

IDENTITIES ABOUT LEVEL 2 EISENSTEIN SERIES

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ABSTRACT. In this paper we consider certain classes of generalized level 2 Eisenstein series by simple differential calculations of trigonometric functions. In particular, we give four new transformation formulas for some level 2 Eisenstein series. We can find that these level 2 Eisenstein series are reducible to infinite series involving hyperbolic functions. Moreover, some interesting new examples are given.

1. Introduction

Let \mathbb{N} be the set of natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, \mathbb{Z} the ring of integers, \mathbb{Q} the field of rational numbers, \mathbb{R} the field of real numbers, and \mathbb{C} the field of complex numbers. Let $i = \sqrt{-1}$.

The subject of this paper are Eisenstein series and hyperbolic functions. Let τ be a complex number with strictly positive imaginary part, the holomorphic Eisenstein series $G_{2k}(\tau)$ of weight $2k$, where $k \geq 2$ is an integer, is defined by the following series:

$$(1.1) \quad G_{2k}(\tau) := \sum_{m,n \in \mathbb{Z} \setminus (0,0)} \frac{1}{(m+n\tau)^{2k}}.$$

This series absolutely converges to a holomorphic function of τ in the upper half-plane. It is well known that the value of $G_{4k}(i)$ can be expressed as

$$(1.2) \quad G_{4k}(i) = \frac{\Gamma^{8k}(1/4)}{2^{2k}\pi^{4k}} H_{4k} \quad (k \in \mathbb{N}),$$

where H_{4m} are called the Hurwitz numbers (see [1, 7, 11]). When working with the q -expansion of the Eisenstein series, this alternate notation is frequently introduced:

$$(1.3) \quad E_{2k}(\tau) := \frac{G_{2k}(\tau)}{2\zeta(2k)} = 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \frac{n^{2k-1}q^n}{1-q^n},$$

Received December 20, 2018; Accepted May 16, 2019.

2010 *Mathematics Subject Classification.* 11M41, 11M99.

Key words and phrases. Eisenstein series, trigonometric function, hyperbolic function, Gamma function.

where $q = \exp(2\pi\tau)$, and $\zeta(s)$ denotes the Riemann zeta function. In Ramanujan's notation, the three relevant Eisenstein series are defined for $|q| < 1$ by

$$(1.4) \quad P(q) := 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n},$$

$$(1.5) \quad Q(q) := 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n},$$

$$(1.6) \quad R(q) := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}.$$

Thus, for $q = \exp(2\pi i\tau)$, $E_4(\tau) = Q(q)$ and $E_6(\tau) = R(q)$, which have weights 4 and 6, respectively. Since (1.3) does not converge for $k = 1$, the Eisenstein series $E_2(\tau)$ must be defined differently, which is defined by

$$(1.7) \quad E_2(\tau) = P(q) - \frac{3}{\pi \operatorname{Im}\tau}.$$

The functions P, Q and R were thoroughly studied in a famous paper [6] by Ramanujan. Berndt [2, 3] found a lot of identities about infinite series involving hyperbolic functions using certain modular transformation formula that originally stems from the gereralized Eisensein series. Further results of infinite series involving hyperbolic functions see Berndt's books [4, 5] and the references therein.

Recently, surprisingly little work has been done on level 2 Eisenstein series involving hyperbolic functions. The motivation for this paper arises from results of Tsumura [11–15] and [8] with Komori and Matsumoto. They studied many level 2 Eisenstein series involving hyperbolic functions. For example, in 2008, Tsumura [11] considered the following two level 2 Eisenstein series of hyperbolic functions

$$\begin{aligned} \mathcal{G}_k(i) &:= \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{(-1)^n}{\sinh(m\pi)(m+ni)^k} \quad \text{and} \\ \mathcal{H}_k(i) &:= \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{(-1)^n}{\cosh(m\pi)(m+ni)^k} \quad (k \in \mathbb{N}). \end{aligned}$$

He proved that $\mathcal{G}_{2k-1}(i)$ and $\mathcal{H}_{2k}(i)$ can be expressed in terms of Γ function and π . Further, in 2009, Tsumura [12] studied the closed form representations of sums

$$\mathcal{C}_k^v := \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{\coth^v(m\pi)}{(m+ni)^k}$$

for $k \in \mathbb{N}$ with $k \geq 3$ and $v \in \mathbb{Z}$. Specially, he showed that

$$\mathcal{C}_k^v \in \mathbb{Q} \left[\frac{1}{\pi}, \pi, \frac{\Gamma^8(1/4)}{\pi^2} \right]$$

for $k \geq 3$ and $v \in \mathbb{N}_0$ with $k \equiv v \pmod{2}$.

In this paper, continuing Tsumura et al.'s work, we study the four level 2 Eisenstein series

$$\begin{aligned} & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{(m + ani)^k}, \quad \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{(m + a(2n+1)i)^k}, \\ & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m + ani)^p}, \quad \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m + a(2n+1)i)^p}, \end{aligned}$$

where $k \in \mathbb{N}$, $p \in \mathbb{N} \setminus \{1\}$, $a \in \mathbb{R} \setminus \{0\}$ and if $m \rightarrow \infty$

$$\begin{aligned} f(m) &= o(1), \\ g(m) &= o(1/m), \quad p = 2 \quad \text{and} \quad g(m) = o(1), \quad p > 2. \end{aligned}$$

We prove that these double series can be expressed by single infinite series involving hyperbolic functions. Moreover, we consider some special cases. We can find that many level 2 Eisenstein series involving hyperbolic functions can be expressed in terms of Γ function and π .

2. Differential formulas of trigonometric functions

Let

$$|\mathbf{r}|_l := r_0 + r_1 + \cdots + r_l \quad (r_j \in \mathbb{N}_0).$$

Lemma 2.1. *Let $I_{k,m}$ be a sequence and k and m are positive integers (including zero). If $I_{k,m}$ satisfies a recurrence relation in the form*

$$(2.1) \quad I_{m,k} = a_m I_{k-1,m+1} - b_m I_{k-1,m},$$

then

$$(2.2) \quad \begin{aligned} I_{k,m} &= \sum_{l=0}^{k-1} \left(\prod_{j=0}^l a_{m+j} \right) \left(\sum_{|\mathbf{r}|_{l+1}=k-l-1} \prod_{h=0}^{l+1} b_{m+h}^{r_h} \right) I_{0,m+l+1} (-1)^{k-l+1} \\ &\quad + (-1)^k b_m^k I_{0,m}, \end{aligned}$$

where a_m and b_m are constants.

