alpha^{y+n}\tau \boldsymbol{\cdot} \alpha^{y_1}\tau \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} \alpha^{y_{n-3}}\tau$, where $n \geq 3$, $y, y_1, \ldots, y_{n-3} \in [0, 2n-1]$ such that $2y \not\equiv 2x \pmod{2n}$, $2y_i \not\equiv 2x \pmod{2n}$ for all i, and $(y_1 + \cdots + y_{n-3}) + 3y + n + x \equiv (n+1)(x+n) \pmod{2n}$.*

(b) *$S_0 = (\alpha^y\tau)^{[n]}$, where $y \in [0, 2n-1]$ such that $2y \not\equiv 2x \pmod{2n}$ and $ny + x \equiv (n+1)(x+n) \pmod{2n}$.*

In particular, there are no minimal product-one sequences S over G such that $S = S_1 \boldsymbol{\cdot} S_2$ for some $S_1 \in \mathcal{F}\big(\langle\alpha\rangle\big)$ and $S_2 \in \mathcal{F}\big(G \setminus \langle\alpha\rangle\big)$ of length $|S_2| \geq 2n+4$.

Proof. For every $x \in \mathbb{Z}$, we set $\overline{x} = x + 2n\mathbb{Z} \in \mathbb{Z}/2n\mathbb{Z}$. Let $S = \prod^{\bullet}_{i \in [1,|S|]} \alpha^{x_i}\tau \in \mathcal{A}(G)$ be of length $|S| \ge 2n+2$ with $\alpha^{x_1}\tau \cdots \alpha^{x_{|S|}}\tau = 1_G$, where $x_1, \ldots, x_{|S|} \in [0, 2n-1]$. Since $S \in \mathcal{A}(G)$, it follows that $|S|$ is even, and after renumbering if necessary, we set

$$W = \overline{x_1} \cdot \ldots \cdot \overline{x_{|S|}} = W_1 \cdot W_2 \in \mathcal{F}(\mathbb{Z}/2n\mathbb{Z}),$$

where $W_1 = \prod^{\bullet}_{i \in [1,|S|/2]} \overline{x_{2i-1}}$, and $W_2 = \prod^{\bullet}_{i \in [1,|S|/2]} \overline{x_{2i}}$. Thus we have that $\sigma(W_1) = \sigma(W_2) + |W_1|\overline{n}$. If we shift the sequence W by $\overline{y}$ for some $y \in \mathbb{Z}$, then the corresponding sequence $S' = \prod^{\bullet}_{i \in [1,|S|]} \alpha^{x_i + y}\tau$ is still a minimal product-one sequence. If S' has the asserted structure, then the same is true for S whence we may shift the sequence W whenever this is convenient. For every subsequence $U = \overline{y_1} \cdot \ldots \cdot \overline{y_v}$ of W, we denote by $\psi(U) = \alpha^{y_1}\tau \cdot \ldots \cdot \alpha^{y_v}\tau$ the corresponding subsequence of S.

A1. *Let $U = U_1 \cdot U_2$ be a subsequence of W such that $|U_1| = |U_2|$ and $\sigma(U_1) = \sigma(U_2) + |U_1|\overline{n}$. Then $\psi(U)$ is a product-one sequence.*

Proof of **A1**. Suppose that $U_1 = \overline{y_1} \cdot \ldots \cdot \overline{y_{|U_1|}}$ and $U_2 = \overline{z_1} \cdot \ldots \cdot \overline{z_{|U_1|}}$. Since $\sigma(U_1) = \sigma(U_2) + |U_1|\overline{n}$, it follows that

$$\alpha^{z_1}\tau\alpha^{y_1}\tau \cdots \alpha^{z_{|U_1|}}\tau\alpha^{y_{|U_1|}}\tau = \alpha^{(z_1 + \cdots + z_{|U_1|}) - (y_1 + \cdots + y_{|U_1|}) + |U_1|n} = 1_G,$$

whence $\psi(U)$ is a product-one sequence. $\square$

If $\mathrm{supp}(W_1) \cap \big(\mathrm{supp}(W_2) + \overline{n}\big) \ne \emptyset$, say $\overline{x_1} = \overline{x_2} + \overline{n}$, then since $\sigma(W_1) = \sigma(W_2) + |W_1|\overline{n}$, it follows by **A1** that $\psi(\overline{x_1} \cdot \overline{x_2})$ and $\psi\big(W \cdot (\overline{x_1} \cdot \overline{x_2})^{[-1]}\big)$ are both product-one sequences, a contradiction. Therefore $\mathrm{supp}(W_1) \cap \big(\mathrm{supp}(W_2) + \overline{n}\big) = \emptyset$, and since $|S| \ge 2n+2$, it follows that $\mathsf{h}(W) \ge 2$.

A2. $\min\big\{\mathsf{v}_{2\overline{g}}(2W_1), \mathsf{v}_{2\overline{g}}(2W_2)\big\} \le 1$ *for every $\overline{g} \in \mathbb{Z}/2n\mathbb{Z}$.*

Proof of **A2**. Assume to the contrary that there exists $\overline{g} \in \mathbb{Z}/2n\mathbb{Z}$ such that $\min\big\{\mathsf{v}_{2\overline{g}}(2W_1), \mathsf{v}_{2\overline{g}}(2W_2)\big\} \ge 2$. Then, for each $i \in [1,2]$, we have $\mathsf{v}_{\overline{g}}(W_i) + \mathsf{v}_{\overline{g}+\overline{n}}(W_i) = \mathsf{v}_{2\overline{g}}(2W_i) \ge 2$. We may assume without loss of generality that $\mathsf{v}_{\overline{g}}(W_1) \ge 1$. Since $\mathrm{supp}(W_1) \cap \big(\mathrm{supp}(W_2) + \overline{n}\big) = \emptyset$, we must have $\mathsf{v}_{\overline{g}+\overline{n}}(W_2) = 0$, whence $\mathsf{v}_{\overline{g}}(W_2) \ge 2$. Since $\mathrm{supp}(W_1) \cap \big(\mathrm{supp}(W_2) + \overline{n}\big) = \emptyset$, we must have $\mathsf{v}_{\overline{g}+\overline{n}}(W_1) = 0$, whence $\mathsf{v}_{\overline{g}}(W_1) \ge 2$. We set $U_1 = U_2 = \overline{g} \cdot \overline{g}$. It follows that $U_1 \mid W_1$ and $U_2 \mid W_2$ such that $|U_1| = |U_2|$ with $\sigma(U_1) = \sigma(U_2) + |U_1|\overline{n}$, and $|W_1 \cdot U_1^{[-1]}| = |W_2 \cdot U_2^{[-1]}|$ with $\sigma\big(W_1 \cdot U_1^{[-1]}\big) = \sigma\big(W_2 \cdot U_2^{[-1]}\big) + |W_1 \cdot U_1^{[-1]}|\overline{n}$. Thus **A1** ensures that $\psi(U_1 \cdot U_2)$ and $\psi\big(W \cdot (U_1 \cdot U_2)^{[-1]}\big)$ are both product-one sequences, a contradiction. $\square$

CASE 1. There exists $\overline{y} \in \mathrm{supp}(W)$ such that $\mathsf{v}_{\overline{y}}(W) \ge 2$ and $\overline{y} + \overline{n} \in \mathrm{supp}(W)$.

In view of $\mathrm{supp}(W_1) \cap \big(\mathrm{supp}(W_2) + \overline{n}\big) = \emptyset$, we may assume without loss of generality that $\overline{y} \boldsymbol{\cdot} (\overline{y} + \overline{n}) \mid W_1$. Let

$$W' = W \boldsymbol{\cdot} \big(\overline{y} \boldsymbol{\cdot} (\overline{y} + \overline{n})\big)^{[-1]} \qquad \text{and} \qquad \ell = \frac{|W'|}{2} = \frac{|W|}{2} - 1 \,.$$

If $\sum_\ell(2W') = 2(\mathbb{Z}/2n\mathbb{Z})$, then since $\sigma(W') + \ell\overline{n} = 2\sigma(W_2) + 2\ell\overline{n} - 2\overline{y} \in 2(\mathbb{Z}/2n\mathbb{Z})$, it follows that there exists a subsequence $T \mid W'$ of length $|T| = \ell$ such that $2\sigma(T) = \sigma(W') + \ell\overline{n}$. Hence we infer that $\sigma(T) = \sigma(W' \boldsymbol{\cdot} T^{[-1]}) + |T|\overline{n}$ and $|T| = |W' \boldsymbol{\cdot} T^{[-1]}|$. Thus **A1** ensures that $\psi\big(\overline{y} \boldsymbol{\cdot} (\overline{y} + \overline{n})\big)$ and $\psi(W')$ are both product-one sequences, a contradiction. Therefore $\sum_\ell(2W') \subsetneq 2(\mathbb{Z}/2n\mathbb{Z})$.

Let $H = \mathsf{H}\big(\sum_\ell(2W')\big)$. By Lemma 3.1, we obtain that

$$|\Sigma_\ell(2W')| \ \ge \ \left(\sum_{g \in (2(\mathbb{Z}/2n\mathbb{Z}))/H} \min\{\ell, \mathsf{v}_g\big(\phi_H(2W')\big)\} - \ell + 1 \right) |H| \,.$$

If $\mathsf{h}\big(\phi_H(2W')\big) \le \ell$, then

$$|\Sigma_\ell(2W')| \ \ge \ \big(|2W'| - \ell + 1\big)|H| \ \ge \ n \ = \ |2(\mathbb{Z}/2n\mathbb{Z})| \,,$$

a contradiction. If there exist distinct $g_1, g_2 \in (2(\mathbb{Z}/2n\mathbb{Z}))/H$ such that $\ell < \mathsf{v}_{g_k}\big(\phi_H(2W')\big)$ for all $k \in [1,2]$, then

$$|\Sigma_\ell(2W')| \ \ge \ (2\ell - \ell + 1)|H| \ \ge \ n \ = \ |2(\mathbb{Z}/2n\mathbb{Z})| \,,$$

a contradiction. Thus there exists only one element, say $g \in (2(\mathbb{Z}/2n\mathbb{Z}))/H$, such that $\mathsf{v}_g\big(\phi_H(2W')\big) > \ell$, which implies that

$$\mathsf{v}_g\big(\phi_H(2W')\big) \ \ge \ |2W'| + 1 - \frac{|\Sigma_\ell(2W')|}{|H|} \ \ge \ |W'| + 2 - \frac{n}{|H|} \,.$$

SUBCASE 1.1. H is non-trivial.

If $[2(\mathbb{Z}/2n\mathbb{Z}) : H] = 2$, then $\mathsf{v}_g\big(\phi_H(2W')\big) \ge |W'|$. We may assume by shifting if necessary that $\mathrm{supp}(2W') \subset H$, and hence $\mathrm{supp}(W') \subset 2(\mathbb{Z}/2n\mathbb{Z})$. Since $\mathsf{v}_{\overline{y}}(W) \ge 2$, it follows that $\overline{y} \in \mathrm{supp}(W') \subset 2(\mathbb{Z}/2n\mathbb{Z})$, whence $\sigma(W') + \ell\overline{n} = 2\sigma(W_2) - 2\overline{y} \in H$. Thus there exists $T \mid W'$ of length $|T| = \ell$ such that $2\sigma(T) = \sigma(W') + \ell\overline{n}$, and hence we infer that $\sigma(T) = \sigma(W' \boldsymbol{\cdot} T^{[-1]}) + |T|\overline{n}$ and $|T| = |W' \boldsymbol{\cdot} T^{[-1]}|$. It follows by **A1** that $\psi\big(\overline{y} \boldsymbol{\cdot} (\overline{y} + \overline{n})\big)$ and $\psi(W')$ are both product-one sequences, a contradiction.

Therefore $[2(\mathbb{Z}/2n\mathbb{Z}) : H] \ge 3$, and hence $|H| \le \frac{n}{3}$. Since $\ell \ge n$, we have

$$\mathsf{v}_g\big(\phi_H(2W')\big) \ \ge \ \ell + 1 + (n+1) - \frac{n}{|H|} \ \ge \ \ell + 2 + 3|H| - 3 \,.$$

Then $\min\big\{\mathsf{v}_g\big(\phi_H(2W_1)\big), \mathsf{v}_g\big(\phi_H(2W_2)\big)\big\} \ge 3|H| - 2$, for otherwise, we obtain that

$$\mathsf{v}_g\big(\phi_H(2W')\big) \ \le \ \mathsf{v}_g\big(\phi_H(2W_1)\big) + \mathsf{v}_g\big(\phi_H(2W_2)\big) \ \le \ (\ell+1) + (3|H| - 3) \,,$$

a contradiction. Since $g \in (2(\mathbb{Z}/2n\mathbb{Z}))/H$, by shifting if necessary, we can assume that $g = H$, whence $|(2W_i)_H| \ge 3|H| - 2$ for all $i \in [1,2]$. It follows by

$\mathsf{s}(H) = 2|H|-1$ that there exist $U_1 \mid W_1$ and $V_1 \mid W_2$ of length $|U_1| = |V_1| = |H|$ such that $\sigma(U_1), \sigma(V_1) \in \{\overline{0}, \overline{n}\}$. Therefore $|\big(2W_1 \boldsymbol{\cdot} (2U_1)^{[-1]}\big)_H| \ge 2|H|-2$ and $|\big(2W_2 \boldsymbol{\cdot} (2V_1)^{[-1]}\big)_H| \ge 2|H|-2$.

Suppose that there exist $U_2 \mid W_1 \boldsymbol{\cdot} U_1^{[-1]}$ and $V_2 \mid W_2 \boldsymbol{\cdot} V_1^{[-1]}$ with $|U_2| = |V_2| = |H|$ and $\sigma(U_2), \sigma(V_2) \in \{\overline{0}, \overline{n}\}$. If there exists $i \in [1,2]$ such that $\sigma(U_i) = \sigma(V_i) + |H|\overline{n}$, then **A1** implies that $\psi(U_i \boldsymbol{\cdot} V_i)$ and $\psi\big(W \boldsymbol{\cdot} (U_i \boldsymbol{\cdot} V_i)^{[-1]}\big)$ are both product-one sequences, a contradiction. Otherwise, we have $\sigma(U_1 \boldsymbol{\cdot} U_2) = \sigma(V_1 \boldsymbol{\cdot} V_2) + 2|H|\overline{n}$, whence **A1** ensures that $\psi(U_1 \boldsymbol{\cdot} U_2 \boldsymbol{\cdot} V_1 \boldsymbol{\cdot} V_2)$ and $\psi\big(W \boldsymbol{\cdot} (U_1 \boldsymbol{\cdot} U_2 \boldsymbol{\cdot} V_1 \boldsymbol{\cdot} V_2)^{[-1]}\big)$ are both product-one sequences, a contradiction.