Proof. The result (2.2) can be proved by mathematical induction. \square

Theorem 2.2. *For integers $k \geq 0$, $m \geq 1$ and complex number $s \in \mathbb{C} \setminus \mathbb{N}_0$, we have*

$$(2.3) \quad \frac{d^{2k}}{ds^{2k}} \left(\frac{1}{\sin^{2m-1}(\pi s)} \right)$$

$$\begin{aligned}
&= \pi^{2k} \sum_{l=0}^k \left(\frac{(2m+2l-2)!}{(2m-2)!} \sum_{|\mathbf{r}|_l=k-l} \prod_{h=0}^l (2m+2h-1)^{2r_h} \right) \frac{(-1)^{k-l}}{\sin^{2m+2l-1}(\pi s)}, \\
(2.4) \quad &\frac{d^{2k+1}}{ds^{2k+1}} \left(\frac{1}{\sin^{2m-1}(\pi s)} \right) \\
&= \pi^{2k+1} \sum_{l=0}^k \left(\frac{(2m+2l-1)!}{(2m-2)!} \sum_{|\mathbf{r}|_l=k-l} \prod_{h=0}^l (2m+2h-1)^{2r_h} \right) \\
&\times \frac{(-1)^{k+1-l} \cos(\pi s)}{\sin^{2m+2l}(\pi s)}.
\end{aligned}$$

Proof. An elementary calculation gives

$$\begin{aligned}
\frac{d^{2k}}{ds^{2k}} \left(\frac{1}{\sin^{2m-1}(\pi s)} \right) &= 2m(2m-1)\pi^2 \frac{d^{2k-2}}{ds^{2k-2}} \left(\frac{1}{\sin^{2m+1}(\pi s)} \right) \\
(2.5) \quad &- (2m-1)^2 \pi^2 \frac{d^{2k-2}}{ds^{2k-2}} \left(\frac{1}{\sin^{2m-1}(\pi s)} \right).
\end{aligned}$$

So, setting $I_{k,m} = d^{2k}(1/\sin^{2m-1}(\pi s))/ds^{2k}$, $a_m = 2m(2m-1)\pi^2$ and $b_m = (2m-1)^2\pi^2$ in (2.2) and combining (2.5) yield the desired result (2.3). Then, differentiating (2.3) with respect to s , we may deduce the evaluation (2.4). \square

In (2.3), let

$$(2.6) \quad A_{k,m}(l) := (-1)^{k-l} \frac{(2m+2l-2)!}{(2m-2)!} \sum_{|\mathbf{r}|_l=k-l} \prod_{h=0}^l (2m+2h-1)^{2r_h},$$

we have

$$(2.7) \quad \frac{d^{2k}}{ds^{2k}} \left(\frac{1}{\sin^{2m-1}(\pi s)} \right) = \pi^{2k} \sum_{l=0}^k \frac{A_{k,m}(l)}{\sin^{2m+2l-1}(\pi s)}.$$

To evaluate $A_{k,m}(l)$, we differentiate both sides of either equation in (2.7) twice and equate the coefficients. The following recurrence relation is obtained, for $1 \leq l \leq k-1$ and $m \geq 1$:

$$\begin{aligned}
(2.8) \quad A_{k,m}(l) &= (2m+2l-3)(2m+2l-2)A_{k-1,m}(l-1) \\
&- (2m+2l-1)^2 A_{k-1,m}(l).
\end{aligned}$$

In particular, for $k \geq 0$ and $m \geq 1$,

$$A_{k,m}(0) = (-1)^k (2m-1)^{2k} \quad \text{and} \quad A_{k,m}(k) = \frac{(2m+2k-2)!}{(2m-2)!}.$$

Hence, in below, we have

$$\sum_{|\mathbf{r}|_l=k-l} \prod_{h=0}^l (2h+1)^{2r_h} = (-1)^{k-l} \frac{A_{k,1}(l)}{(2l)!}.$$

We give some values of coefficient $A_{k,1}(l)$ with the help of Mathematica.

TABLE 1. Coefficient $A_{k,1}(l)$

$A_{k,1}(l) \setminus k$	0	1	2	3	4	5	6	7
l	1	-1	1	-1	1	-1	1	-1
0	1	2	-20	182	-1640	14762	-132860	1195742
1	0	0	24	-840	23184	-599280	15159144	-380572920
2	0	0	0	720	-60480	3659040	-197271360	10121070960
3	0	0	0	0	40320	-6652800	743783040	-71293622400
4	0	0	0	0	0	362880	-1037836800	192518726400
5	0	0	0	0	0	362880	-1037836800	192518726400

Theorem 2.3. For integer $k \in \mathbb{N}_0$ and complex number $s \in \mathbb{C} \setminus \mathbb{N}_0$, we have

$$(2.9) \quad \frac{d^{2k}}{ds^{2k}} (\cot(\pi s)) = \pi^{2k} \sum_{0 \leq l \leq j \leq k} \left\{ \binom{2k}{2j} (2l)! - \binom{2k}{2j+1} (2l+1)! \right\} \\ \times \left\{ \sum_{|\mathbf{r}|_l=j-l} \prod_{h=0}^l (2h+1)^{2r_h} \right\} \frac{(-1)^{k-l} \cos(\pi s)}{\sin^{2l+1}(\pi s)},$$

$$(2.10) \quad \frac{d^{2k+1}}{ds^{2k+1}} (\cot(\pi s)) \\ = \pi^{2k+1} \sum_{1 \leq l \leq j \leq k+1} \left\{ \binom{2k+2}{2j} (2l-1)! - \binom{2k+2}{2j+1} (2l-1)!(2l+1)! \right\} \\ \times \left\{ \sum_{|\mathbf{r}|_l=j-l} \prod_{h=0}^l (2h+1)^{2r_h} \right\} \frac{(-1)^{k-l}}{\sin^{2l}(\pi s)},$$

where

$$\binom{n}{k} := \frac{n!}{k!(n-k)!},$$

if $k > n$, then $\binom{n}{k} = 0$.