Assume that either $\big(2W_1 \boldsymbol{\cdot} (2U_1)^{[-1]}\big)_H$ or $\big(2W_2 \boldsymbol{\cdot} (2V_1)^{[-1]}\big)_H$ dose not contain a zero-sum subsequence of length $|H|$, say $2W_1 \boldsymbol{\cdot} (2U_1)^{[-1]}$, which then forces $|\big(2W_1 \boldsymbol{\cdot} (2U_1)^{[-1]}\big)_H| = 2|H|-2$. By [16, Proposition 5.1.12], there exist $h_1, h_2 \in H$ with $\mathrm{ord}(h_1 - h_2) = |H|$ such that $\Big(2W_1 \boldsymbol{\cdot} (2U_1)^{[-1]}\Big)_H = h_1^{[|H|-1]} \boldsymbol{\cdot} h_2^{[|H|-1]}$. Then $\mathrm{ord}(h_1 - h_2) = |H|$ ensures that

$$H \;=\; \underbrace{\{h_1, h_2\} + \cdots + \{h_1, h_2\}}_{|H|-1} \;=\; \Sigma_{|H|-1}\big(h_1^{[|H|-1]} \boldsymbol{\cdot} h_2^{[|H|-1]}\big)\,.$$

Thus we infer that there exist subsequences $2U_3 \mid 2W_1 \boldsymbol{\cdot} (2U_1)^{[-1]}$ and $2V_3 \mid 2W_2 \boldsymbol{\cdot} (2V_1)^{[-1]}$ such that $|2U_3| = |2V_3| = |H|-1$ and $\sigma(2U_3) = \sigma(2V_3)$. Hence $\sigma(U_3) = \sigma(V_3)$ or $\sigma(U_3) = \sigma(V_3) + \overline{n}$. If there exists $i \in \{1,3\}$ such that $\sigma(U_i) = \sigma(V_i) + |U_i|\overline{n}$, then **A1** implies that $\psi(U_i \boldsymbol{\cdot} V_i)$ and $\psi\big(W \boldsymbol{\cdot} (U_i \boldsymbol{\cdot} V_i)^{[-1]}\big)$ are both product-one sequences, a contradiction. Otherwise, we have $\sigma(U_1 \boldsymbol{\cdot} U_3) = \sigma(V_1 \boldsymbol{\cdot} V_3) + (2|H|-1)\overline{n}$, whence **A1** ensures that $\psi(U_1 \boldsymbol{\cdot} U_3 \boldsymbol{\cdot} V_1 \boldsymbol{\cdot} V_3)$ and $\psi\big(W \boldsymbol{\cdot} (U_1 \boldsymbol{\cdot} U_3 \boldsymbol{\cdot} V_1 \boldsymbol{\cdot} V_3)^{[-1]}\big)$ are both product-one sequences, a contradiction.

SUBCASE 1.2. H is trivial.

Since $\ell = \frac{|W'|}{2} \ge n$, it follows that $\mathsf{v}_g(2W') \ge |W'| + 2 - n \ge \ell + 2$. Hence **A2** ensures that $\min\big\{\mathsf{v}_g(2W_1), \mathsf{v}_g(2W_2)\big\} = 1$. If $g = 2\overline{y}$, it follows by $\overline{y} \boldsymbol{\cdot} (\overline{y} + \overline{n}) \mid W_1$ that $\mathsf{v}_g(2W_2) = 1$, whence $\ell + 2 \le \mathsf{v}_g(2W') = \mathsf{v}_g(2W_1) - 2 + 1 \le \ell$, a contradiction. Thus $g \ne 2\overline{y}$. Since

$$\ell + 2 \;\le\; \mathsf{v}_g(2W') \;=\; \mathsf{v}_g\Big(2\big(W_1 \boldsymbol{\cdot} (\overline{y} \boldsymbol{\cdot} (\overline{y} + \overline{n}))^{[-1]}\big)\Big) + \mathsf{v}_g(2W_2)\,,$$

we have $\mathsf{v}_g(2W_1) = 1$ and $\mathsf{v}_g(2W_2) = \ell + 1$. Then $\mathsf{v}_g(2W') = \ell + 2$. If $|W| \ge 2n+4$, then $\ell \ge n+1$, and hence $\mathsf{v}_g(2W') \ge |W'| + 2 - n \ge \ell + 3$, a contradiction. Therefore $|W| = 2n+2$, $\ell = n$, $2W_2 = (2\overline{x})^{[n+1]}$, and $\mathsf{v}_{2\overline{x}}(2W_1) = 1$ for some $\overline{x} \in \mathbb{Z}/2n\mathbb{Z}$ with $2\overline{x} = g \ne 2\overline{y}$.

Since $\mathrm{supp}(W_1) \cap \big(\mathrm{supp}(W_2) + \overline{n}\big) = \emptyset$, we may assume that $W_2 = \overline{x}^{[n+1]}$ and $\mathsf{v}_{\overline{x}}(W_1) = 1$. It follows by $\mathsf{v}_{\overline{y}}(W) \ge 2$ and $|W_1| = n+1$ that $\overline{x} \boldsymbol{\cdot} \overline{y} \boldsymbol{\cdot} \overline{y} \boldsymbol{\cdot} (\overline{y} + \overline{n}) \mid W_1$. Then $n \ge 3$ and

$$W \;=\; W_1 \boldsymbol{\cdot} W_2 \;=\; (\overline{x} \boldsymbol{\cdot} T) \boldsymbol{\cdot} \overline{x}^{[n+1]}\,,$$

where $T \in \mathcal{F}\big(\mathbb{Z}/2n\mathbb{Z}\big)$ with $|T| = n$ such that $2\overline{x} \notin \operatorname{supp}(2T)$ and $\overline{y}^{[2]} \boldsymbol{\cdot} (\overline{y}+\overline{n}) \mid T$. Since $\sigma(W_1) = \sigma(W_2) + |W_1|\overline{n}$, it follows that $\sigma(T) + \overline{x} = (n+1)\overline{x} + (n+1)\overline{n}$. Therefore $\psi(W)$ is the desired sequence for (a).

CASE 2. For every $\overline{x} \in \operatorname{supp}(W)$ with $\mathsf{v}_{\overline{x}}(W) \ge 2$, we have that $\overline{x} + \overline{n} \notin \operatorname{supp}(W)$.

If $\mathsf{h}(2W) \le 2$, then we have

$$2n + 2 \le |W| = |2W| \le \mathsf{h}(2W)|2(\mathbb{Z}/2n\mathbb{Z})| \le 2n\,,$$

a contradiction, and from the case hypothesis, we have $\mathsf{h}(W) = \mathsf{h}(2W) \ge 3$. Let $\overline{x} \in \operatorname{supp}(W)$ be an element with $\mathsf{v}_{\overline{x}}(W) = \mathsf{h}(W) \ge 3$, and assume without loss of generality that

$$\mathsf{v}_{\overline{x}}(W_1) \ge \mathsf{v}_{\overline{x}}(W_2) \quad \text{with} \quad \mathsf{v}_{\overline{x}}(W_1) \ge 2\,.$$

If $\mathsf{h}(W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]}) \le 1$, then it follows by the case hypothesis that

$$2n \le |W| - 2 = |W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]}| \le |(\mathbb{Z}/2n\mathbb{Z}) \setminus \{\overline{x} + \overline{n}\}| = 2n - 1\,,$$

a contradiction, whence $\mathsf{h}(W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]}) \ge 2$. Let $\overline{y} \in \operatorname{supp}(W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]})$ be an element with $\mathsf{v}_{\overline{y}}(W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]}) \ge 2$, and let

$$W' = W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x} \boldsymbol{\cdot} \overline{y} \boldsymbol{\cdot} \overline{y})^{[-1]} \quad \text{and} \quad \ell = \frac{|W'|}{2} = \frac{|W|}{2} - 2\,.$$

Suppose in addition that $\overline{y}$ is chosen to satisfy either that $\mathsf{v}_{\overline{y}}(W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]}) = \mathsf{h}(W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]})$, or that both $\mathsf{v}_{\overline{y}}(W_2) \ge 3$ and $\mathsf{h}(W) \le \ell + 2$.

If $\sum_\ell(2W') = 2(\mathbb{Z}/2n\mathbb{Z})$, then since $\sigma(W') + \ell\overline{n} = 2\sigma(W_2) + (2\ell+2)\overline{n} - 2\overline{x} - 2\overline{y} \in 2(\mathbb{Z}/2n\mathbb{Z})$, it follows that there exists a subsequence $T \mid W'$ of length $|T| = \ell$ such that $2\sigma(T) = \sigma(W') + \ell\overline{n}$. Hence we infer $\sigma(T) = \sigma(W' \boldsymbol{\cdot} T^{[-1]}) + |T|\overline{n}$ and $|T| = |W' \boldsymbol{\cdot} T^{[-1]}|$. Thus **A1** ensures that $\psi\big(\overline{x}^{[2]} \boldsymbol{\cdot} \overline{y}^{[2]}\big)$ and $\psi(W')$ are both product-one sequences, a contradiction. Therefore $\sum_\ell(2W') \subsetneq 2(\mathbb{Z}/2n\mathbb{Z})$.

Let $H = \mathsf{H}\big(\sum_\ell(2W')\big)$. As at the start of the proof of **CASE 1**, it follows by Lemma 3.1 that there exists only one element, say $g \in (2(\mathbb{Z}/2n\mathbb{Z}))/H$, such that $\mathsf{v}_g\big(\phi_H(2W')\big) \ge \ell + 1$, which implies that

$$\mathsf{v}_g\big(\phi_H(2W')\big) \ge |2W'| + 1 - \frac{|\Sigma_\ell(2W')|}{|H|} \ge |W'| + 2 - \frac{n}{|H|}\,.$$

SUBCASE 2.1. H is non-trivial.

If $n = 2$, then $H \subset 2(\mathbb{Z}/4\mathbb{Z}) \cong C_2$ implies that $H = 2(\mathbb{Z}/4\mathbb{Z})$, whence $\sum_\ell(2W') = 2(\mathbb{Z}/4\mathbb{Z})$, a contradiction. Thus we can assume that $n \ge 3$.

If $[2(\mathbb{Z}/2n\mathbb{Z}) : H] = 2$, then $\mathsf{v}_g\big(\phi_H(2W')\big) \ge |W'|$. We may assume by shifting if necessary that $\operatorname{supp}(2W') \subset H$, and hence $\operatorname{supp}(W') \subset 2(\mathbb{Z}/2n\mathbb{Z})$. We assert that $\sigma(W') + \ell\overline{n} = 2\sigma(W_2) - 2\overline{x} - 2\overline{y} \in H$. Clearly this holds true for $\overline{x} = \overline{y}$. Suppose $\overline{x} \ne \overline{y}$. Since $\mathsf{v}_{\overline{x}}(W) = \mathsf{h}(W) \ge 3$, it follows that $\overline{x} \in \operatorname{supp}(W') \subset 2(\mathbb{Z}/2n\mathbb{Z})$. If $\mathsf{v}_{\overline{y}}(W_2) \ge 3$, then $\overline{y} \in \operatorname{supp}(W') \subset 2(\mathbb{Z}/2n\mathbb{Z})$.

Suppose that $\mathsf{v}_{\overline{y}}(W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]}) = \mathsf{h}(W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]})$, and we need to verify $\overline{y} \in \operatorname{supp}(W') \subset 2(\mathbb{Z}/2n\mathbb{Z})$. If $\mathsf{h}(2W') \le 2$, then

$$2n-2 \le |W'| = |2W'| \le \mathsf{h}(2W')|H| \le n\,,$$

a contradiction to $n \ge 3$. Hence, in view of the main case hypothesis, we have $\mathsf{h}(W') = \mathsf{h}(2W') \ge 3$. Since $\mathsf{h}(W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]}) \ge \mathsf{h}(W') \ge 3$, it follows that $\overline{y} \in \operatorname{supp}(W') \subset 2(\mathbb{Z}/2n\mathbb{Z})$. Thus $\sigma(W') + \ell\overline{n} \in H$, which implies that there exists a subsequence $T \mid W'$ of length $|T| = \ell$ such that $2\sigma(T) = \sigma(W') + \ell\overline{n}$. Then $\sigma(T) = \sigma(W' \boldsymbol{\cdot} T^{[-1]}) + |T|\overline{n}$ and $|T| = |W' \boldsymbol{\cdot} T^{[-1]}|$. It follows by **A1** that $\psi\big(\overline{x}^{[2]} \boldsymbol{\cdot} \overline{y}^{[2]}\big)$ and $\psi(W')$ are both product-one sequences, a contradiction.

Therefore $[2(\mathbb{Z}/2n\mathbb{Z}) : H] \ge 3$, and hence $|H| \le \frac{n}{3}$. Since $\ell = \frac{|W'|}{2} \ge n-1$, we have

$$\mathsf{v}_g\big(\phi_H(2W')\big) \ge \ell + 1 + n - \frac{n}{|H|} \ge \ell + 1 + 3|H| - 3\,.$$

We assert that $\min\big\{\mathsf{v}_g\big(\phi_H(2W_1)\big), \mathsf{v}_g\big(\phi_H(2W_2)\big)\big\} \ge 3|H| - 2$. Assume to the contrary that $\min\big\{\mathsf{v}_g\big(\phi_H(2W_1)\big), \mathsf{v}_g\big(\phi_H(2W_2)\big)\big\} \le 3|H| - 3$. If $\mathsf{v}_g\big(\phi_H(2W_2)\big) \le \ell$, then $\mathsf{v}_{\overline{x}}(W_1) \ge 2$ implies that

$$\mathsf{v}_g\big(\phi_H(2W')\big) \le \mathsf{v}_g\Big(\phi_H\big(2(W_1 \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]})\big)\Big) + \mathsf{v}_g\big(\phi_H(2W_2)\big) \le \ell + 3|H| - 3\,,$$

a contradiction. Thus $\mathsf{v}_g\big(\phi_H(2W_2)\big) \ge \ell + 1 \ge n$, and hence $\mathsf{h}(2W_2) \ge \frac{n}{|H|} \ge 3$. The main case hypothesis ensures that $\mathsf{h}(W_2) = \mathsf{h}(2W_2) \ge 3$. If $\mathsf{v}_{\overline{y}}(W_2) \ge 2$, then $\mathsf{v}_g\big(\phi_H(2W')\big) = \mathsf{v}_g\Big(\phi_H\big(2(W_1 \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]})\big)\Big) + \mathsf{v}_g\Big(\phi_H\big(2(W_2 \boldsymbol{\cdot} (\overline{y} \boldsymbol{\cdot} \overline{y})^{[-1]})\big)\Big) \le \ell + 3|H| - 3$, a contradiction. Suppose that $\mathsf{v}_{\overline{y}}(W_2) \le 1$. Then we infer that $\mathsf{v}_{\overline{y}}(W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]}) = \mathsf{h}(W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]})$. It follows by $\mathsf{h}(W_2) \ge 3$ that there exists $\overline{z} \in \operatorname{supp}(W_2)$ with $\mathsf{v}_{\overline{z}}(W_2) = \mathsf{h}(W_2) \ge 3$. Then we assert that $\mathsf{v}_{\overline{x}}(W) = \mathsf{h}(W) \le \ell + 2$. Assume to the contrary that $\mathsf{v}_{\overline{x}}(W) = \mathsf{h}(W) \ge \ell + 3$. Since $\mathsf{v}_{\overline{x}}(W_1) \ge 2$, **A2** implies $W_1 = \overline{x}^{[\ell+2]}$ with $\mathsf{v}_{\overline{x}}(W_2) = 1$, whence $\overline{y} = \overline{x}$. By the main case hypothesis, we have $\mathsf{v}_{2\overline{x}}(2W') = \mathsf{v}_{\overline{x}}(W') = \ell - 1$. Since $g \in (2(\mathbb{Z}/2n\mathbb{Z}))/H$ is the only element satisfying $\mathsf{v}_g\big(\phi_H(2W')\big) \ge \ell + 1 \ge 3$, it follows again by the main case hypothesis that $g = 2\overline{z}$, $W_2 = \overline{x} \boldsymbol{\cdot} \overline{z}^{[\ell+1]}$, and $\mathsf{v}_{\overline{z}}(W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]}) = \mathsf{h}(W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]})$. By swapping the role between $\overline{y}$ and $\overline{z}$, the argument used in the case above when $\mathsf{v}_{\overline{y}}(W_2) \ge 2$ leads to a contradiction. Thus $\mathsf{v}_{\overline{x}}(W) = \mathsf{h}(W) \le \ell + 2$, and then the swapping argument again leads to a contradiction. Since $g \in (2(\mathbb{Z}/2n\mathbb{Z}))/H$, by shifting if necessary, we can assume that $g = H$, whence $|(2W_i)_H| \ge 3|H| - 2$ for all $i \in [1,2]$. By the same lines of the proof of **SUBCASE 1.1**, we get a contradiction to $S \in \mathcal{A}(G)$.

SUBCASE 2.2. H is trivial.

Since $\ell = \frac{|W'|}{2} \ge n - 1$, it follows that $\mathsf{v}_g(2W') = \mathsf{v}_g\big(\phi_H(2W')\big) \ge |W'| + 2 - n \ge \ell + 1$, and by **A2**,

$$\mathsf{h}(2W) = \mathsf{v}_{2\overline{x}}(2W) = \mathsf{v}_{2\overline{x}}(2W_1) + \mathsf{v}_{2\overline{x}}(2W_2) \le (\ell+2) + 1 = \ell + 3\,.$$

Thus we have $\mathsf{v}_{2\overline{x}}(2W') \le \mathsf{v}_{2\overline{x}}(2W) - 2 \le \ell + 1$.