Proof. By a direct calculation we find that

$$(2.11) \quad \begin{aligned} \frac{d^{2k}}{ds^{2k}} (\cot(\pi s)) &= \frac{d^{2k}}{ds^{2k}} \left(\frac{\sin(\pi s)}{\cos(\pi s)} \right) \\ &= \sum_{j=0}^k \binom{2k}{2j} \frac{d^{2j}}{ds^{2j}} \left(\frac{1}{\sin(\pi s)} \right) \frac{d^{2k-2j}}{ds^{2k-2j}} (\cos(\pi s)) \\ &\quad - \sum_{j=0}^k \binom{2k}{2j+1} \frac{d^{2j+1}}{ds^{2j+1}} \left(\frac{1}{\sin(\pi s)} \right) \frac{d^{2k-2j-1}}{ds^{2k-2j-1}} (\cos(\pi s)). \end{aligned}$$

Hence, letting $k = j$ and $m = 1$ in (2.3) and (2.4), then substituting it into (2.11) we obtain (2.9). Integrating (2.9) over the interval $(1/2, s)$ with respect to s , a simple calculation gives the formula (2.10). \square

Further, changing s to $1/2 - s$ in Theorems 2.2 and 2.3, we can get the following corollaries.

Corollary 2.4. *For a positive integer k and complex number s with $s \neq \pm 1/2, \pm 3/2, \dots$, we have*

$$(2.12) \quad \frac{d^{2k}}{ds^{2k}} \left(\frac{1}{\cos(\pi s)} \right) = \pi^{2k} \sum_{l=0}^k \left((2l)! \sum_{|\mathbf{r}|_l=k-l} \prod_{h=0}^l (2h+1)^{2r_h} \right) \frac{(-1)^{k-l}}{\cos^{2l+1}(\pi s)},$$

$$(2.13) \quad \begin{aligned} & \frac{d^{2k+1}}{ds^{2k+1}} \left(\frac{1}{\cos(\pi s)} \right) \\ &= \pi^{2k+1} \sum_{l=0}^k \left((2l+1)! \sum_{|\mathbf{r}|_l=k-l} \prod_{h=0}^l (2h+1)^{2r_h} \right) \frac{(-1)^{k-l} \sin(\pi s)}{\cos^{2l+2}(\pi s)}. \end{aligned}$$

Corollary 2.5. *For a positive integer k and complex number s with $s \neq \pm 1/2, \pm 3/2, \dots$, we have*

$$(2.14) \quad \begin{aligned} \frac{d^{2k}}{ds^{2k}} (\tan(\pi s)) &= \pi^{2k} \sum_{0 \leq l \leq j \leq k} \left\{ \binom{2k}{2j} (2l)! - \binom{2k}{2j+1} (2l+1)! \right\} \\ &\times \left\{ \sum_{|\mathbf{r}|_l=j-l} \prod_{h=0}^l (2h+1)^{2r_h} \right\} \frac{(-1)^{k-l} \sin(\pi s)}{\cos^{2l+1}(\pi s)}, \end{aligned}$$

$$(2.15) \quad \begin{aligned} & \frac{d^{2k+1}}{ds^{2k+1}} (\tan(\pi s)) \\ &= \pi^{2k+1} \sum_{1 \leq l \leq j \leq k+1} \left\{ \binom{2k+2}{2j} (2l-1)! - \binom{2k+2}{2j+1} (2l-1)!(2l+1)! \right\} \\ &\times \left\{ \sum_{|\mathbf{r}|_l=j-l} \prod_{h=0}^l (2h+1)^{2r_h} \right\} \frac{(-1)^{k-l+1}}{\cos^{2l}(\pi s)}. \end{aligned}$$

3. Main theorems and corollaries

In this section we consider the following level 2 Eisenstein series

$$\sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)}{(m + ani)^k} (-1)^n, \quad \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)}{(m + a(2n+1)i)^k} (-1)^n$$

and

$$\sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m + ani)^p}, \quad \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m + a(2n+1)i)^p},$$

where $k \in \mathbb{N}$, $p \in \mathbb{N} \setminus \{1\}$, $a \in \mathbb{R} \setminus \{0\}$ and

$$f(m) = o(1), \quad g(m) = o(1/m), \quad m \rightarrow \infty.$$

Note that if $k = 1$ in the first two sums, then

$$\begin{aligned} \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{m + ani} &= \sum_{\substack{m \in \mathbb{Z}, \\ m \neq 0}} \lim_{N \rightarrow \infty} \sum_{-N \leq n \leq N} \frac{f(m)(-1)^n}{m + ani}, \\ \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{m + a(2n+1)i} &= \sum_{\substack{m \in \mathbb{Z}, \\ m \neq 0}} \lim_{N \rightarrow \infty} \sum_{-N \leq n \leq N} \frac{f(m)(-1)^n}{m + a(2n+1)i}. \end{aligned}$$

3.1. Four theorems

According to the partial fraction expansion of trigonometric function

$$\begin{aligned} \frac{\pi}{\sin(\pi s)} &= \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{n + s}, \\ \frac{\pi}{\cos(\pi s)} &= 2 \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{2n + 1 - 2s}, \\ \pi \cot(\pi s) &= \lim_{N \rightarrow \infty} \sum_{-N \leq n \leq N} \frac{1}{n + s}, \\ \pi \tan(\pi s) &= 2 \lim_{N \rightarrow \infty} \sum_{-N \leq n \leq N} \frac{1}{2n + 1 - 2s}, \end{aligned}$$