Suppose that $\ell = 1$. Then $|W| = 6$, $n = 2$, and $|2W'| = 2$. Hence $\mathsf{v}_g(2W') = 2$ and $W = \overline{x}^{[2]} \boldsymbol{\cdot} \overline{y}^{[2]} \boldsymbol{\cdot} \overline{w_1} \boldsymbol{\cdot} \overline{w_2}$ for some $\overline{w_1}, \overline{w_2} \in \mathbb{Z}/2n\mathbb{Z}$ with $2\overline{w_1} = 2\overline{w_2} = g$. If $\overline{w_1} = \overline{w_2} + \overline{n}$, then $\psi(\overline{w_1} \boldsymbol{\cdot} \overline{w_2})$ and $\psi\big(\overline{x}^{[2]} \boldsymbol{\cdot} \overline{y}^{[2]}\big)$ are both product-one sequences, a contradiction. Therefore $\overline{w_1} = \overline{w_2}$. Since $\operatorname{ord}(\alpha^i\tau) = 4$ for all $i \in [0, 2n-1]$ and $\psi(W)$ is a product-one sequence, we obtain that $|\{\overline{x}, \overline{y}, \overline{w_1}\}| \ge 2$. Since $\mathsf{v}_{\overline{x}}(W) = \mathsf{h}(W) \ge 3$ and $\mathsf{h}\big(W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]}\big) \ge 2$, it follows that either $\overline{x} = \overline{y}$ or $\overline{x} = \overline{w_1}$. Since $\sigma(W_1) = \sigma(W_2) + |W_1|\overline{n}$, we have

$$W = W_1 \boldsymbol{\cdot} W_2 = \overline{x}^{[3]} \boldsymbol{\cdot} \big(\overline{x} \boldsymbol{\cdot} \overline{w}^{[2]}\big)$$

for some $\overline{w} \in \mathbb{Z}/4\mathbb{Z}$ with $2\overline{w} \ne 2\overline{x}$. Thus $\psi(W)$ is the desired sequence for (b).

Suppose that $\ell \ge 2$. We assume to the contrary that $\mathsf{v}_{\overline{y}}(W_2) \ge 3$ and $\mathsf{h}(W) \le \ell + 2$. Since $\mathsf{v}_{2\overline{x}}(2W) = \mathsf{v}_{\overline{x}}(W) \le \ell + 2$, it follows that $\mathsf{v}_{2\overline{x}}(2W') \le \ell$, whence $g \ne 2\overline{x}$. In view of $\mathsf{v}_{\overline{x}}(W_1) \ge 2$, $\mathsf{v}_{\overline{y}}(W_2) \ge 2$, and **A2**, we must have $2\overline{y} \ne 2\overline{x}$. Let $g = 2\overline{z}$ for some $\overline{z} \in \mathbb{Z}/2n\mathbb{Z}$. If $g \ne 2\overline{y}$, then by the main case hypothesis, $\overline{x}$, $\overline{y}$ and $\overline{z}$ are all distinct elements with $\mathsf{v}_{\overline{x}}(W) \ge \mathsf{v}_{\overline{z}}(W) \ge \ell + 1$ and $\mathsf{v}_{\overline{y}}(W) \ge 3$, implying $2\ell + 4 = |W| \ge 2(\ell+1) + 3 = 2\ell + 5$, a contradiction. Thus $g = 2\overline{y}$, and again by the main case hypothesis, we have $\overline{z} = \overline{y}$. Hence $\mathsf{v}_{\overline{y}}\big(W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]}\big) = \mathsf{v}_{\overline{z}}(W') + 2 \ge \ell + 3$, contradicting that $\mathsf{h}(W) \le \ell + 2$.

Therefore $\mathsf{v}_{\overline{y}}\big(W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]}\big) = \mathsf{h}\big(W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]}\big)$, and in view of the main case hypothesis, we have

$$3 \le \ell + 1 \le \mathsf{v}_g(2W') \le \mathsf{v}_{2\overline{y}}\big(2\big(W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]}\big)\big) \le \mathsf{v}_{2\overline{x}}(2W)\,.$$

Then it follows by $|2W| = 2\ell + 4$ and $\mathsf{h}(2W) \le \ell + 3$ that $|\{2\overline{x}, 2\overline{y}, g\}| = 2$. If $2\overline{y} = g$, then $2\overline{x} \ne 2\overline{y}$ and $\mathsf{v}_{2\overline{x}}(2W) \ge \mathsf{v}_{2\overline{y}}(2W) \ge \ell + 3$, whence $2\ell + 4 = |2W| \ge 2\ell + 6$, a contradiction. Thus $2\overline{y} \ne g$.

If $2\overline{x} = 2\overline{y}$, then $\mathsf{v}_{2\overline{x}}(2W) = 2 + \mathsf{v}_{2\overline{y}}\big(2\big(W \boldsymbol{\cdot} (\overline{x} \boldsymbol{\cdot} \overline{x})^{[-1]}\big)\big) \ge \ell + 3$ implies that $\mathsf{v}_{2\overline{x}}(2W) = \ell + 3$ and $\mathsf{v}_g(2W') = \ell + 1$. If $|W| \ge 2n + 4$, then $\ell \ge n$, and hence $\ell + 1 = \mathsf{v}_g(2W') \ge |W'| + 2 - n \ge \ell + 2$, a contradiction. Thus $|W| = 2n + 2$ and $\ell = n - 1$. Since $\mathsf{v}_{\overline{x}}(W_1) \ge \mathsf{v}_{\overline{x}}(W_2)$, we have $\mathsf{v}_{2\overline{x}}(2W_1) \ge \mathsf{v}_{2\overline{x}}(2W_2)$, and hence **A2** ensures that $\mathsf{v}_{2\overline{x}}(2W_2) = 1$. It follows in view of the main case hypothesis that

$$W = W_1 \boldsymbol{\cdot} W_2 = \overline{x}^{[n+1]} \boldsymbol{\cdot} \big(\overline{x} \boldsymbol{\cdot} \overline{z}^{[n]}\big)\,,$$

where $\overline{z} \in \mathbb{Z}/2n\mathbb{Z}$ with $2\overline{z} = g \ne 2\overline{x}$. Since $\sigma(W_1) = \sigma(W_2) + |W_1|\overline{n}$, we have $nz + x \equiv (n+1)(x+n) \pmod{2n}$. Therefore $\psi(W)$ is the desired sequence for (b).

If $2\overline{x} = g$, then $\mathsf{v}_{2\overline{x}}(2W) \ge 2 + \mathsf{v}_g(2W') \ge \ell + 3$ implies that $\mathsf{v}_{2\overline{x}}(2W) = \ell + 3$ and $\mathsf{v}_{2\overline{y}}(2W) = \ell + 1$. The same argument as shown above ensures that $W = W_1 \boldsymbol{\cdot} W_2 = \overline{x}^{[n+1]} \boldsymbol{\cdot} \big(\overline{x} \boldsymbol{\cdot} \overline{y}^{[n]}\big)$, where $\overline{x}, \overline{y} \in \mathbb{Z}/2n\mathbb{Z}$ with $2\overline{x} \ne 2\overline{y}$, and $ny + x \equiv (n+1)(x+n) \pmod{2n}$. Thus $\psi(W)$ is the desired sequence for (b).

To prove the "In particular" statement, we assume to the contrary that there exists a minimal product-one sequence S such that $S = S_1 \boldsymbol{\cdot} S_2$, where $S_1 \in \mathcal{F}\big(\langle\alpha\rangle\big)$ and $S_2 \in \mathcal{F}\big(G \setminus \langle\alpha\rangle\big)$ of length $|S_2| \ge 2n+4$. Then we suppose that $S_2 = \prod^\bullet_{i\in[1,|S_2|]} \alpha^{x_i}\tau$ and $S_1 = T_1 \boldsymbol{\cdot} T_2$ such that $\pi^*(T_1)(\alpha^{x_1}\tau)\pi^*(T_2)(\alpha^{x_2}\tau \cdots \alpha^{x_{|S_2|}}\tau)$

$= 1_G$. Since $S \in \mathcal{A}(G)$, it follows that

$$S'' = \big(\pi^*(T_1)\alpha^{x_1}\tau\big) \cdot \big(\pi^*(T_2)\alpha^{x_2}\tau\big) \cdot \Big({\prod}^\bullet_{i\in[3,|S_2|]} \alpha^{x_i}\tau\Big) \in \mathcal{A}\big(G \setminus \langle\alpha\rangle\big)$$

and $|S''| = |S_2| \geq 2n+4$, a contradiction to the main statement. $\square$

4. The main results

Theorem 4.1. *Let G be a dihedral group of order $2n$, where $n \in \mathbb{N}_{\geq 3}$ is odd. A sequence S over G of length $\mathsf{D}(G)$ is a minimal product-one sequence if and only if it has one of the following two forms:*

(a) *There exist $\alpha, \tau \in G$ such that $G = \langle \alpha, \tau \mid \alpha^n = \tau^2 = 1_G \ \text{ and } \ \tau\alpha = \alpha^{-1}\tau\rangle$ and $S = \alpha^{[2n-2]} \cdot \tau^{[2]}$.*

(b) *There exist $\alpha, \tau \in G$ and $i, j \in [0, n-1]$ with $\gcd(i-j, n) = 1$ such that $G = \langle \alpha, \tau \mid \alpha^n = \tau^2 = 1_G \ \text{ and } \ \tau\alpha = \alpha^{-1}\tau\rangle$ and $S = (\alpha^i\tau)^{[n]} \cdot (\alpha^j\tau)^{[n]}$.*

Proof. We fix $\alpha, \tau \in G$ such that $G = \langle \alpha, \tau \mid \alpha^n = \tau^2 = 1_G \ \text{ and } \ \tau\alpha = \alpha^{-1}\tau\rangle$. Then

$$G = \big\{\alpha^i \mid i \in [0, n-1]\big\} \cup \big\{\alpha^i\tau \mid i \in [0, n-1]\big\}\,.$$

Let $G_0 = G \setminus \langle\alpha\rangle$. If $|S_{G_0}| = 0$, then $S \in \mathcal{F}\big(\langle\alpha\rangle\big)$, and since $|S| = 2n > \mathsf{D}\big(\langle\alpha\rangle\big) = n$, it follows that S is not a minimal product-one sequence, a contradiction. Since S is a product-one sequence, we have that $|S_{G_0}|$ is even. We distinguish three cases depending on $|S_{G_0}|$.

CASE 1. $|S_{G_0}| = 2$.

Then we may assume by changing generating set if necessary that $S = T_1 \cdot \tau \cdot T_2 \cdot (\alpha^x\tau)$ with $\pi^*(T_1)(\tau)\pi^*(T_2)(\alpha^x\tau) = 1_G$, where $x \in [0, n-1]$ and $T_1, T_2 \in \mathcal{F}\big(\langle\alpha\rangle\big)$. Since $S \in \mathcal{A}(G)$, it follows that T_1 and T_2 must be both product-one free sequences, and thus $|T_1| = |T_2| = n-1$. Then we may assume by Lemma 2.2.1 that

$$T_1 = \alpha^{[n-1]} \qquad \text{and} \qquad T_2 = (\alpha^j)^{[n-1]}\,,$$

where $j \in [0, n-1]$ with $\gcd(j, n) = 1$. Since $\pi^*(T_1)(\tau)\pi^*(T_2)(\alpha^x\tau) = 1_G$, it follows that $-1 \equiv -j + x \pmod n$, and thus it suffices to show that $x = 0$; Indeed, if this holds, then $j = 1$, whence $S = \alpha^{[2n-2]} \cdot \tau^{[2]}$ which is the desired sequence for (a).

Assume to the contrary that $x \in [1, n-1]$ so that $j \neq 1$.

SUBCASE 1.1. j is even.

Let $S_1 = \alpha^j \cdot \alpha^{[n-j]} \in \mathcal{B}(G)$. Since j is even and n is odd, $-1 \equiv -j + x \pmod n$ implies that

$$S_2 = \alpha^{[\frac{j-2}{2}]} \cdot (\alpha^j)^{[\frac{n-3}{2}]} \cdot (\alpha^j \cdot \tau) \cdot \alpha^{[\frac{j-2}{2}]} \cdot (\alpha^j)^{[\frac{n-3}{2}]} \cdot (\alpha \cdot \alpha^x\tau) \in \mathcal{B}(G)\,,$$

whence $S = S_1 \cdot S_2$ contradicts that $S \in \mathcal{A}(G)$.

SUBCASE 1.2. j is odd.

Since $-1 \equiv -j + x \pmod n$, we obtain that $x = j - 1$, whence x is even. Then $n - 1 - x$ is even, and we obtain that

$$\Big(\alpha^{[\frac{n-1-x}{2}]}(\alpha^j)^{[\frac{n-1}{2}]}\alpha^{[x]}\Big)\tau\Big(\alpha^{[\frac{n-1-x}{2}]}(\alpha^j)^{[\frac{n-1}{2}]}\Big)\alpha^x\tau \;=\; 1_G\,.$$

Let $S_1 = \alpha^{[\frac{n-1-x}{2}]} \boldsymbol{\cdot} (\alpha^j)^{[\frac{n-1}{2}]} \boldsymbol{\cdot} \alpha^{[x]} \in \mathcal{F}\big(\langle\alpha\rangle\big)$. Since x is even, it follows that $|S_1| = n-1+\frac{x}{2} \ge n$, and hence S_1 has a product-one subsequence W. Thus W and $S \boldsymbol{\cdot} W^{[-1]}$ are both product-one sequences, contradicting that $S \in \mathcal{A}(G)$.

CASE 2. $|S_{G_0}| \in [4, 2n-2]$.

Then we may assume by changing generating set if necessary that $S = T_1 \boldsymbol{\cdot} \tau \boldsymbol{\cdot} T_2 \boldsymbol{\cdot} T_3 \boldsymbol{\cdot} (\alpha^x\tau)$, where $x \in [0, n-1]$, $T_1, T_2 \in \mathcal{F}\big(\langle\alpha\rangle\big)$, and $T_3 \in \mathcal{F}(G_0)$ with $|T_3| = |S_{G_0}| - 2$. Moreover, we suppose that $\pi^*(T_1)(\tau)\pi^*\big(T_2 \boldsymbol{\cdot} T_3'\big)(\alpha^x\tau) = 1_G$, where $T_3 = \prod^\bullet_{i\in[1,|T_3|]} g_i$ is an ordered sequence and $T_3' = \prod^\bullet_{i\in[1,|T_3|/2]}(g_{2i-1}g_{2i}) \in \mathcal{F}\big(\langle\alpha\rangle\big)$. Then T_1 and $T_2 \boldsymbol{\cdot} T_3'$ are both product-one free sequences and

$$|T_1 \boldsymbol{\cdot} T_2 \boldsymbol{\cdot} T_3'| \;=\; \Big(2n - |S_{G_0}|\Big) + \frac{|S_{G_0}| - 2}{2} \;\ge\; n\,.$$

Let $T_1 = p_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} p_{|T_1|}$, $T_2 = f_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} f_{|T_2|}$, and $T_3' = q_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} q_{|T_3'|}$. Then we consider

- $H_1 = \big\{p_1,\ p_1p_2,\ \ldots,\ (p_1\cdots p_{|T_1|})\big\}$, and
- $H_2 = \big\{q_1,\ q_1q_2,\ \ldots,\ (q_1\cdots q_{|T_3'|}),\ (q_1\cdots q_{|T_3'|}f_1),\ (q_1\cdots q_{|T_3'|}f_1f_2),$ $(q_1\cdots q_{|T_3'|}f_1f_2f_3),\ \ldots,\ (q_1\cdots q_{|T_3'|}f_1\cdots f_{|T_2|})\big\}$.