elementary calculations show that

$$\begin{aligned} &\sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)}{(m + ani)^k} (-1)^n \\ &= \frac{(-1)^{k-1} i^k}{a^k (k-1)!} \sum_{m=1}^{\infty} (f(m) + (-1)^k f(-m)) \frac{d^{k-1}}{ds^{k-1}} \left(\frac{\pi}{\sin(\pi s)} \right)_{s=mi/a}, \\ &\sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m + ani)^p} \\ &= \frac{(-1)^{p-1} i^p}{a^p (p-1)!} \sum_{m=1}^{\infty} (g(m) + (-1)^p g(-m)) \frac{d^{p-1}}{ds^{p-1}} (\pi \cot(\pi s))_{s=mi/a}, \end{aligned}$$

$$\begin{aligned}
& \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{(m + a(2n+1)i)^k} \\
&= \frac{(-1)^k i^k}{(2a)^k (k-1)!} \sum_{m=1}^{\infty} (f(m) + (-1)^{k-1} f(-m)) \frac{d^{k-1}}{ds^{k-1}} \left(\frac{\pi}{\cos(\pi s)} \right)_{s=mi/2a}, \\
& \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m + a(2n+1)i)^p} \\
&= \frac{(-1)^p i^p}{(2a)^p (p-1)!} \sum_{m=1}^{\infty} (g(m) + (-1)^p g(-m)) \frac{d^{p-1}}{ds^{p-1}} (\pi \tan(\pi s))_{s=mi/2a}.
\end{aligned}$$

In general, we have

$$\begin{aligned}
(3.1) \quad & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{(bm + c + ani)^k} \\
&= \frac{(-1)^{k-1} i^k}{a^k (k-1)!} \sum_{m=1}^{\infty} \left\{ \begin{array}{l} f(m) \frac{d^{k-1}}{ds^{k-1}} \left(\frac{\pi}{\sin(\pi s)} \right)_{s=(bm+c)i/a} \\ + (-1)^k f(-m) \frac{d^{k-1}}{ds^{k-1}} \left(\frac{\pi}{\sin(\pi s)} \right)_{s=(bm-c)i/a} \end{array} \right\},
\end{aligned}$$

$$\begin{aligned}
(3.2) \quad & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(bm + c + ani)^p} \\
&= \frac{(-1)^{p-1} i^p}{a^p (p-1)!} \sum_{m=1}^{\infty} \left\{ \begin{array}{l} g(m) \frac{d^{p-1}}{ds^{p-1}} (\pi \cot(\pi s))_{s=(bm+c)i/a} \\ + (-1)^p g(-m) \frac{d^{p-1}}{ds^{p-1}} (\pi \cot(\pi s))_{s=(bm-c)i/a} \end{array} \right\},
\end{aligned}$$

$$\begin{aligned}
(3.3) \quad & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{(bm + c + a(2n+1)i)^k} \\
&= \frac{(-1)^k i^k}{(2a)^k (k-1)!} \sum_{m=1}^{\infty} \left\{ \begin{array}{l} f(m) \frac{d^{k-1}}{ds^{k-1}} \left(\frac{\pi}{\cos(\pi s)} \right)_{s=(bm+c)i/2a} \\ + (-1)^{k-1} f(-m) \frac{d^{k-1}}{ds^{k-1}} \left(\frac{\pi}{\cos(\pi s)} \right)_{s=(bm-c)i/2a} \end{array} \right\},
\end{aligned}$$

$$\begin{aligned}
(3.4) \quad & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(bm + c + a(2n+1)i)^p} \\
&= \frac{(-1)^p i^p}{(2a)^p (p-1)!} \sum_{m=1}^{\infty} \left\{ \begin{array}{l} g(m) \frac{d^{p-1}}{ds^{p-1}} (\pi \tan(\pi s))_{s=(bm+c)i/2a} \\ + (-1)^p g(-m) \frac{d^{p-1}}{ds^{p-1}} (\pi \tan(\pi s))_{s=(bm-c)i/2a} \end{array} \right\},
\end{aligned}$$

where $a, b \in \mathbb{R} \setminus \{0\}$ and $c \in \mathbb{R}$. Then with the help of Theorems 2.2, 2.3 and Corollaries 2.4, 2.5, we can get the following theorems.

Theorem 3.1. *For a positive integer k and real $a \in \mathbb{R} \setminus \{0\}$, we have*

$$(3.5) \quad \begin{aligned} & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{(m + ani)^{2k}} \\ &= \frac{\pi^{2k}}{a^{2k}(2k-1)!} \sum_{l=0}^{k-1} (2l+1)! \left\{ \sum_{|\mathbf{r}|_l=k-1-l} \prod_{h=0}^l (2h+1)^{2r_h} \right\} \\ & \times \sum_{m=1}^{\infty} \frac{(f(m) + f(-m)) \cosh(m\pi/a)}{\sinh^{2l+2}(m\pi/a)}, \end{aligned}$$

$$(3.6) \quad \begin{aligned} & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{(m + ani)^{2k-1}} \\ &= \frac{\pi^{2k-1}}{a^{2k-1}(2k-2)!} \sum_{l=0}^{k-1} (2l)! \left\{ \sum_{|\mathbf{r}|_l=k-1-l} \prod_{h=0}^l (2h+1)^{2r_h} \right\} \\ & \times \sum_{m=1}^{\infty} \frac{(f(m) - f(-m))}{\sinh^{2l+1}(m\pi/a)}. \end{aligned}$$

Theorem 3.2. *For a positive integer k and $a \in \mathbb{R} \setminus \{0\}$, we have*

$$(3.7) \quad \begin{aligned} & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m + ani)^{2k}} \\ &= \frac{\pi^{2k}}{a^{2k}(2k-1)!} \sum_{1 \leq l \leq j \leq k} \left\{ \binom{2k}{2j} (2l-1)! - \binom{2k}{2j+1} (2l-1)!(2l+1)! \right\} \\ & \times \left\{ \sum_{|\mathbf{r}|_l=j-l} \prod_{h=0}^l (2h+1)^{2r_h} \right\} \sum_{m=1}^{\infty} \frac{g(m) + g(-m)}{\sinh^{2l}(m\pi/a)}, \end{aligned}$$

$$(3.8) \quad \begin{aligned} & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m + ani)^{2k+1}} \\ &= \frac{\pi^{2k+1}}{a^{2k+1}(2k)!} \sum_{0 \leq l \leq j \leq k} \left\{ \binom{2k}{2j} (2l)! - \binom{2k}{2j+1} (2l+1)! \right\} \\ & \times \left\{ \sum_{|\mathbf{r}|_l=j-l} \prod_{h=0}^l (2h+1)^{2r_h} \right\} \sum_{m=1}^{\infty} \frac{(g(m) - g(-m)) \cosh(m\pi/a)}{\sinh^{2l+1}(m\pi/a)}. \end{aligned}$$