Since both T_1 and $T_2 \boldsymbol{\cdot} T_3'$ are product-one free, it follows that $H_1, H_2 \subset \langle\alpha\rangle\setminus\{1_G\}$ with $|H_1| = |T_1|$, $|H_2| = |T_2 \boldsymbol{\cdot} T_3'|$, and $|H_1| + |H_2| = |T_1 \boldsymbol{\cdot} T_2 \boldsymbol{\cdot} T_3'| \ge n$. Since $|\langle\alpha\rangle| = n$, we obtain that $H_1 \cap H_2 \ne \emptyset$, and hence we infer that there exist $W_1 \,|\, T_1$, $W_2 \,|\, T_2$, and $W_3' \,|\, T_3'$ such that W_3' is a non-trivial sequence and $\pi^*(W_1) = \pi^*(W_2 \boldsymbol{\cdot} W_3')$. Let W_3 denote the corresponding subsequence of T_3 and assume that $W_3 = (\alpha^{y_1}\tau) \boldsymbol{\cdot} (\alpha^{y_2}\tau) \boldsymbol{\cdot} W_3''$. Then $Z = W_2 \boldsymbol{\cdot} (\alpha^{y_1}\tau) \boldsymbol{\cdot} W_1 \boldsymbol{\cdot} (\alpha^{y_2}\tau) \boldsymbol{\cdot} W_3''$ and $S \boldsymbol{\cdot} Z^{[-1]}$ are both product-one sequences, contradicting that $S \in \mathcal{A}(G)$.

CASE 3. $|S_{G_0}| = 2n$.

Since $|S| = 2n = |S_{G_0}|$, we may assume that

$$S \;=\; \alpha^{k_1}\tau \boldsymbol{\cdot} \alpha^{\ell_1}\tau \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} \alpha^{k_n}\tau \boldsymbol{\cdot} \alpha^{\ell_n}\tau \quad \text{with} \quad \alpha^{k_1}\tau\alpha^{\ell_1}\tau\cdots\alpha^{k_n}\tau\alpha^{\ell_n}\tau \;=\; 1_G\,,$$

where $k_1, \ldots, k_n, \ell_1, \ldots, \ell_n \in [0, n-1]$. Then we set $S' = a^{k_1-\ell_1} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} a^{k_n-\ell_n} \in \mathcal{B}\big(\langle\alpha\rangle\big)$ of length $|S'| = n$. Since $S \in \mathcal{A}(G)$, it follows that $S' \in \mathcal{A}\big(\langle\alpha\rangle\big)$, and by applying Lemma 2.2.1,

$$k_1 - \ell_1 \;\equiv\; k_2 - \ell_2 \;\equiv\; \cdots \;\equiv\; k_n - \ell_n \pmod n \tag{4.1}$$

with $\gcd(k_i - \ell_i, n) = 1$ for all $i \in [1, n]$. Let $j \in [1, n-1]$. Then we observe that

$$\alpha^{k_j}\tau\alpha^{\ell_j}\tau\alpha^{k_{j+1}}\tau \;=\; \alpha^{k_j-\ell_j+k_{j+1}}\tau \;=\; \alpha^{k_{j+1}}\tau\alpha^{\ell_j}\tau\alpha^{k_j}\tau\,.$$

By swapping the role between $\alpha^{k_j}\tau$ and $\alpha^{k_{j+1}}\tau$, we obtain that

$$S'' \;=\; \alpha^{k_1-\ell_1} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} \alpha^{k_{j+1}-\ell_j} \boldsymbol{\cdot} \alpha^{k_j-\ell_{j+1}} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} \alpha^{k_n-\ell_n} \;\in\; \mathcal{A}\big(\langle\alpha\rangle\big)$$

of length $|S''| = n$. Hence it follows again by applying Lemma 2.2.1 that

$$k_1 - \ell_1 \equiv \cdots \equiv k_{j+1} - \ell_j \equiv k_j - \ell_{j+1} \equiv \cdots \equiv k_n - \ell_n \pmod n,$$

and thus (4.1) ensures that $k_j = k_{j+1}$, whence $k_1 = k_2 = \cdots = k_n$. Similarly we also obtain that $\ell_1 = \ell_2 = \cdots = \ell_n$, whence $S = (\alpha^{k_1}\tau)^{[n]} \boldsymbol{\cdot} (\alpha^{\ell_1}\tau)^{[n]}$ with $\gcd(k_1 - \ell_1, n) = 1$, which is the desired sequence for (b). $\square$

Theorem 4.2. *Let G be a dihedral group of order $2n$, where $n \in \mathbb{N}_{\ge 4}$ is even. A sequence S over G of length $\mathsf{D}(G)$ is a minimal product-one sequence if and only if there exist α, $\tau \in G$ such that $G = \langle \alpha, \tau \mid \alpha^n = \tau^2 = 1_G \text{ and } \tau\alpha = \alpha^{-1}\tau \rangle$ and $S = \alpha^{[n+\frac{n}{2}-2]} \boldsymbol{\cdot} \tau \boldsymbol{\cdot} (\alpha^{\frac{n}{2}}\tau)$.*

Proof. We fix $\alpha, \tau \in G$ such that $G = \langle \alpha, \tau \mid \alpha^n = \tau^2 = 1_G \text{ and } \tau\alpha = \alpha^{-1}\tau \rangle$. Then

$$G = \big\{ \alpha^i \mid i \in [0, n-1] \big\} \cup \big\{ \alpha^i \tau \mid i \in [0, n-1] \big\}.$$

Let $G_0 = G \setminus \langle \alpha \rangle$. If $|S_{G_0}| = 0$, then $S \in \mathcal{F}\big(\langle \alpha \rangle\big)$, and since $|S| = n + \frac{n}{2} > \mathsf{D}\big(\langle \alpha \rangle\big) = n$, it follows that S is not a minimal product-one sequence, a contradiction. Since S is a product-one sequence, Proposition 3.2 ensures that $|S_{G_0}| \in [2, n]$ is even. We distinguish two cases depending on $|S_{G_0}|$.

CASE 1. $|S_{G_0}| = 2$.

Then we may assume by changing generating set if necessary that $S = T_1 \boldsymbol{\cdot} \tau \boldsymbol{\cdot} T_2 \boldsymbol{\cdot} (\alpha^x \tau)$ with $\pi^*(T_1)(\tau)\pi^*(T_2)(\alpha^x\tau) = 1_G$, where $x \in [0, n-1]$ and $T_1, T_2 \in \mathcal{F}\big(\langle \alpha \rangle\big)$. Since $S \in \mathcal{A}(G)$, it follows that T_1 and T_2 must be both product-one free sequences.

If $|T_1| \ge \frac{n}{2}$ and $|T_2| \ge \frac{n}{2}$, then T_1^2 and $T_2^2 \in \mathcal{F}\big(\langle \alpha^2 \rangle\big)$ (see (3.1)) with $|T_1^2| \ge \frac{n}{2}$ and $|T_2^2| \ge \frac{n}{2}$, and it follows by $\mathsf{D}\big(\langle \alpha^2 \rangle\big) = \frac{n}{2}$ that there exist $W_1 \mid T_1$ and $W_2 \mid T_2$ such that W_1^2 and W_2^2 are product-one sequences over $\langle \alpha^2 \rangle$. Since T_1 and T_2 are product-one free, we obtain that $\pi^*(W_1) = \alpha^{\frac{n}{2}} = \pi^*(W_2)$. Therefore $W_1 \boldsymbol{\cdot} W_2$ and $S \boldsymbol{\cdot} (W_1 \boldsymbol{\cdot} W_2)^{[-1]}$ are both product-one sequences, contradicting that $S \in \mathcal{A}(G)$.

Thus either $|T_1| \le \frac{n}{2} - 1$ or $|T_2| \le \frac{n}{2} - 1$, and we may assume that $|T_1| = \frac{n}{2} - 1$ and $|T_2| = n - 1$. Then Lemma 2.2.1 implies that $T_2 = (\alpha^j)^{[n-1]}$ for some odd $j \in [1, n-1]$. Then we may assume by changing generating set if necessary that $j = 1$ so that $S = T_3 \boldsymbol{\cdot} \tau \boldsymbol{\cdot} \alpha^{[n-1]} \boldsymbol{\cdot} (\alpha^y \tau)$, where $y \in [0, n-1]$ and $T_3 \in \mathcal{F}\big(\langle \alpha \rangle\big)$. Since $T_3 \boldsymbol{\cdot} \alpha \boldsymbol{\cdot} \tau \boldsymbol{\cdot} (\alpha^y \tau)$ is a product-one sequence, we have that

$$T_3 \boldsymbol{\cdot} \alpha^{[\frac{n}{2}]} \boldsymbol{\cdot} \tau \boldsymbol{\cdot} \alpha^{[\frac{n}{2}-1]} \boldsymbol{\cdot} (\alpha^y \tau) \in \mathcal{B}(G).$$

It follows that $T_3 \boldsymbol{\cdot} \alpha^{[\frac{n}{2}]}$ is a product-one free sequence of length $n-1$, and again by Lemma 2.2.1 that $T_3 = \alpha^{[\frac{n}{2}-1]}$. Since $(\frac{n}{2} - 1) \equiv (n-1) + y \pmod n$, we infer that $y = \frac{n}{2}$, and the assertion follows.

CASE 2. $|S_{G_0}| \in [4, n]$.

SUBCASE 2.1. $n = 4$.

Then we may assume by changing generating set if necessary that $S = \alpha^{r_1} \cdot \alpha^{r_2} \cdot \tau \cdot \alpha^x\tau \cdot \alpha^y\tau \cdot \alpha^z\tau$ for some $r_1, r_2 \in [1,3]$ and $x, y, z \in [0,3]$. If $\alpha^{r_1}\alpha^{r_2}\tau\alpha^x\tau\alpha^y\tau\alpha^z\tau = 1_G$, then $S' = \alpha^{r_1} \cdot \alpha^{r_2} \cdot \alpha^{-x} \cdot \alpha^{y-z} \in \mathcal{A}\big(\langle\alpha\rangle\big)$, and hence it follows by applying Lemma 2.2.1 that $r_1 \equiv r_2 \equiv -x \equiv y - z \equiv j \pmod 4$ for some odd $j \in [1,3]$. Thus $S = S_1 \cdot S_2$, where $S_1 = \tau \cdot \alpha^{r_1} \cdot \alpha^x\tau$ and $S_2 = \alpha^y\tau \cdot \alpha^{r_2} \cdot \alpha^z\tau$ are both product-one sequences, contradicting that $S \in \mathcal{A}(G)$. Thus we can assume that $\alpha^{r_1}\tau\alpha^{r_2}\alpha^x\tau\alpha^y\tau\alpha^z\tau = 1_G$, and we consider

$$S'' = \alpha^{r_1} \cdot \alpha^{-r_2} \cdot \tau \cdot \alpha^x\tau \cdot \alpha^y\tau \cdot \alpha^z\tau \in \mathcal{B}(G)\,.$$

Then, by the same argument as shown above, we obtain that $S'' \notin \mathcal{A}(G)$. Let $S'' = U_1 \cdot U_2$ for some $U_1, U_2 \in \mathcal{B}(G)$. Since S is a minimal product-one sequence, we must have that

$$U_1 = \alpha^{r_1} \cdot \alpha^{-r_2} \quad \text{and} \quad U_2 = \tau \cdot \alpha^x\tau \cdot \alpha^y\tau \cdot \alpha^z\tau$$

are both minimal product-one sequences, whence we obtain that $r_1 = r_2$. Since $U_2 \in \mathcal{A}(G)$, Proposition 3.2 implies that

$$U_2 = \tau \cdot \alpha\tau \cdot \alpha^2\tau \cdot \alpha^3\tau \quad \text{or} \quad U_2 = (\alpha^{x_1}\tau)^{[2]} \cdot \alpha^{y_1}\tau \cdot \alpha^{y_1+2}\tau\,,$$

where $x_1, y_1 \in [0,3]$ with $x_1 \equiv y_1 + 1 \pmod 2$. Since $S \in \mathcal{A}(G)$, we obtain that either $r_1 = r_2 = 1$ or $r_1 = r_2 = 3$. If $r_1 = r_2 = 1$, then

$$S = (\alpha\cdot\tau\cdot\alpha\tau)\cdot(\alpha\cdot\alpha^2\tau\cdot\alpha^3\tau) \quad \text{or} \quad S = (\alpha\cdot\alpha^{x_1}\tau\cdot\alpha^{y_1}\tau)\cdot(\alpha^{x_1}\tau\cdot\alpha\cdot\alpha^{y_1+2}\tau)\,,$$

contradicting that $S \in \mathcal{A}(G)$. If $r_1 = r_3 = 3$, then

$$S = (\tau\cdot\alpha^3\cdot\alpha\tau)\cdot(\alpha^2\tau\cdot\alpha^3\cdot\alpha^3\tau) \quad \text{or} \quad S = (\alpha^3\cdot\alpha^{x_1}\tau\cdot\alpha^{y_1}\tau)\cdot(\alpha^{x_1}\tau\cdot\alpha^3\cdot\alpha^{y_1+2}\tau)\,,$$

contradicting that $S \in \mathcal{A}(G)$.

SUBCASE 2.2. $n \geq 6$.

Then we may assume by changing generating set if necessary that $S = T_1 \cdot \tau \cdot T_2 \cdot T_3 \cdot (\alpha^x\tau)$, where $x \in [0, n-1]$, $T_1, T_2 \in \mathcal{F}\big(\langle\alpha\rangle\big)$ with $|T_2| \geq |T_1| \geq 0$, and $T_3 \in \mathcal{F}(G_0)$ with $|T_3| = |S_{G_0}| - 2$. Moreover, we suppose that $\pi^*(T_1)(\tau)\pi^*\big(T_2 \cdot T_3'\big)(\alpha^x\tau) = 1_G$, where $T_3 = \prod^\bullet_{i\in[1,|T_3|]} g_i$ is an ordered sequence and $T_3' = \prod^\bullet_{i\in[1,|T_3|/2]}(g_{2i-1}g_{2i}) \in \mathcal{F}\big(\langle\alpha\rangle\big)$. Then T_1 and $T_2 \cdot T_3'$ are both product-one free sequences and

$$|T_1 \cdot T_2 \cdot T_3'| = \Big(n + \frac{n}{2} - |S_{G_0}|\Big) + \frac{|S_{G_0}| - 2}{2} \geq n - 1\,.$$

If $|T_1\cdot T_2\cdot T_3'| \geq n$, then we infer that there exist subsequences $W_1 \mid T_1$, $W_2 \mid T_2$, and $W_3' \mid T_3'$ such that W_3' is a non-trivial sequence (this follows by the same argument as used in **CASE 2** of Theorem 4.1) and $\pi^*(W_1) = \pi^*\big(W_2 \cdot W_3'\big)$. Let W_3 denote the corresponding subsequence of T_3 and assume that $W_3 = (\alpha^{y_1}\tau) \cdot (\alpha^{y_2}\tau) \cdot W_3''$. Then $Z = W_2 \cdot (\alpha^{y_1}\tau) \cdot W_1 \cdot (\alpha^{y_2}\tau) \cdot W_3''$ and $S \cdot Z^{[-1]}$ are both product-one sequences, contradicting that $S \in \mathcal{A}(G)$.

Suppose that $|T_1 \cdot T_2 \cdot T_3'| = n - 1$. Then $|T_3'| = \frac{n}{2} - 1$ and $|T_2| \geq \frac{n}{4}$. Since $T_2 \cdot T_3'$ is a product-one free sequence with $|T_2 \cdot T_3'| \geq \frac{3n}{4} - 1 \geq \frac{n+1}{2}$, it follows by Lemma 2.2 that $T_2 \cdot T_3'$ is g-smooth for some $g \in \langle\alpha\rangle$ with $\mathrm{ord}(g) = n$, and for every $z \in \Pi(T_2 \cdot T_3')$, there exists a subsequence $W \mid T_2 \cdot T_3'$ with $g \mid W$ such that $\pi^*(W) = z$. Since $|T_3'| = \frac{n}{2} - 1$, Lemma 2.2.3 implies that $g \mid T_3'$.

If $\Pi(T_1) \cap \Pi(T_2 \cdot T_3') \neq \emptyset$, then there exist subsequences $W_1 \mid T_1$, $W_2 \mid T_2$, and $W_3' \mid T_3'$ such that W_3' is a non-trivial sequence (this follows from the above paragraph that we can choose $W_2 \cdot W_3' \mid T_2 \cdot T_3'$ such that $g \mid W_2 \cdot W_3'$ and $g \mid T_3'$) and $\pi^*(W_1) = \pi^*(W_2 \cdot W_3')$. Let W_3 denote the corresponding subsequence of T_3 and assume that $W_3 = (\alpha^{y_1}\tau) \cdot (\alpha^{y_2}\tau) \cdot W_3''$. Then $Z = W_2 \cdot (\alpha^{y_1}\tau) \cdot W_1 \cdot (\alpha^{y_2}\tau) \cdot W_3''$ and $S \cdot Z^{[-1]}$ are both product-one sequences, contradicting that $S \in \mathcal{A}(G)$. Hence $\Pi(T_1) \cap \Pi(T_2 \cdot T_3') = \emptyset$, and it follows that $T_1^{-1} \cdot T_2 \cdot T_3'$ is a product-one free sequence of length $n-1$. By Lemma 2.2.1, there exists an odd $j \in [1, n-1]$ such that

$$T_1^{-1} \cdot T_2 \cdot T_3' = \big(\alpha^j\big)^{[n-1]},$$

and we may assume by changing generating set if necessary that $j = 1$ so that $x = 1$. If $|T_1| \geq 1$, then

$$\big(\alpha \cdot \alpha^{-1}\big)^{[|T_1|]} \qquad \text{and} \qquad \alpha^{[1+\frac{n-2-2|T_1|}{2}]} \cdot \tau \cdot \alpha^{[\frac{n-2-2|T_1|}{2}]} \cdot \alpha\tau$$

are both product-one sequences, contradicting that $S \in \mathcal{A}(G)$. Thus $|T_1| = 0$, and then we obtain that $T_3 = \big(\alpha^{r+1}\tau \cdot \alpha^r\tau\big)^{[\frac{n}{2}-1]}$ for some $r \in [0, n-1]$ (this follows by the swapping argument as used in **CASE 3** of Theorem 4.1). This implies that $S = (\alpha \cdot \tau \cdot \alpha\tau) \cdot (\alpha^{r+1}\tau \cdot \alpha \cdot \alpha^r\tau)^{[\frac{n}{2}-1]}$, contradicting that $S \in \mathcal{A}(G)$. □

Theorem 4.3. *Let G be a dicyclic group of order $4n$, where $n \geq 2$. A sequence S over G of length $\mathsf{D}(G)$ is a minimal product-one sequence if and only if there exist α, $\tau \in G$ such that $G = \langle \alpha, \tau \mid \alpha^{2n} = 1_G,\ \tau^2 = \alpha^n, \ \text{and} \ \tau\alpha = \alpha^{-1}\tau \rangle$ and $S = \alpha^{[3n-2]} \cdot \tau^{[2]}$.*

Proof. We fix $\alpha, \tau \in G$ such that $G = \langle \alpha, \tau \mid \alpha^{2n} = 1_G, \tau^2 = \alpha^n, \ \text{and} \ \tau\alpha = \alpha^{-1}\tau \rangle$. Then

$$G = \big\{\alpha^i \mid i \in [0, 2n-1]\big\} \cup \big\{\alpha^i\tau \mid i \in [0, 2n-1]\big\}.$$

Let $G_0 = G \setminus \langle \alpha \rangle$. If $|S_{G_0}| = 0$, then $S \in \mathcal{F}\big(\langle \alpha \rangle\big)$, and since $|S| = 3n > \mathsf{D}\big(\langle \alpha \rangle\big) = 2n$, it follows that S is not a minimal product-one sequence, a contradiction. Since S is a product-one sequence, Proposition 3.3 ensures that $|S_{G_0}| \in [2, 2n+2]$ is even. We distinguish two cases depending on $|S_{G_0}|$.

CASE 1. $|S_{G_0}| = 2$.

Then we may assume by changing generating set if necessary that $S = T_1 \cdot \tau \cdot T_2 \cdot (\alpha^x\tau)$ with $\pi^*(T_1)(\tau)\pi^*(T_2)(\alpha^x\tau) = 1_G$, where $x \in [0, 2n-1]$ and $T_1, T_2 \in \mathcal{F}\big(\langle \alpha \rangle\big)$. Since $S \in \mathcal{A}(G)$, it follows that T_1 and T_2 must be both product-one free sequences.

If $|T_1| \geq n$ and $|T_2| \geq n$, then T_1^2 and $T_2^2 \in \mathcal{F}\big(\langle \alpha^2 \rangle\big)$ (see (3.1)) with $|T_1^2| \geq n$ and $|T_2^2| \geq n$, and it follows by $\mathsf{D}\big(\langle \alpha^2 \rangle\big) = n$ that there exist $W_1 \mid T_1$ and $W_2 \mid T_2$ such that W_1^2 and W_2^2 are product-one sequence over $\langle \alpha^2 \rangle$. Since T_1 and T_2 are product-one free, we obtain that $\pi^*(W_1) = \alpha^n = \pi^*(W_2)$. Therefore

$W_1 \cdot W_2$ and $S \cdot (W_1 \cdot W_2)^{[-1]}$ are both product-one sequences, contradicting that $S \in \mathcal{A}(G)$.

Thus either $|T_1| \le n-1$ or $|T_2| \le n-1$, and we may assume that $|T_1| = n-1$ and $|T_2| = 2n-1$. Then Lemma 2.2.1 implies that $T_2 = (\alpha^j)^{[2n-1]}$ for some odd $j \in [1, 2n-1]$. Then we may assume by changing generating set if necessary that $j = 1$ so that $S = T_3 \cdot \tau \cdot \alpha^{[2n-1]} \cdot (\alpha^y \tau)$, where $y \in [0, 2n-1]$ and $T_3 \in \mathcal{F}\big(\langle\alpha\rangle\big)$. Since $T_3 \cdot \alpha \cdot \tau \cdot (\alpha^y \tau)$ is a product-one sequence, we have that

$$T_3 \cdot \alpha^{[n]} \cdot \tau \cdot \alpha^{[n-1]} \cdot (\alpha^y \tau) \in \mathcal{B}(G)\,.$$

It follows that $T_3 \cdot \alpha^{[n]}$ is a product-one free sequence of length $2n-1$, and again by Lemma 2.2.1 that $T_3 = \alpha^{[n-1]}$. Since $(n-1) \equiv (2n-1) + y + n \pmod{2n}$, we infer that $y = 0$, and the assertion follows.

CASE 2. $|S_{G_0}| \in [4, 2n+2]$.

SUBCASE 2.1. $n = 2$.

Then $G = Q_8$ is the quaternion group. If $|S_{G_0}| = 6$, then by Proposition 3.3, we have that

$$S = (\alpha^x \tau)^{[4]} \cdot (\alpha^y \tau)^{[2]}$$

where $x, y \in [0,3]$ such that $2x \not\equiv 2y \pmod 4$ and $2y + x \equiv 3(x+2) \pmod 4$. Since $2y \equiv 2x + 2 \pmod 4$, it follows by letting $\alpha_1 = \alpha^x \tau$ and $\tau_1 = \alpha^y \tau$ that $S = \alpha_1^{[4]} \cdot \tau_1^{[2]}$, where $G = \langle \alpha_1, \tau_1 \mid \alpha_1^4 = 1_G,\, \tau_1^2 = \alpha_1^2, \text{ and } \tau_1 \alpha_1 = \alpha_1^{-1} \tau_1 \rangle$, whence the assertion follows.

Suppose that $|S_{G_0}| = 4$, and we may assume by changing generating set if necessary that $S = \alpha^{r_1} \cdot \alpha^{r_2} \cdot \tau \cdot \alpha^x \tau \cdot \alpha^y \tau \cdot \alpha^z \tau$ for some $r_1, r_2 \in [1,3]$ and $x, y, z \in [0,3]$. If $\alpha^{r_1}\alpha^{r_2}\tau\alpha^x\tau\alpha^y\tau\alpha^z\tau = 1_G$, then $S' = \alpha^{r_1} \cdot \alpha^{r_2} \cdot \alpha^{-x+2} \cdot \alpha^{y-z+2} \in \mathcal{A}\big(\langle\alpha\rangle\big)$, and hence it follows by applying Lemma 2.2.1 that $r_1 \equiv r_2 \equiv -x+2 \equiv y-z+2 \equiv j \pmod 4$ for some odd $j \in [1,3]$. Thus $S = S_1 \cdot S_2$, where $S_1 = \alpha^{r_1} \cdot \alpha^x \tau \cdot \tau$ and $S_2 = \alpha^{r_2} \cdot \alpha^z \tau \cdot \alpha^y \tau$ are both product-one sequences, contradicting that $S \in \mathcal{A}(G)$. Hence we can assume that $\alpha^{r_1}\tau\alpha^{r_2}\alpha^x\tau\alpha^y\tau\alpha^z\tau = 1_G$, and we consider

$$S'' = \alpha^{r_1} \cdot \alpha^{-r_2} \cdot \tau \cdot \alpha^x \tau \cdot \alpha^y \tau \cdot \alpha^z \tau \in \mathcal{B}(G)\,.$$

Then, by the same argument as shown above, we obtain that $S'' \notin \mathcal{A}(G)$. Let $S'' = U_1 \cdot U_2$ for some $U_1, U_2 \in \mathcal{B}(G)$. Since S is a minimal product-one sequence, we must have that

$$U_1 = \alpha^{r_1} \cdot \alpha^{-r_2} \qquad \text{and} \qquad U_2 = \tau \cdot \alpha^x \tau \cdot \alpha^y \tau \cdot \alpha^z \tau$$

are both minimal product-one sequences. Then $r_1 = r_2$, and we may assume that $\tau\alpha^x\tau\alpha^y\tau\alpha^z\tau = 1_G$. Then $U_2 \in \mathcal{A}(G)$ implies that $\alpha^{-x+2} \cdot \alpha^{y-z+2} \in \mathcal{A}\big(\langle\alpha\rangle\big)$, whence $x \equiv y - z \pmod 4$. Since $(\alpha^2\tau \cdot \tau) \cdot (\alpha^2\tau \cdot \tau)$ is not a minimal product-one sequence, it follows by case distinction on x, y, z that we have

$$U_2 \in \big\{\tau^{[4]},\ \tau^{[2]} \cdot (\alpha\tau)^{[2]},\ \tau^{[2]} \cdot (\alpha^3\tau)^{[2]},\ \tau^{[2]} \cdot \alpha\tau \cdot \alpha^3\tau,\\ \tau \cdot (\alpha\tau)^{[2]} \cdot \alpha^2\tau,\ \tau \cdot \alpha^2\tau \cdot (\alpha^3\tau)^{[2]},\ \tau \cdot \alpha\tau \cdot \alpha^2\tau \cdot \alpha^3\tau\big\}\,.$$

Since $S \in \mathcal{A}(G)$, we can assume by changing the generator α for α^3 if necessary that $r_1 = r_2 = 1$, and thus we must have $U_2 = \tau^{[4]}$, for otherwise, S is the product of two product-one sequences, a contradiction. By letting $\alpha_1 = \tau$ and $\tau_1 = \alpha^{r_1}$, we obtain that $S = \alpha_1^{[4]} \boldsymbol{\cdot} \tau_1^{[2]}$, where $G = \langle \alpha_1, \tau_1 \mid \alpha_1^4 = 1_G,\, \tau_1^2 = \alpha_1^2, \text{ and } \tau_1\alpha_1 = \alpha_1^{-1}\tau_1 \rangle$, whence the assertion follows.

SUBCASE 2.2. $n \ge 3$.

Then we may assume by changing generating set if necessary that $S = T_1 \boldsymbol{\cdot} \tau \boldsymbol{\cdot} T_2 \boldsymbol{\cdot} T_3 \boldsymbol{\cdot} \alpha^x\tau$, where $x \in [0, 2n-1]$, $T_1, T_2 \in \mathcal{F}\big(\langle\alpha\rangle\big)$ with $|T_2| \ge |T_1| \ge 0$, and $T_3 \in \mathcal{F}(G_0)$ with $|T_3| = |S_{G_0}| - 2$. Moreover, we suppose that $\pi^*(T_1)(\tau)\pi^*\big(T_2 \boldsymbol{\cdot} T_3'\big)(\alpha^x\tau) = 1_G$, where $T_3 = \prod^\bullet_{i\in[1,|T_3|]} g_i$ is an ordered sequence and $T_3' = \prod^\bullet_{i\in[1,|T_3|/2]}(g_{2i-1}g_{2i}) \in \mathcal{F}\big(\langle\alpha\rangle\big)$. Then T_1 and $T_2 \boldsymbol{\cdot} T_3'$ are both product-one free sequences and

$$|T_1 \boldsymbol{\cdot} T_2 \boldsymbol{\cdot} T_3'| \,=\, \Big(3n - |S_{G_0}|\Big) + \frac{|S_{G_0}| - 2}{2} \,\ge\, 2n - 2\,.$$

If $|T_1 \boldsymbol{\cdot} T_2 \boldsymbol{\cdot} T_3'| \ge 2n$, then we infer that there exists a product-one subsequence Z of S such that $S \boldsymbol{\cdot} Z^{[-1]}$ is again a product-one sequence (this follows by the same line of the proof as used in **SUBCASE 2.2** of Theorem 4.2), contradicting that $S \in \mathcal{A}(G)$.

Suppose that $|T_1 \boldsymbol{\cdot} T_2 \boldsymbol{\cdot} T_3'| = 2n - 1$. Then $|T_3'| = n - 1$ and $|T_2| \ge \frac{n}{2}$. Since $T_2 \boldsymbol{\cdot} T_3'$ is a product-one free sequence with $|T_2 \boldsymbol{\cdot} T_3'| \ge \frac{3n}{2} - 1 \ge \frac{2n+1}{2}$, it follows by Lemma 2.2 that $T_2 \boldsymbol{\cdot} T_3'$ is g-smooth for some $g \in \langle\alpha\rangle$ with $\mathrm{ord}(g) = 2n$, and for every $z \in \Pi(T_2 \boldsymbol{\cdot} T_3')$, there exists a subsequence $W \mid T_2 \boldsymbol{\cdot} T_3'$ with $g \mid W$ such that $\pi^*(W) = z$. Since $|T_3'| = n - 1$, Lemma 2.2.3 implies that $g \mid T_3'$.

If $\Pi(T_1) \cap \Pi(T_2 \boldsymbol{\cdot} T_3') \ne \emptyset$, then there exist subsequences $W_1 \mid T_1$, $W_2 \mid T_2$, and $W_3' \mid T_3'$ such that W_3' is a non-trivial sequence (as argued in similar cases) and $\pi^*(W_1) = \pi^*(W_2 \boldsymbol{\cdot} W_3')$. Let W_3 denote the corresponding subsequence of T_3 and assume that $W_3 = (\alpha^{y_1}\tau) \boldsymbol{\cdot} (\alpha^{y_2}\tau) \boldsymbol{\cdot} W_3''$. Then $Z = W_2 \boldsymbol{\cdot} (\alpha^{y_1}\tau) \boldsymbol{\cdot} W_1 \boldsymbol{\cdot} (\alpha^{y_2}\tau) \boldsymbol{\cdot} W_3''$ and $S \boldsymbol{\cdot} Z^{[-1]}$ are both product-one sequences, contradicting that $S \in \mathcal{A}(G)$. Hence $\Pi(T_1) \cap \Pi(T_2 \boldsymbol{\cdot} T_3') = \emptyset$, and it follows that $T_1^{-1} \boldsymbol{\cdot} T_2 \boldsymbol{\cdot} T_3'$ is a product-one free sequence of length $2n - 1$. By Lemma 2.2.1, there exists an odd $j \in [1, 2n-1]$ such that

$$T_1^{-1} \boldsymbol{\cdot} T_2 \boldsymbol{\cdot} T_3' \,=\, \big(\alpha^j\big)^{[2n-1]}\,,$$

and we may assume by changing generating set if necessary that $j = 1$ so that $x \equiv 1 + n \pmod{2n}$. Note that $2n - 2 - 2|T_1| \ge 0$ is even. If $|T_1| \ge 1$, then

$$\big(\alpha \boldsymbol{\cdot} \alpha^{-1}\big)^{[|T_1|]} \qquad \text{and} \qquad \alpha^{[1 + \frac{2n-2-2|T_1|}{2}]} \boldsymbol{\cdot} \tau \boldsymbol{\cdot} \alpha^{[\frac{2n-2-2|T_1|}{2}]} \boldsymbol{\cdot} (\alpha^x\tau)$$

are both product-one sequences, contradicting that $S \in \mathcal{A}(G)$. Thus $|T_1| = 0$, and we obtain that $T_3 = \big(\alpha^{r+1}\tau \boldsymbol{\cdot} \alpha^{r+n}\tau\big)^{[n-1]}$ for some $r \in [0, 2n-1]$ (as argued in similar cases). Since $x \equiv 1 + n \pmod{2n}$, we obtain that $S = (\alpha \boldsymbol{\cdot} \tau \boldsymbol{\cdot} \alpha^x\tau) \boldsymbol{\cdot} (\alpha \boldsymbol{\cdot} \alpha^{r+n}\tau \boldsymbol{\cdot} \alpha^{r+1}\tau)^{[n-1]}$, contradicting that $S \in \mathcal{A}(G)$.