Theorem 3.3. For a positive integer k and $a \in \mathbb{R} \setminus \{0\}$, we have

$$(3.9) \quad \begin{aligned} & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{(m + a(2n+1)i)^{2k}} \\ &= \frac{\pi^{2k} i}{(2a)^{2k}(2k-1)!} \sum_{l=0}^{k-1} (-1)^{l-1} (2l+1)! \left\{ \sum_{|\mathbf{r}|_l=k-1-l} \prod_{h=0}^l (2h+1)^{2r_h} \right\} \\ & \times \sum_{m=1}^{\infty} \frac{(f(m) - f(-m)) \sinh(m\pi/2a)}{\cosh^{2l+2}(m\pi/2a)}, \end{aligned}$$

$$(3.10) \quad \begin{aligned} & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{(m + a(2n+1)i)^{2k-1}} \\ &= \frac{\pi^{2k-1} i}{(2a)^{2k-1}(2k-2)!} \sum_{l=0}^{k-1} (-1)^{l-1} (2l)! \left\{ \sum_{|\mathbf{r}|_l=k-1-l} \prod_{h=0}^l (2h+1)^{2r_h} \right\} \\ & \times \sum_{m=1}^{\infty} \frac{(f(m) + f(-m))}{\cosh^{2l+1}(m\pi/2a)}. \end{aligned}$$

Theorem 3.4. For a positive integer k and $a \in \mathbb{R} \setminus \{0\}$, we have

$$(3.11) \quad \begin{aligned} & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m + a(2n+1)i)^{2k}} \\ &= \frac{\pi^{2k}}{(2a)^{2k}(2k-1)!} \sum_{1 \leq l \leq j \leq k} \left\{ \binom{2k}{2j} (2l-1)! - \binom{2k}{2j+1} (2l-1)!(2l+1) \right\} \\ & \times (-1)^l \left\{ \sum_{|\mathbf{r}|_l=j-l} \prod_{h=0}^l (2h+1)^{2r_h} \right\} \sum_{m=1}^{\infty} \frac{g(m) + g(-m)}{\cosh^{2l}(m\pi/a)}, \end{aligned}$$

$$(3.12) \quad \begin{aligned} & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m + a(2n+1)i)^{2k+1}} \\ &= \frac{\pi^{2k+1}}{(2a)^{2k+1}(2k)!} \sum_{0 \leq l \leq j \leq k} \left\{ \binom{2k}{2j} (2l)! - \binom{2k}{2j+1} (2l+1)! \right\} (-1)^l \\ & \times \left\{ \sum_{|\mathbf{r}|_l=j-l} \prod_{h=0}^l (2h+1)^{2r_h} \right\} \sum_{m=1}^{\infty} \frac{(g(m) - g(-m)) \sinh(m\pi/a)}{\cosh^{2l+1}(m\pi/a)}. \end{aligned}$$

From Theorems 3.1-3.4, we establish many relations of level 2 Eisenstein series. For instance, let

$$(3.13) \quad a_{k,l} := (2l-1) \frac{\pi^{2k} (-1)^{k+l}}{a^{2k} (2k-1)!} A_{k-1,1}(l-1), \quad k, l \in \mathbb{N}$$

and

$$(3.14) \quad b_{k,l} := \frac{\pi^{2k} (-1)^l}{a^{2k} (2k-1)!} \sum_{j=l}^k \left\{ \frac{1}{2l} \binom{2k}{2j} - \binom{2k}{2j+1} \frac{2l+1}{2l} \right\} (-1)^j A_{j,1}(l), \quad k, l \in \mathbb{N}.$$

Then, the formulas (3.5) and (3.7) can be rewritten as

$$(3.15) \quad \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{(m+ani)^{2k}} = \sum_{l=1}^k a_{k,l} \sum_{m=1}^{\infty} \frac{(f(m) + f(-m)) \cosh(m\pi/a)}{\sinh^{2l}(m\pi/a)},$$

$$(3.16) \quad \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m+ani)^{2k}} = \sum_{l=1}^k b_{k,l} \sum_{m=1}^{\infty} \frac{g(m) + g(-m)}{\sinh^{2l}(m\pi/a)}.$$

Define two square matrix \mathbf{A}_k and \mathbf{B}_k by

$$(3.17) \quad \mathbf{A}_k := \{a_{i,j}\}_{k \times k} \quad \text{and} \quad \mathbf{B}_k := \{b_{i,j}\}_{k \times k},$$

and let

$$(3.18) \quad F_k(f(\cdot), a) := \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{(m+ani)^{2k}},$$

$$G_k(g(\cdot), a) := \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m+ani)^{2k}},$$

$$(3.19) \quad \bar{F}_l(f(\cdot), a) := \sum_{m=1}^{\infty} \frac{(f(m) + f(-m)) \cosh(m\pi/a)}{\sinh^{2l}(m\pi/a)},$$

$$\bar{G}_l(g(\cdot), a) := \sum_{m=1}^{\infty} \frac{g(m) + g(-m)}{\sinh^{2l}(m\pi/a)},$$

then we have

$$(3.20) \quad \mathbf{F}_k(f(\cdot), a) = \mathbf{A}_k \cdot \bar{\mathbf{F}}_k(f(\cdot), a) \quad \text{and} \quad \mathbf{G}_k(g(\cdot), a) = \mathbf{B}_k \cdot \bar{\mathbf{G}}_k(g(\cdot), a),$$

where

$$\mathbf{F}_k(f(\cdot), a) := (F_1(f(\cdot), a), F_2(f(\cdot), a), \dots, F_k(f(\cdot), a))^T,$$