Suppose that $|T_1 \boldsymbol{\cdot} T_2 \boldsymbol{\cdot} T_3'| = 2n - 2$. Then $|T_3'| = n$ and $|T_2| \ge \frac{n}{2} - 1$. Since $T_2 \boldsymbol{\cdot} T_3'$ is a product-one free sequence with $|T_2 \boldsymbol{\cdot} T_3'| \ge \frac{3n}{2} - 1 \ge \frac{2n+1}{2}$, it follows by Lemma 2.2 that $T_2 \boldsymbol{\cdot} T_3'$ is g-smooth for some $g \in \langle \alpha \rangle$ with $\mathrm{ord}(g) = 2n$, and for every $z \in \Pi(T_2 \boldsymbol{\cdot} T_3')$, there exists a subsequence $W \mid T_2 \boldsymbol{\cdot} T_3'$ with $g \mid W$ such that $\pi^*(W) = z$. Since $|T_3'| \ge n-1$, Lemma 2.2.3 implies that $g \mid T_3'$.

If $\Pi(T_1) \cap \Pi(T_2 \boldsymbol{\cdot} T_3') \ne \emptyset$, then there exist subsequences $W_1 \mid T_1$, $W_2 \mid T_2$, and $W_3' \mid T_3'$ such that W_3' is a non-trivial sequence (as argued in similar cases) and $\pi^*(W_1) = \pi^*(W_2 \boldsymbol{\cdot} W_3')$. Let W_3 denote the corresponding subsequence of T_3 and assume that $W_3 = (\alpha^{y_1}\tau) \boldsymbol{\cdot} (\alpha^{y_2}\tau) \boldsymbol{\cdot} W_3''$. Then $Z = W_2 \boldsymbol{\cdot} (\alpha^{y_1}\tau) \boldsymbol{\cdot} W_1 \boldsymbol{\cdot} (\alpha^{y_2}\tau) \boldsymbol{\cdot} W_3''$ and $S \boldsymbol{\cdot} Z^{[-1]}$ are both product-one sequences, contradicting that $S \in \mathcal{A}(G)$. Hence $\Pi(T_1) \cap \Pi(T_2 \boldsymbol{\cdot} T_3') = \emptyset$, and it follows that $T_1^{-1} \boldsymbol{\cdot} T_2 \boldsymbol{\cdot} T_3'$ is a product-one free sequence of length $2n-2$. By Lemma 2.2.2, there exists an odd $j \in [1, 2n-1]$ such that either

$$T_1^{-1} \boldsymbol{\cdot} T_2 \boldsymbol{\cdot} T_3' = \big(\alpha^j\big)^{[2n-3]} \boldsymbol{\cdot} \alpha^{2j} \quad \text{or} \quad T_1^{-1} \boldsymbol{\cdot} T_2 \boldsymbol{\cdot} T_3' = \big(\alpha^j\big)^{[2n-2]},$$

and we may assume by changing generating set if necessary that $j = 1$ so that either

$$T_1^{-1} \boldsymbol{\cdot} T_2 \boldsymbol{\cdot} T_3' = \alpha^{[2n-3]} \boldsymbol{\cdot} \alpha^2, \quad \text{whence } x \equiv 1 + n \pmod{2n},$$

or else

$$T_1^{-1} \boldsymbol{\cdot} T_2 \boldsymbol{\cdot} T_3' = \alpha^{[2n-2]}, \quad \text{whence } x \equiv 2 + n \pmod{2n}.$$

Suppose that $T_1^{-1} \boldsymbol{\cdot} T_2 \boldsymbol{\cdot} T_3' = \alpha^{[2n-3]} \boldsymbol{\cdot} \alpha^2$ and $x \equiv 1 + n \pmod{2n}$. If $|T_1| \ge 1$ and $\alpha^{-2} \in \mathrm{supp}(T_1)$, then

$$\big(\alpha^{-2} \boldsymbol{\cdot} \alpha \boldsymbol{\cdot} \alpha\big) \boldsymbol{\cdot} \big(\alpha \boldsymbol{\cdot} \alpha^{-1}\big)^{[|T_1|-1]} \quad \text{and} \quad \alpha^{[1+\frac{2n-4-2|T_1|}{2}]} \boldsymbol{\cdot} \tau \boldsymbol{\cdot} \alpha^{[\frac{2n-4-2|T_1|}{2}]} \boldsymbol{\cdot} \alpha^x\tau$$

are both product-one sequences, contradicting that $S \in \mathcal{A}(G)$. If $|T_1| \ge 1$ and $\alpha^{-2} \notin \mathrm{supp}(T_1)$, then

$$\big(\alpha \boldsymbol{\cdot} \alpha^{-1}\big)^{[|T_1|]} \quad \text{and} \quad \alpha^2 \boldsymbol{\cdot} \alpha^{[\frac{2n-4-2|T_1|}{2}]} \boldsymbol{\cdot} \tau \boldsymbol{\cdot} \alpha^{[1+\frac{2n-4-2|T_1|}{2}]} \boldsymbol{\cdot} \alpha^x\tau$$

are both product-one sequences, contradicting that $S \in \mathcal{A}(G)$. Thus we obtain that $|T_1| = 0$.

If $\alpha^2 \in \mathrm{supp}(T_2)$, then $T_3 = \big(\alpha^{r+1}\tau \boldsymbol{\cdot} \alpha^{r+n}\tau\big)^{[n]}$ for some $r \in [0, 2n-1]$ (as argued in similar cases). Since $x \equiv 1 + n \pmod{2n}$, we obtain that

$$S_1 = \alpha^{r+1}\tau \boldsymbol{\cdot} \alpha^{r+n}\tau \boldsymbol{\cdot} \alpha^x\tau \boldsymbol{\cdot} \alpha^2 \boldsymbol{\cdot} \tau,\ S_2 = \big(\alpha^{r+1}\tau \boldsymbol{\cdot} \alpha^{r+n}\tau\big)^{[2]},\ S_3 = \alpha^{r+n}\tau \boldsymbol{\cdot} \alpha^{r+1}\tau \boldsymbol{\cdot} \alpha$$

are all product-one sequences, whence $S = S_1 \boldsymbol{\cdot} S_2 \boldsymbol{\cdot} S_3^{[n-3]}$, contradicting that $S \in \mathcal{A}(G)$. If $\alpha^2 \in \mathrm{supp}(T_3')$, then $T_3 = \big(\alpha^{r_1+1}\tau \boldsymbol{\cdot} \alpha^{r_1+n}\tau\big)^{[n-1]} \boldsymbol{\cdot} \big(\alpha^{r_2+2}\tau \boldsymbol{\cdot} \alpha^{r_2+n}\tau\big)$ for some $r_1, r_2 \in [0, 2n-1]$ (as argued in similar cases). Since $x \equiv 1 + n \pmod{2n}$, we obtain that

$$S_1 = \alpha^{r_2+n}\tau \boldsymbol{\cdot} \alpha^{r_2+2}\tau \boldsymbol{\cdot} \alpha^{r_1+1}\tau \boldsymbol{\cdot} \alpha^{r_1+n}\tau \boldsymbol{\cdot} \alpha^x\tau \boldsymbol{\cdot} \tau \quad \text{and} \quad S_2 = \alpha^{r_1+n}\tau \boldsymbol{\cdot} \alpha^{r_1+1}\tau \boldsymbol{\cdot} \alpha$$

are both product-one sequences, whence $S = S_1 \boldsymbol{\cdot} S_2^{[n-2]}$, contradicting that $S \in \mathcal{A}(G)$.

Suppose that $T_1^{-1} \cdot T_2 \cdot T_3' = \alpha^{[2n-2]}$ and $x \equiv 2+n \pmod{2n}$. If $|T_1| \geq 1$, then

$$\left(\alpha \cdot \alpha^{-1}\right)^{[|T_1|]} \quad \text{and} \quad \alpha^{[2+\frac{2n-4-2|T_1|}{2}]} \cdot \tau \cdot \alpha^{[\frac{2n-4-2|T_1|}{2}]} \cdot \alpha^x \tau$$

are both product-one sequences, contradicting that $S \in \mathcal{A}(G)$. Thus $|T_1| = 0$, and we obtain that $T_3 = \left(\alpha^{r+1}\tau \cdot \alpha^{r+n}\tau\right)^{[n]}$ for some $r \in [0, 2n-1]$ (as argued in similar cases). Since $x \equiv 2+n \pmod{2n}$, we obtain that

$$S_1 = \tau \cdot \alpha^x \tau \cdot \left(\alpha^{r+1}\tau \cdot \alpha^{r+n}\tau\right)^{[2]} \quad \text{and} \quad S_2 = \alpha^{r+n}\tau \cdot \alpha^{r+1}\tau \cdot \alpha$$

are both product-one sequences, whence $S = S_1 \cdot S_2^{[n-2]}$, contradicting that $S \in \mathcal{A}(G)$. □

5. Unions of sets of lengths

In this section, we study sets of lengths and their unions in the monoid $\mathcal{B}(G)$ of product-one sequences over dihedral and dicyclic groups. To do so, we briefly gather the required concepts in the setting of atomic monoids.

Let H be an atomic monoid, this means a commutative, cancellative semigroup with unit element such that every non-unit element can be written as a finite product of atoms. If $a = u_1 \cdot \ldots \cdot u_k \in H$, where $k \in \mathbb{N}$ and $u_1, \ldots, u_k$ are atoms of H, then k is called the length of the factorization and

$$\mathsf{L}(a) = \{k \in \mathbb{N} \mid a \text{ has a factorization of length } k\} \subset \mathbb{N}$$

is the *set of lengths* of a. As usual we set $\mathsf{L}(a) = \{0\}$ if a is invertible, and then

$$\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$$

denotes the *system of sets of lengths* of H. If $k \in \mathbb{N}$ and H is not a group, then

$$\mathcal{U}_k(H) = \bigcup_{k \in L, L \in \mathcal{L}(H)} L \subset \mathbb{N}$$

denotes the *union of sets of lengths* containing k. For every $k \in \mathbb{N}$, $\rho_k(H) = \sup \mathcal{U}_k(H)$ is the *kth-elasticity* of H, and we denote by $\lambda_k(H) = \inf \mathcal{U}_k(H)$. Moreover,

$$\rho(H) = \sup\left\{\frac{\rho_k(H)}{k} \mid k \in \mathbb{N}\right\} = \lim_{k \to \infty} \frac{\rho_k(H)}{k}$$

is the *elasticity* of H. Unions of sets of lengths have been studied in settings ranging from power monoids to Mori domains and to local quaternion orders (for a sample of recent results we refer to [1, 11, 12, 19, 32]).

Let G be a finite group. The monoid $\mathcal{B}(G)$ of product-one sequences over G is a finitely generated reduced monoid, and it is a Krull monoid if and only if G is abelian ([27, Proposition 3.4]). If G is abelian, then most features of the arithmetic of a general Krull monoid having class group G and prime divisors in all classes can be studied in the monoid $\mathcal{B}(G)$. For this reason, $\mathcal{B}(G)$ has received extensive investigations (see [31] for a survey). If G is non-abelian, then $\mathcal{B}(G)$ fails to be Krull but it is still a C-monoid ([8, Theorem

3.2]). Thus it shares all arithmetical finiteness properties valid for abstract C-monoids ([18, 20]). Investigations aiming at precise results for arithmetical invariants were started in [27, 28]. We continue them in this section and obtain explicit upper and lower bounds in the case of dihedral and dicyclic groups. As usual, we set

$$\mathcal{L}(G) = \mathcal{L}\big(\mathcal{B}(G)\big), \mathcal{U}_k(G) = \mathcal{U}_k\big(\mathcal{B}(G)\big), \rho_k(G) = \rho_k\big(\mathcal{B}(G)\big), \rho(G) = \rho\big(\mathcal{B}(G)\big)$$

for every $k \in N$. It is well-known that $\mathcal{U}_k(G) = \{k\}$ for all $k \in \mathbb{N}$ if and only if $|G| \le 2$. Thus, whenever convenient, we will assume that $|G| \ge 3$. It is already known that the sets $\mathcal{U}_k(G)$ are intervals ([27, Theorem 5.5.1]). Our study of the minima $\lambda_k(G)$ runs along the lines of what was done in the abelian case ([16, Section 3.1]). The study of the maxima $\rho_k(G)$ substantially uses the results of Section 4.

Lemma 5.1. *Let G be a finite group with $|G| \ge 3$ and let $k \in \mathbb{N}$.*

1. $\rho_k(G) \le \frac{k\mathsf{D}(G)}{2}$ *and* $\rho_{2k}(G) = k\mathsf{D}(G)$. *In particular,* $\rho(G) = \frac{\mathsf{D}(G)}{2}$.
2. *If $j, l \in \mathbb{N}_0$ such that $l\mathsf{D}(G) + j \ge 1$, then*

$$2l + \frac{2j}{\mathsf{D}(G)} \;\le\; \lambda_{l\mathsf{D}(G)+j}(G) \;\le\; 2l + j\,.$$

In particular, $\lambda_{l\mathsf{D}(G)}(G) = 2l$ for every $l \in \mathbb{N}$.

Proof. 1. [27, Proposition 5.6].

2. Let $j, l \in \mathbb{N}_0$ such that $l\mathsf{D}(G) + j \ge 1$. Note that there is some $L \in \mathcal{L}(G)$ with $k, \lambda_k(G) \in L$, and it follows that

$$k \le \max L \le \rho(G) \min L = \rho(G)\lambda_k(G)\,.$$

Hence we obtain that

$$2l + \frac{2j}{\mathsf{D}(G)} \;=\; \rho(G)^{-1}(l\mathsf{D}(G) + j) \;\le\; \lambda_{l\mathsf{D}(G)+j}\,.$$

Since $2 \le \mathsf{D}(G)$, it follows by 1. that

$$\lambda_{2l+j}(G) \;\le\; 2l + j \;\le\; l\mathsf{D}(G) + j \;\le\; \rho_{2l}(G) + \rho_j(G) \;\le\; \rho_{2l+j}(G)\,,$$

whence $l\mathsf{D}(G) + j \in \mathcal{U}_{2l+j}(G)$ (by [27, Theorem 5.5.1]), equivalently $2l + j \in \mathcal{U}_{l\mathsf{D}(G)+j}(G)$. Therefore

$$2l + \frac{2j}{\mathsf{D}(G)} \;\le\; \lambda_{l\mathsf{D}(G)+j} \;\le\; 2l + j\,.$$

If $j = 0$, then $\lambda_{l\mathsf{D}(G)}(G) = 2l$. □

Lemma 5.2. *Let G be a finite group with $|G| \ge 3$. For every $j \in \mathbb{N}_{\ge 2}$, the following statements are equivalent*:

(a) *There exists some $L \in \mathcal{L}(G)$ with $\{2, j\} \subset L$.*

(b) *$j \le \mathsf{D}(G)$.*

Proof. (a) $\Rightarrow$ (b) If $L \in \mathcal{L}(G)$ with $\{2, j\} \subset L$, then Lemma 5.1.1 implies that $j \le \sup L \le \rho_2(G) = \mathsf{D}(G)$.

(b) $\Rightarrow$ (a) If $j \le \mathsf{D}(G)$, then there exists some $U \in \mathcal{A}(G)$ with $|U| = \ell \ge j$, say $U = g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_\ell$ with $g_1 g_2 \cdots g_\ell = 1_G$. Then $V = g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_{j-1} \boldsymbol{\cdot} (g_j \cdots g_\ell) \in \mathcal{A}(G)$, and $\{2, j\} \subset \mathsf{L}\big(V \boldsymbol{\cdot} V^{-1}\big)$. $\square$

Proposition 5.3. *Let G be a finite group with $|G| \ge 3$. For every $l \in \mathbb{N}_0$, we have*

$$\lambda_{l\mathsf{D}(G)+j}(G) = \begin{cases} 2l & \text{for } j = 0, \\ 2l+1 & \text{for } j \in [1, \rho_{2l+1}(G) - l\mathsf{D}(G)], \\ 2l+2 & \text{for } j \in [\rho_{2l+1}(G) - l\mathsf{D}(G) + 1, \mathsf{D}(G) - 1], \end{cases}$$

provided that $l\mathsf{D}(G) + j \ge 1$.

Proof. Let $l \in \mathbb{N}_0$ and $j \in [0, \mathsf{D}(G) - 1]$ such that $l\mathsf{D}(G) + j \ge 1$. For $j = 0$, the assertion follows from Lemma 5.1.2. Let $j \in [1, \mathsf{D}(G) - 1]$. Then Lemma 5.1.2 implies that

$$2l + \frac{2j}{\mathsf{D}(G)} \ = \ \frac{l\mathsf{D}(G) + j}{\rho(G)} \ \le \ \lambda_{l\mathsf{D}(G)+j}(G) \ \le \ 2l + j\,.$$

For the $j = 1$ case, note that $\rho_{2\ell+1}(G) \ge \rho_{2\ell}(G) + 1 = \ell\mathsf{D}(G) + 1$, so $j = 1$ forces the second of the three cases to hold, and thus we may assume that $j \ge 2$. Then Lemma 5.2 implies that $\{2, j\} \subset \mathsf{L}(U)$ for some $U \in \mathcal{B}(G)$, whence $\lambda_j(G) = 2$. Thus we have

$$\lambda_{l\mathsf{D}(G)+j}(G) \ \le \ \lambda_{l\mathsf{D}(G)}(G) + \lambda_j(G) \ = \ 2l + 2\,,$$

and hence $\lambda_{l\mathsf{D}(G)+j}(G) \in [2l+1, 2l+2]$.

If $j \in [2, \rho_{2l+1}(G) - l\mathsf{D}(G)]$, then $l \ge 1$, and by [27, Theorem 5.5.1], we obtain that $l\mathsf{D}(G) + j \in \mathcal{U}_{2l+1}(G)$, equivalently $2l+1 \in \mathcal{U}_{l\mathsf{D}(G)+j}(G)$. Therefore $\lambda_{l\mathsf{D}(G)+j}(G) \le 2l+1$ and thus $\lambda_{l\mathsf{D}(G)+j} = 2l+1$.

If $j > \rho_{2l+1}(G) - l\mathsf{D}(G)$, then $l\mathsf{D}(G) + j > \rho_{2l+1}(G)$, and by [27, Theorem 5.5.1], we obtain that $l\mathsf{D}(G) + j \notin \mathcal{U}_{2l+1}(G)$, and that $\lambda_{l\mathsf{D}(G)+j}(G) > 2l+1$. Therefore $\lambda_{l\mathsf{D}(G)+j}(G) = 2l+2$. $\square$

Theorem 5.4. *Let G be a dihedral group of order $2n$, where $n \in \mathbb{N}_{\ge 3}$ is odd. Then, for every $k \in \mathbb{N}_{\ge 2}$ and every $l \in \mathbb{N}_0$, we have $\mathcal{U}_k(G) = [\lambda_k(G), \rho_k(G)]$,*

$$\rho_k(G) = kn\,, \quad \textit{and} \quad \lambda_{2ln+j}(G) = \begin{cases} 2l+j & \text{for } j \in [0,1], \\ 2l+2 & \text{for } j \ge 2 \text{ and } l = 0, \\ 2l+1 & \text{for } j \in [2, n] \text{ and } l \ge 1, \\ 2l+2 & \text{for } j \in [n+1, 2n-1] \text{ and } l \ge 1, \end{cases}$$

provided that $2ln + j \ge 1$.

Proof. We obtain that $\mathcal{U}_k(G) = [\lambda_k(G), \rho_k(G)]$ by [27, Theorem 5.5.1]. We prove the assertion on $\rho_k(G)$, and then the assertion on $\lambda_{2ln+j}(G)$ follows from Proposition 5.3.

Let $k \in \mathbb{N}$. If k is even, the assertion follows from Lemma 5.1.1. For odd k, it is sufficient to show that $\rho_3(G) \ge 3n$. Indeed Lemma 5.1.1 implies that

$$3n + 2kn \le \rho_3(G) + \rho_{2k}(G) \le \rho_{2k+3}(G) \le \frac{(2k+3)2n}{2} = 3n + 2kn\,,$$

and hence the assertion follows.

Since $n \in \mathbb{N}_{\ge 3}$ is odd, it follows by letting $G = \langle \alpha, \tau \rangle$ that

$$U = (\alpha\tau)^{[n]} \cdot \tau^{[n]}\,, \quad V = (\alpha^2\tau)^{[n]} \cdot (\alpha\tau)^{[n]}\,, \ \text{ and } \ W = (\alpha^2\tau)^{[n]} \cdot \tau^{[n]}$$

are the minimal product-one sequences of length $\mathsf{D}(G)$ (Theorem 4.1). Thus we obtain that $\{3, 3n\} \subset \mathsf{L}(U \cdot V \cdot W)$, whence $\rho_3(G) \ge 3n$. $\square$

Theorem 5.5. *Let G be either a dihedral group D_{2n} of order $2n$ or a dicyclic group Q_{4m} of order $4m$, where $n \in \mathbb{N}_{\ge 4}$ is even and $m \in \mathbb{N}_{\ge 2}$. Then, for every $k \in \mathbb{N}$, we have*

$$k\mathsf{D}(G) + 2 \overset{(a)}{\le} \rho_{2k+1}(G) \overset{(b)}{\le} k\mathsf{D}(G) + \frac{\mathsf{D}(G)}{2} - 1\,.$$

In particular, if G is isomorphic to D_8 or to Q_8, then $\rho_{2k+1}(G) = k\mathsf{D}(G) + 2$ for every $k \in \mathbb{N}$.

Proof. 1. Let $n \in \mathbb{N}_{\ge 4}$ be even, and $G = \langle \alpha, \tau \mid \alpha^n = \tau^2 = 1_G$ and $\tau\alpha = \alpha^{-1}\tau \rangle$. To show the inequality (a), we take three minimal product-one sequences

$$U = \alpha^{[n+\frac{n}{2}-2]} \cdot \tau \cdot \alpha^{\frac{n}{2}}\tau,\ V = \left(\alpha^{-1}\right)^{[n+\frac{n}{2}-2]} \cdot \alpha\tau \cdot \alpha^{\frac{n}{2}+1}\tau,\ W = \tau \cdot \alpha^{\frac{n}{2}}\tau \cdot \alpha\tau \cdot \alpha^{\frac{n}{2}+1}\tau$$

of length $|U| = |V| = \mathsf{D}(G)$ (Theorem 4.2) and $|W| = 4$. Then it follows by $\{3, \mathsf{D}(G)+2\} \subset \mathsf{L}(U \cdot V \cdot W)$ that $\mathsf{D}(G) + 2 \le \rho_3(G)$, whence we obtain that, for every $k \ge 2$,

$$k\mathsf{D}(G) + 2 = (k-1)\mathsf{D}(G) + \big(\mathsf{D}(G) + 2\big) \le \rho_{2k-2}(G) + \rho_3(G) \le \rho_{2k+1}(G)\,.$$

To show the inequality (b), we assume to the contrary that $\rho_{2k+1}(G) = \left\lfloor \frac{(2k+1)\mathsf{D}(G)}{2} \right\rfloor$. Then there exist $U_1, \ldots, U_{2k+1} \in \mathcal{A}(G)$ with $|U_1| \ge \cdots \ge |U_{2k+1}|$ such that $\rho = \rho_{2k+1}(G) \in \mathsf{L}\big(U_1 \cdot \ldots \cdot U_{2k+1}\big)$. Hence we have that

$$U_1 \cdot \ldots \cdot U_{2k+1} = W_1 \cdot \ldots \cdot W_\rho\,,$$

where $W_1, \ldots, W_\rho \in \mathcal{A}(G)$ with $|W_1| \le \cdots \le |W_\rho|$. Let $H_0 = \langle \alpha \rangle \setminus \{1_G, \alpha^{\frac{n}{2}}\}$. For every $g \in H_0$ and every sequence $S \in \mathcal{F}(G)$, we define

$$\psi_g(S) = \mathsf{v}_g(S) - \mathsf{v}_{g^{-1}}(S)\,.$$

Then, for every $g \in H_0$, we have $|\psi_g(T)| \le |T|$ and $|\psi_g(W)| = 0$ for sequences $T \in \mathcal{F}(G)$ and $W \in \mathcal{A}(G)$ with $|W| = 2$.

CASE 1. $|U_1| = \cdots = |U_{2k+1}| = \mathsf{D}(G)$.

Then we obtain that either $|W_1| = \cdots = |W_\rho| = 2$, or else $|W_1| = \cdots = |W_{\rho-1}| = 2$ and $|W_\rho| = 3$. Since $2k+1$ is odd, it follows by Theorem 4.2 that there exists $g_0 \in H_0$ with $\mathrm{ord}(g_0) = n$ such that the absolute value $|\psi_{g_0}(U_1 \cdot$

$\ldots \boldsymbol{\cdot} U_{2k+1})|$ is $t(\frac{3n}{2}-2)$ for some $t \in \mathbb{N}$. Since $\psi_{g_0}(W_i)=0$ for all $i \in [1,\rho-1]$, we obtain that

$$\begin{aligned} 4 \;\le\; \Big(\frac{3n}{2}-2\Big) &\le |\psi_{g_0}(U_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} U_{2k+1})| \\ &= |\psi_{g_0}(W_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_\rho)| \\ &\le |\psi_{g_0}(W_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_{\rho-1})| + |\psi_{g_0}(W_\rho)| \;\le\; 3\,, \end{aligned}$$

a contradiction.

CASE 2. $|U_1| = \cdots = |U_{2k}| = \mathsf{D}(G)$ and $|U_{2k+1}| = \mathsf{D}(G)-1$.

Then we obtain that $|W_1| = \cdots = |W_\rho| = 2$ and hence

$$\psi_g(U_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} U_{2k}) + \psi_g(U_{2k+1}) = \psi_g(U_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} U_{2k+1}) = \psi_g(W_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_\rho) = 0$$

for every $g \in H_0$. Let $U_{2k+1} = T_1 \boldsymbol{\cdot} T_2$, where $T_1 \in \mathcal{F}\big(\langle \alpha \rangle\big)$ and $T_2 \in \mathcal{F}\big(G \setminus \langle \alpha \rangle\big)$. If $|T_1| = 0$, then it follows by Proposition 3.2 that $\frac{3n}{2}-1 = |U_{2k+1}| = |T_2| \le n$, contradicting that $n \ge 4$. If $|T_2| = 0$, then $\mathsf{D}\big(\langle \alpha \rangle\big) = n$ ensures that $\frac{3n}{2}-1 = |U_{2k+1}| = |T_1| \le n$, again a contradiction. Thus T_1 and T_2 are both nontrivial sequences, and we show that they are product-one sequences to get a contradiction.

First, we prove that T_1 is a product-one sequence. Note that $\psi_g(U_{2k+1}) = \psi_g(T_1)$ for all $g \in H_0$. If there exists $g_0 \in H_0$ such that $\psi_{g_0}(T_1) \neq 0$, then $|\psi_{g_0}(U_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} U_{2k})| = |\psi_{g_0}(T_1)| \ge 1$. Thus Theorem 4.2 ensures that $|\psi_{g_0}(U_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} U_{2k})| = t(\frac{3n}{2}-2)$ for some $t \in \mathbb{N}$. Since $|T_2| \ge 2$, it follows that

$$\frac{3n}{2} - 1 \;=\; |U_{2k+1}| \;=\; |T_2| + |T_1| \;\ge\; 2 + |\psi_{g_0}(T_1)| \;=\; 2 + t\Big(\frac{3n}{2}-2\Big) \;\ge\; \frac{3n}{2}\,,$$

a contradiction. Thus $\psi_g(U_{2k+1}) = \psi_g(T_1) = 0$ for all $g \in H_0$. Since $\alpha^{\frac{n}{2}} \in \mathsf{Z}(G)$, we have $\mathsf{v}_{\alpha^{\frac{n}{2}}}(U) \le 1$ for any $U \in \mathcal{A}(G)$ with $|U| \ge 3$. Hence Theorem 4.2 ensures that $\alpha^{\frac{n}{2}} \notin \mathrm{supp}(U_i)$ for all $i \in [1,2k]$, and hence $\mathsf{v}_{\alpha^{\frac{n}{2}}}(U_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} U_{2k+1}) = \mathsf{v}_{\alpha^{\frac{n}{2}}}(U_{2k+1}) \le 1$. Since $\mathsf{v}_{\alpha^{\frac{n}{2}}}(W_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_\rho)$ must be even, we obtain $\mathsf{v}_{\alpha^{\frac{n}{2}}}(U_{2k+1}) = 0$, and therefore $T_1 = \prod^\bullet_{i \in [1, |T_1|/2]} (g_i \boldsymbol{\cdot} g_i^{-1}) \in \mathcal{B}\big(H_0\big)$.

Next, we show that T_2 is a product-one sequence. Let $U_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} U_{2k} = Z_1 \boldsymbol{\cdot} Z_2$, where $Z_1 \in \mathcal{F}\big(\langle \alpha \rangle\big)$ and $Z_2 \in \mathcal{F}\big(G \setminus \langle \alpha \rangle\big)$. Then Theorem 4.2 implies that

$$Z_2 \;=\; V_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} V_{2k}\,,$$

where for each $i \in [1,2k]$, $V_i = \alpha^{r_i}\tau \boldsymbol{\cdot} \alpha^{\frac{n}{2}+r_i}\tau$ for some $r_i \in [0,n-1]$. Choose $I \subset [1,2k]$ to be maximal such that $\prod^\bullet_{i \in I} V_i$ is a product of minimal product-one sequences of length 2. Then both $|I|$ and $|[1,2k] \setminus I|$ are even, and thus $Z_2' = \prod^\bullet_{j \in [1,2k] \setminus I} V_j$ is a product-one sequence.

Since $T_1 \boldsymbol{\cdot} Z_1$ is a product of minimal product-one sequences of length 2, it follows that $T_2 \boldsymbol{\cdot} Z_2$ is also a product of minimal product-one sequences of length 2. Let T_2' be a subsequence of T_2 obtained by deleting all minimal product-one subsequences of length 2. Then $T_2' \boldsymbol{\cdot} Z_2'$ is again a product of minimal product-one sequences of length 2. Since T_2' and Z_2' are both squarefree sequences, we obtain that $T_2' = Z_2'$ is a product-one sequence, whence $T_2 = \big(T_2 \boldsymbol{\cdot} (T_2')^{[-1]}\big) \boldsymbol{\cdot} T_2' \in \mathcal{B}(G)$.