$$\bar{\mathbf{F}}_k(f(\cdot), a) := (\bar{F}_1(f(\cdot), a), \bar{F}_2(f(\cdot), a), \dots, \bar{F}_k(f(\cdot), a))^T,$$

$$\mathbf{G}_k(g(\cdot), a) := (G_1(g(\cdot), a), G_2(g(\cdot), a), \dots, G_k(g(\cdot), a))^T,$$

$$\bar{\mathbf{G}}_k(g(\cdot), a) := (\bar{G}_1(g(\cdot), a), \bar{G}_2(g(\cdot), a), \dots, \bar{G}_k(g(\cdot), a))^T,$$

where \mathbf{A}^T is a transposed matrix of \mathbf{A} . We note that if $f(m) = g(m)/\cosh(m\pi/a)$, then

$$\bar{F}_l \left(\frac{g(\cdot)}{\cosh((\cdot)\pi/a)}, a \right) = \bar{G}_l(g(\cdot), a) \quad \text{and} \quad \bar{\mathbf{F}}_l \left(\frac{g(\cdot)}{\cosh((\cdot)\pi/a)}, a \right) = \bar{\mathbf{G}}_l(g(\cdot), a).$$

Hence, by (3.20) one obtain

$$(3.21) \quad \mathbf{A}_k^{-1} \mathbf{F}_k \left(\frac{g(\cdot)}{\cosh((\cdot)\pi/a)}, a \right) = \mathbf{B}_k^{-1} \mathbf{G}_k(g(\cdot), a),$$

where \mathbf{A}^{-1} is the inversion of \mathbf{A} . From (3.21) we obtain the relations between $F_k \left(\frac{g(\cdot)}{\cosh((\cdot)\pi/a)}, a \right)$ and $G_k(g(\cdot), a)$. For example,

$$\begin{aligned} & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)(-1)^n}{\cosh(m\pi/a)(m+ani)^2} = \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m+ani)^2}, \\ & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)(-1)^n}{\cosh(m\pi/a)(m+ani)^4} \\ &= \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m+ani)^4} - \frac{\pi^2}{2a^2} \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m+ani)^2}, \\ & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)(-1)^n}{\cosh(m\pi/a)(m+ani)^4} + \frac{\pi^2}{2a^2} \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)(-1)^n}{\cosh(m\pi/a)(m+ani)^2} \\ &= \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m+ani)^4}. \end{aligned}$$

3.2. Corollaries

From Theorems 3.1-3.4 we give the following corollaries.

Corollary 3.5. *For $a \in \mathbb{R} \setminus \{0\}$, we have*

$$(3.22) \quad \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{m+ani} = \frac{\pi}{a} \sum_{m=1}^{\infty} \frac{f(m) - f(-m)}{\sinh(m\pi/a)},$$

$$(3.23) \quad \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{(m+ani)^2} = \frac{\pi^2}{a^2} \sum_{m=1}^{\infty} \frac{(f(m) + f(-m)) \cosh(m\pi/a)}{\sinh^2(m\pi/a)},$$

$$(3.24) \quad \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{(m+ani)^3} = \frac{\pi^3}{2a^3} \sum_{m=1}^{\infty} \left(\frac{f(m) - f(-m)}{\sinh(m\pi/a)} + 2 \frac{f(m) - f(-m)}{\sinh^3(m\pi/a)} \right),$$

$$(3.25) \quad \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{(m+ani)^4} = \frac{\pi^4}{6a^4} \sum_{m=1}^{\infty} (f(m) + f(-m)) \left(\frac{\cosh(m\pi/a)}{\sinh^2(m\pi/a)} + 6 \frac{\cosh(m\pi/a)}{\sinh^4(m\pi/a)} \right).$$

Corollary 3.6. For $a \in \mathbb{R} \setminus \{0\}$, we have

$$(3.26) \quad \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m+ani)^2} = \frac{\pi^2}{a^2} \sum_{m=1}^{\infty} \frac{g(m) + g(-m)}{\sinh^2(m\pi/a)},$$

$$(3.27) \quad \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m+ani)^3} = \frac{\pi^3}{a^3} \sum_{m=1}^{\infty} \frac{(g(m) - g(-m)) \cosh(m\pi/a)}{\sinh^3(m\pi/a)},$$

$$(3.28) \quad \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m+ani)^4} = \frac{2\pi^4}{3a^4} \sum_{m=1}^{\infty} \frac{g(m) + g(-m)}{\sinh^2(m\pi/a)} + \frac{\pi^4}{a^4} \sum_{m=1}^{\infty} \frac{g(m) + g(-m)}{\sinh^4(m\pi/a)}.$$

Corollary 3.7. For $a \in \mathbb{R} \setminus \{0\}$, we have

$$(3.29) \quad \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{m + a(2n+1)i} = -\frac{\pi i}{2a} \sum_{m=1}^{\infty} \frac{f(m) + f(-m)}{\cosh(m\pi/2a)},$$

$$(3.30) \quad \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{(m + a(2n+1)i)^2} = -\frac{\pi^2 i}{4a^2} \sum_{m=1}^{\infty} \frac{(f(m) - f(-m)) \sinh(m\pi/2a)}{\cosh^2(m\pi/2a)},$$

$$(3.31) \quad \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{(m + a(2n+1)i)^3} = -\frac{\pi^3 i}{16a^2} \sum_{m=1}^{\infty} \left(\frac{f(m) + f(-m)}{\cosh(m\pi/2a)} - 2 \frac{f(m) + f(-m)}{\cosh^3(m\pi/2a)} \right),$$

$$(3.32) \quad \begin{aligned} & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{f(m)(-1)^n}{(m + a(2n+1)i)^4} \\ &= -\frac{\pi^4 i}{96a^4} \sum_{m=1}^{\infty} (f(m) - f(-m)) \left(\frac{\sinh(m\pi/2a)}{\cosh^2(m\pi/2a)} - 6 \frac{\sinh(m\pi/2a)}{\cosh^4(m\pi/2a)} \right). \end{aligned}$$

Corollary 3.8. For $a \in \mathbb{R} \setminus \{0\}$, we have

$$(3.33) \quad \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m + a(2n+1)i)^2} = -\frac{\pi^2}{4a^2} \sum_{m=1}^{\infty} \frac{g(m) + g(-m)}{\cosh^2(m\pi/2a)},$$

$$(3.34) \quad \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m + a(2n+1)i)^3} = -\frac{\pi^3}{8a^3} \sum_{m=1}^{\infty} \frac{(g(m) - g(-m)) \sinh(m\pi/2a)}{\cosh^3(m\pi/2a)},$$

$$(3.35) \quad \begin{aligned} & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{g(m)}{(m + a(2n+1)i)^4} \\ &= -\frac{\pi^4}{48a^4} \sum_{m=1}^{\infty} \left(2 \frac{g(m) + g(-m)}{\cosh^2(m\pi/2a)} - 3 \frac{g(m) + g(-m)}{\cosh^4(m\pi/2a)} \right). \end{aligned}$$

3.3. Examples

Since the four infinite series involving hyperbolic functions

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{\sinh^{2m}(n\pi)}, \quad \sum_{n=1}^{\infty} \frac{1}{\cosh^{2m}(n\pi)}, \quad \sum_{n=1}^{\infty} \frac{n^{2p}}{\sinh^{2k}(n\pi)} \quad \text{and} \\ & \sum_{n=1}^{\infty} \frac{n^{2p}}{\cosh^{2k}(n\pi)} \quad (m, p, k \in \mathbb{N}, p \geq k) \end{aligned}$$

can be evaluated by Gamma function and π (explicit evaluations see [9, 16]). For example, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{\sinh^2(n\pi)} = \frac{1}{6} - \frac{1}{2\pi}, \\ & \sum_{n=1}^{\infty} \frac{1}{\sinh^4(n\pi)} = -\frac{11}{90} + \frac{1}{3\pi} + \frac{\Gamma^8(1/4)}{960\pi^2}, \\ & \sum_{n=1}^{\infty} \frac{1}{\cosh^2(n\pi)} = -\frac{1}{2} + \frac{1}{2\pi} + \frac{\Gamma^4(1/4)}{16\pi^3}, \\ & \sum_{n=1}^{\infty} \frac{1}{\cosh^4(n\pi)} = -\frac{1}{2} + \frac{1}{3\pi} + \frac{\Gamma^4(1/4)}{24\pi^3} + \frac{\Gamma^8(1/4)}{192\pi^6}. \end{aligned}$$

Hence, from Corollaries 3.5-3.8, we give some well-known and new results of level 2 Eisenstein series involving hyperbolic functions

$$\begin{aligned} & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{m^2(-1)^n}{\sinh(m\pi)(m+ni)} = -\frac{1}{4\pi} + \frac{\Gamma^8(1/4)}{768\pi^5}, \\ & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{m^4(-1)^n}{\sinh(m\pi)(m+ni)} = \frac{\Gamma^8(1/4)}{640\pi^6}, \\ & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{m^6(-1)^n}{\sinh(m\pi)(m+ni)} = \frac{\Gamma^{16}(1/4)}{57344\pi^{11}}, \end{aligned}$$

$$\begin{aligned}
& \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{m^8(-1)^n}{\sinh(m\pi)(m+ni)} = \frac{3\Gamma^{16}(1/4)}{40960\pi^{12}}, \\
& \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{m^2(-1)^n}{\cosh(m\pi)(m+(2n+1)i/2)} = i \left(\frac{\Gamma^4(1/4)}{32\pi^3} - \frac{\Gamma^8(1/4)}{768\pi^5} \right), \\
& \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{m^4(-1)^n}{\cosh(m\pi)(m+(2n+1)i/2)} = i \left(\frac{3\Gamma^8(1/4)}{5120\pi^6} - \frac{\Gamma^{12}(1/4)}{8192\pi^8} \right), \\
& \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{m^6(-1)^n}{\cosh(m\pi)(m+(2n+1)i/2)} = -i \left(\frac{9\Gamma^{12}(1/4)}{32768\pi^9} + \frac{3\Gamma^{16}(1/4)}{1835008\pi^{11}} \right), \\
& \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{m^8(-1)^n}{\cosh(m\pi)(m+(2n+1)i/2)} = -i \left(\frac{21\Gamma^{16}(1/4)}{5242880\pi^{12}} + \frac{33\Gamma^{20}(1/4)}{8388608\pi^{14}} \right), \\
& \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{m^2}{\cosh^2(m\pi/2)(m+2ni)^2} = -\frac{1}{4} + \frac{\Gamma^8(1/4)}{768\pi^4}, \\
& \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{1}{\sinh^2(m\pi)(m+ni)^2} = -\frac{11}{45}\pi^2 + \frac{2}{3}\pi + \frac{\Gamma^8(1/4)}{960\pi^4}, \\
& \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{1}{\sinh(2m\pi)(m+ni)^3} = -\frac{11}{90}\pi^2 + \frac{1}{3}\pi^2 + \frac{\Gamma^8(1/4)}{1920\pi^4}, \\
& \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{(-1)^n}{\sinh(m\pi)(m+(2n+1)i/2)^2} = i \left(\pi^2 - \pi - \frac{\Gamma^4(1/4)}{8\pi} \right), \\
& \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{(-1)^n}{\sinh(m\pi)(m+(2n+1)i/2)^4} = i \left(\frac{\pi^3}{2} - \frac{5}{6}\pi^4 + \frac{\pi}{16}\Gamma^4(1/4) + \frac{\Gamma^8(1/4)}{96\pi^2} \right), \\
& \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{(-1)^n}{\cosh(m\pi)(m+(2n+1)i/2)} = i \left(\pi - 1 - \frac{\Gamma^4(1/4)}{8\pi^2} \right), \\
& \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{(-1)^n}{\cosh(m\pi)(m+(2n+1)i/2)^3} = i \left(\frac{\pi^2}{6} - \frac{\pi^3}{2} + \frac{\Gamma^4(1/4)}{48} + \frac{\Gamma^8(1/4)}{96\pi^3} \right), \\
& \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{1}{\sinh^2(m\pi/2)(m+(2n+1)i)^2} = \pi - \frac{\pi^2}{3}, \\
& \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{1}{\cosh^2(m\pi)(m+(2n+1)i/2)^4} = \frac{4}{45}\pi^3 - \frac{\pi^4}{3} + \frac{\pi}{90}\Gamma^4(1/4) + \frac{\Gamma^8(1/4)}{288\pi^2} + \frac{\Gamma^{12}(1/4)}{1280\pi^5},
\end{aligned}$$

$$\sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{1}{\sinh(2m\pi)(m + (2n+1)i/2)^3} = \frac{\pi^3}{2} - \frac{\pi^2}{3} - \frac{\Gamma^4(1/4)}{24} - \frac{\Gamma^8(1/4)}{192\pi^3}.$$

The tenth equation appear as example of Example 3 in the [8]. It should be emphasized that the reference [8] also contains many other types of double Eisenstein series.