2. Let $m \ge 2$, and $G = \langle \alpha, \tau \mid \alpha^{2m} = 1_G,\, \tau^2 = \alpha^m, \text{ and } \tau\alpha = \alpha^{-1}\tau \rangle$. To show the inequality (a), we take three minimal product-one sequences

$$U = \alpha^{[3m-2]} \boldsymbol{\cdot} \tau^{[2]}, \quad V = (\alpha^{-1})^{[3m-2]} \boldsymbol{\cdot} (\alpha\tau)^{[2]}, \quad W = (\alpha^m\tau \boldsymbol{\cdot} \alpha^{m+1}\tau)^{[2]}$$

of length $|U| = |V| = \mathsf{D}(G)$ (Theorem 4.3) and $|W| = 4$. Then it follows by $\{3, \mathsf{D}(G)+2\} \subset \mathsf{L}(U \boldsymbol{\cdot} V \boldsymbol{\cdot} W)$ that $\mathsf{D}(G) + 2 \le \rho_3(G)$, whence we obtain that, for every $k \ge 2$,

$$k\mathsf{D}(G) + 2 = (k-1)\mathsf{D}(G) + \big(\mathsf{D}(G)+2\big) \le \rho_{2k-2}(G) + \rho_3(G) \le \rho_{2k+1}(G).$$

To show the inequality (b), we assume to the contrary that $\rho_{2k+1}(G) = \lfloor \frac{(2k+1)\mathsf{D}(G)}{2} \rfloor$. Then there exist $U_1, \ldots, U_{2k+1} \in \mathcal{A}(G)$ with $|U_1| \ge \cdots \ge |U_{2k+1}|$ such that $\rho = \rho_{2k+1}(G) \in \mathsf{L}\big(U_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} U_{2k+1}\big)$. Hence we have that

$$U_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} U_{2k+1} = W_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_\rho,$$

where $W_1, \ldots, W_\rho \in \mathcal{A}(G)$ with $|W_1| \le \cdots \le |W_\rho|$. Let $H_0 = \langle \alpha \rangle \setminus \{1_G, \alpha^m\}$. For every $g \in H_0$ and every sequence $S \in \mathcal{F}(G)$, we define

$$\psi_g(S) = \mathsf{v}_g(S) - \mathsf{v}_{g^{-1}}(S).$$

Then, for every $g \in H_0$, we have $|\psi_g(T)| \le |T|$ and $|\psi_g(W)| = 0$ for sequences $T \in \mathcal{F}(G)$ and $W \in \mathcal{A}(G)$ with $|W| = 2$.

CASE 1. $|U_1| = \cdots = |U_{2k+1}| = \mathsf{D}(G)$.

Then we obtain that either $|W_1| = \cdots = |W_\rho| = 2$, or else $|W_1| = \cdots = |W_{\rho-1}| = 2$ and $|W_\rho| = 3$. Since $2k+1$ is odd, it follows by Theorem 4.3 that there exists $g_0 \in H_0$ with $\operatorname{ord}(g_0) = 2m$ such that the absolute value $|\psi_{g_0}(U_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} U_{2k+1})|$ is $t(3m-2)$ for some $t \in \mathbb{N}$. Since $\psi_{g_0}(W_i) = 0$ for all $i \in [1, \rho-1]$, we obtain that

$$\begin{aligned} 4 \le 3m-2 &\le |\psi_{g_0}(U_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} U_{2k+1})| \\ &= |\psi_{g_0}(W_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_\rho)| \\ &\le |\psi_{g_0}(W_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_{\rho-1})| + |\psi_{g_0}(W_\rho)| \le 3, \end{aligned}$$

a contradiction.

CASE 2. $|U_1| = \cdots = |U_{2k}| = \mathsf{D}(G)$ and $|U_{2k+1}| = \mathsf{D}(G) - 1$.

Then we obtain that $|W_1| = \cdots = |W_\rho| = 2$, and hence

$$\psi_g(U_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} U_{2k}) + \psi_g(U_{2k+1}) = \psi_g(U_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} U_{2k+1}) = \psi_g(W_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_\rho) = 0$$

for every $g \in H_0$. Let $U_{2k+1} = T_1 \boldsymbol{\cdot} T_2$, where $T_1 \in \mathcal{F}\big(\langle \alpha \rangle\big)$ and $T_2 \in \mathcal{F}\big(G \setminus \langle \alpha \rangle\big)$. If $|T_2| = 0$, then $\mathsf{D}\big(\langle \alpha \rangle\big) = 2m$ ensures that $3m-1 = |U_{2k+1}| = |T_1| \le 2m$, a contradiction to $m \ge 2$. Thus T_2 is a non-trivial sequence. We show that T_1 and T_2 are both product-one sequences, and it will be shown that $T_2 \notin \mathcal{A}(G)$ when $|T_1| = 0$.

First, we prove that T_1 is a product-one sequence. Note that $\psi_g(U_{2k+1}) = \psi_g(T_1)$ for all $g \in H_0$. If there exists $g_0 \in H_0$ such that $\psi_{g_0}(T_1) \neq 0$, then $|\psi_{g_0}(U_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} U_{2k})| = |\psi_{g_0}(T_1)| \geq 1$. Thus Theorem 4.3 ensures that $|\psi_{g_0}(U_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} U_{2k})| = t(3m-2)$ for some $t \in \mathbb{N}$. Since $|T_2| \geq 2$, it follows that

$$3m-1 \,=\, |U_{2k+1}| \,=\, |T_2| + |T_1| \,\geq\, 2 + |\psi_{g_0}(T_1)| \,=\, 2 + t(3m-2) \,\geq\, 3m\,,$$

a contradiction. Thus $\psi_g(U_{2k+1}) = \psi_g(T_1) = 0$ for all $g \in H_0$. Since $\alpha^m \in \mathsf{Z}(G)$, we have $\mathsf{v}_{\alpha^m}(U) \leq 1$ for any $U \in \mathcal{A}(G)$ with $|U| \geq 3$. Hence Theorem 4.3 ensures that $\alpha^m \notin \operatorname{supp}(U_i)$ for all $i \in [1, 2k]$, and thus $\mathsf{v}_{\alpha^m}(U_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} U_{2k+1}) = \mathsf{v}_{\alpha^m}(U_{2k+1}) \leq 1$. Since $\mathsf{v}_{\alpha^m}(W_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_\rho)$ must be even, we obtain $\mathsf{v}_{\alpha^m}(U_{2k+1}) = 0$, and therefore $T_1 = \prod^\bullet_{i \in [1, |T_1|/2]}(g_i \boldsymbol{\cdot} g_i^{-1}) \in \mathcal{B}(H_0)$.

Next, we show that T_2 is a product-one sequence, which is not a minimal product-one sequence when $|T_1| = 0$. Let $U_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} U_{2k} = Z_1 \boldsymbol{\cdot} Z_2$, where $Z_1 \in \mathcal{F}\big(\langle\alpha\rangle\big)$ and $Z_2 \in \mathcal{F}\big(G \setminus \langle\alpha\rangle\big)$. Then Theorem 4.3 implies that

$$Z_2 \,=\, V_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} V_{2k}\,,$$

where for each $i \in [1, 2k]$, $V_i = (\alpha^{r_i}\tau)^{[2]}$ for some $r_i \in [0, 2m-1]$. Choose $I \subset [1, 2k]$ to be maximal such that $\prod^\bullet_{i \in I} V_i$ is a product of minimal product-one sequences of length 2. Then both $|I|$ and $|[1, 2k] \setminus I|$ are even, and thus $Z_2' = \prod^\bullet_{j \in [1,2k]\setminus I} V_j$ is a product-one sequence, which is in fact a product of product-one subsequences of length at most 4.

Since $T_1 \boldsymbol{\cdot} Z_1$ is a product of minimal product-one sequences of length 2, it follows that $T_2 \boldsymbol{\cdot} Z_2$ is also a product of minimal product-one sequences of length 2. Let T_2' be a subsequence of T_2 obtained by deleting all minimal product-one subsequences of length 2. Then $T_2' \boldsymbol{\cdot} Z_2'$ is again a product of minimal product-one sequences of length 2. Since both T_2' and Z_2' have no product-one subsequences of length 2 and $\alpha^m \in \mathsf{Z}(G)$, it follows that $1_G \in \pi(Z_2') = \pi(T_2')$, whence $T_2 = \big(T_2 \boldsymbol{\cdot} (T_2')^{[-1]}\big) \boldsymbol{\cdot} T_2' \in \mathcal{B}(G)$. To conclude the proof, we may assume that $|T_1| = 0$. Then $U_{2k+1} = T_2$, and it follows that either that T_2' is trivial, or that $U_{2k+1} = T_2'$. In the former case, U_{2k+1} is a product of product-one subsequences of length 4 (as this is the case for Z_2' with the terms of Z_2' and T_2' pairing up), so $U_{2k+1} \in \mathcal{A}(G)$ forces $3m-1 = |U_{2k+1}| \leq 4$, contradicting that $m \geq 2$. In the latter case, U_{2k+1} is a product of product-one sequences of length 2 by definition of T_2', whence $U_{2k+1} \in \mathcal{A}(G)$ forces $3m-1 = |U_{2k+1}| \leq 2$, again a contradiction. □

Acknowledgments. We thank the referee for very careful reading and for providing valuable suggestions. The referee pointed out some mistakes in a previous manuscript, and hence we realized and corrected them. We thank Alfred Geroldinger and David Grynkiewicz for feedback on a preliminary version.

References

[1] N. R. Baeth and D. Smertnig, *Arithmetical invariants of local quaternion orders*, Acta Arith. **186** (2018), no. 2, 143–177. https://doi.org/10.4064/aa170601-13-8

[2] F. E. Brochero Martínez and S. Ribas, *Extremal product-one free sequences in dihedral and dicyclic groups*, Discrete Math. **341** (2018), no. 2, 570–578. https://doi.org/10.1016/j.disc.2017.09.024

[3] ———, *The $\{1, s\}$-weighted Davenport constant in $\mathbb{Z}_n$ and an application in an inverse problem*, submitted; https://arxiv.org/abs/1803.09705.

[4] F. Chen and S. Savchev, *Long minimal zero-sum sequences in the groups $C_2^{r-1} \oplus C_{2k}$*, Integers **14** (2014), Paper No. A23, 29 pp.

[5] K. Cziszter, *The Noether number of p-groups*, J. Algebra Appl. **18** (2019), no. 4, 1950066, 14 pp. https://doi.org/10.1142/S021949881950066X

[6] K. Cziszter and M. Domokos, *On the generalized Davenport constant and the Noether number*, Cent. Eur. J. Math. **11** (2013), no. 9, 1605–1615. https://doi.org/10.2478/s11533-013-0259-z

[7] ———, *The Noether number for the groups with a cyclic subgroup of index two*, J. Algebra **399** (2014), 546–560. https://doi.org/10.1016/j.jalgebra.2013.09.044

[8] K. Cziszter, M. Domokos, and A. Geroldinger, *The interplay of invariant theory with multiplicative ideal theory and with arithmetic combinatorics*, in Multiplicative ideal theory and factorization theory, 43–95, Springer Proc. Math. Stat., 170, Springer, 2016. https://doi.org/10.1007/978-3-319-38855-7_3

[9] K. Cziszter, M. Domokos, and I. Szöllősi, *The Noether numbers and the Davenport constants of the groups of order less than 32*, J. Algebra **510** (2018), 513–541. https://doi.org/10.1016/j.jalgebra.2018.02.040

[10] Y. Fan, W. Gao, and Q. Zhong, *On the Erdős-Ginzburg-Ziv constant of finite abelian groups of high rank*, J. Number Theory **131** (2011), no. 10, 1864–1874. https://doi.org/10.1016/j.jnt.2011.02.017

[11] Y. Fan, A. Geroldinger, F. Kainrath, and S. Tringali, *Arithmetic of commutative semigroups with a focus on semigroups of ideals and modules*, J. Algebra Appl. **16** (2017), no. 12, 1750234, 42 pp. https://doi.org/10.1142/S0219498817502346

[12] Y. Fan and S. Tringali, *Power monoids: a bridge between factorization theory and arithmetic combinatorics*, J. Algebra **512** (2018), 252–294. https://doi.org/10.1016/j.jalgebra.2018.07.010

[13] W. Gao and A. Geroldinger, *Zero-sum problems in finite abelian groups: a survey*, Expo. Math. **24** (2006), no. 4, 337–369. https://doi.org/10.1016/j.exmath.2006.07.002

[14] W. Gao, A. Geroldinger, and D. J. Grynkiewicz, *Inverse zero-sum problems III*, Acta Arith. **141** (2010), no. 2, 103–152. https://doi.org/10.4064/aa141-2-1

[15] W. Gao, A. Geroldinger, and W. A. Schmid, *Inverse zero-sum problems*, Acta Arith. **128** (2007), no. 3, 245–279. https://doi.org/10.4064/aa128-3-5

[16] A. Geroldinger, *Additive group theory and non-unique factorizations*, in Combinatorial number theory and additive group theory, 1–86, Adv. Courses Math. CRM Barcelona, Birkhäuser Verlag, Basel, 2009. https://doi.org/10.1007/978-3-7643-8962-8

[17] A. Geroldinger and D. J. Grynkiewicz, *The large Davenport constant I: Groups with a cyclic, index 2 subgroup*, J. Pure Appl. Algebra **217** (2013), no. 5, 863–885. https://doi.org/10.1016/j.jpaa.2012.09.004

[18] A. Geroldinger and F. Halter-Koch, *Non-Unique Factorizations*, Pure and Applied Mathematics (Boca Raton), **278**, Chapman & Hall/CRC, Boca Raton, FL, 2006. https://doi.org/10.1201/9781420003208

[19] A. Geroldinger, F. Kainrath, and A. Reinhart, *Arithmetic of seminormal weakly Krull monoids and domains*, J. Algebra **444** (2015), 201–245. https://doi.org/10.1016/j.jalgebra.2015.07.026

[20] A. Geroldinger and Q. Zhong, *A characterization of seminormal C-monoids*, Boll. Unione Mat. Ital., to appear; https://doi.org/10.1007/s40574-019-00194-9

[21] B. Girard, *An asymptotically tight bound for the Davenport constant*, J. Éc. polytech. Math. **5** (2018), 605–611. https://doi.org/10.5802/jep.79

[22] B. Girard and W. A. Schmid, *Inverse zero-sum problems for certain groups of rank three*, submitted; https://arxiv.org/abs/1809.03178.

[23] ———, *Direct zero-sum problems for certain groups of rank three*, J. Number Theory **197** (2019), 297–316. https://doi.org/10.1016/j.jnt.2018.08.016

[24] D. J. Grynkiewicz, *The large Davenport constant II: general upper bounds*, J. Pure Appl. Algebra **217** (2013), no. 12, 2221–2246. https://doi.org/10.1016/j.jpaa.2013.03.002

[25] ———, *Structural Additive Theory*, Developments in Mathematics, **30**, Springer, Cham, 2013. https://doi.org/10.1007/978-3-319-00416-7

[26] D. Han and H. Zhang, *Erdős-Ginzburg-Ziv theorem and Noether number for $C_m \ltimes_\varphi C_{mn}$*, J. Number Theory **198** (2019), 159–175. https://doi.org/10.1016/j.jnt.2018.10.007

[27] J. S. Oh, *On the algebraic and arithmetic structure of the monoid of product-one sequences*, J. Commut. Algebra, to appear; https://projecteuclid.org/euclid.jca/1523433705

[28] ———, *On the algebraic and arithmetic structure of the monoid of product-one sequences II*, Period. Math. Hungar. **78** (2019), 203–230.

[29] W. A. Schmid, *Inverse zero-sum problems II*, Acta Arith. **143** (2010), no. 4, 333–343. https://doi.org/10.4064/aa143-4-2

[30] ———, *The inverse problem associated to the Davenport constant for $C_2 \oplus C_2 \oplus C_{2n}$, and applications to the arithmetical characterization of class groups*, Electron. J. Combin. **18** (2011), no. 1, Paper 33, 42 pp.

[31] ———, *Some recent results and open problems on sets of lengths of Krull monoids with finite class group*, in Multiplicative ideal theory and factorization theory, 323–352, Springer Proc. Math. Stat., 170, Springer, 2016. https://doi.org/10.1007/978-3-319-38855-7_14

[32] S. Tringali, *Structural properties of subadditive families with applications to factorization theory*, Israel J. Math., to appear; https://arxiv.org/abs/1706.03525.

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