It is possible that many other evaluations of double Eisenstein series involving hyperbolic functions can be obtained by using the methods and techniques of the present paper. For example, by Theorems 3.1-3.4, it is clear that the twelve level 2 Eisenstein series involving hyperbolic functions

$$\begin{aligned} & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{\coth^{2p}(m\pi)}{(m+ni)^{2k+2}}, \\ & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{\coth^{2p+1}(m\pi)}{(m+ni)^{2k+1}}, \\ & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{\tanh^{2p}(m\pi)}{(m + (2n+1)i/2)^{2k+2}}, \\ & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{\tanh^{2p+1}(m\pi)}{(m + (2n+1)i/2)^{2k+1}}, \\ & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{(-1)^n}{(m+ni)^{2k} \cosh(m\pi) \sinh^{2p}(m\pi)}, \\ & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{(-1)^n}{(m+ni)^{2k-1} \sinh^{2p+1}(m\pi)}, \\ & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{1}{(m+ni)^{2k+1} \cosh(m\pi) \sinh^{2p+1}(m\pi)}, \\ & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{1}{(m+ni)^{2k} \sinh^{2p+2}(m\pi)}, \\ & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{(-1)^n}{(m + (2n+1)i/2)^{2k} \sinh(m\pi) \cosh^{2p}(m\pi)}, \\ & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{(-1)^n}{(m + (2n+1)i/2)^{2k-1} \cosh^{2p+1}(m\pi)}, \\ & \sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{1}{(m + (2n+1)i/2)^{2k+1} \sinh(m\pi) \cosh^{2p+1}(m\pi)}, \end{aligned}$$

$$\sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{1}{(m + (2n+1)i/2)^{2k} \cosh^{2p+2}(m\pi)}$$

can be represented by $\sum_{n=1}^{\infty} \frac{1}{\sinh^{2l}(n\pi)}$ or $\sum_{n=1}^{\infty} \frac{1}{\cosh^{2l}(n\pi)}$, which implies that these level 2 Eisenstein series can be expressed in terms of Γ function and π . Here $k \in \mathbb{N}$ and $p \in \mathbb{N}_0$. For example, the evaluations of the first and second sums can be obtained by Theorem 3.2. The evaluations of the third and forth sums can be deduced by Theorem 3.4.

Similarly, by Theorems 2.2, 2.3, Corollaries 2.4, 2.5, and formulas (3.1)-(3.4) we can give explicit evaluations of the sums

$$\begin{aligned} & \sum_{m,n \in \mathbb{Z}} \frac{f(m)(-1)^n}{(2bm + b + ani)^k}, \quad \sum_{m,n \in \mathbb{Z}} \frac{g(m)}{(2bm + b + ani)^p}, \\ & \sum_{m,n \in \mathbb{Z}} \frac{f(m)(-1)^n}{(2bm + b + a(2n+1)i)^k}, \quad \sum_{m,n \in \mathbb{Z}} \frac{g(m)}{(2bm + b + a(2n+1)i)^p}, \end{aligned}$$

where $a, b \in \mathbb{R} \setminus \{0\}$. For instance, a simple calculation gives

$$\begin{aligned} (3.36) \quad & \sum_{m,n \in \mathbb{Z}} \frac{f(m)(-1)^n}{(2bm + b + ani)^{2k}} \\ & = \frac{\pi^{2k}}{a^{2k}(2k-1)!} \sum_{l=0}^{k-1} (2l+1)! \left\{ \sum_{|\mathbf{r}|_l=k-1-l} \prod_{h=0}^l (2h+1)^{2r_h} \right\} \\ & \times \sum_{m=1}^{\infty} \frac{(f(m-1) + f(-m)) \cosh((2m-1)b\pi/a)}{\sinh^{2l+2}((2m-1)b\pi/a)}. \end{aligned}$$

Setting $k = 1$ in equation above yields

$$(3.37) \quad \sum_{m,n \in \mathbb{Z}} \frac{f(m)(-1)^n}{(2bm + b + ani)^2} = \frac{\pi^2}{a^2} \sum_{m=1}^{\infty} \frac{(f(m-1) + f(-m)) \cosh((2m-1)b\pi/a)}{\sinh^2((2m-1)b\pi/a)}.$$

Hence, we can deduce the two evaluations

$$\begin{aligned} & \sum_{m,n \in \mathbb{Z}} \frac{(-1)^n}{\cosh((2m+1)\pi/2)(2m+1+2ni)^2} = -\frac{\pi}{4} + \frac{\Gamma^4(1/4)}{32\pi}, \\ & \sum_{m,n \in \mathbb{Z}} \frac{(-1)^n}{\cosh^3((2m+1)\pi/2)(2m+1+2ni)^2} = -\frac{\pi}{2} + \frac{\Gamma^4(1/4)}{32\pi}, \end{aligned}$$

where we used the three well known results (see [9, 10])

$$\sum_{n=1}^{\infty} \frac{1}{\sinh^2(n\pi)} = \frac{1}{6} - \frac{1}{2\pi},$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sinh^2(n\pi)} &= -\frac{1}{6} + \frac{\Gamma^4(1/4)}{32\pi^3}, \\ \sum_{n=1}^{\infty} \frac{1}{\sinh^2((2n-1)\pi/2)} &= -\frac{1}{2\pi} + \frac{\Gamma^4(1/4)}{16\pi^3}. \end{aligned}$$

Note that in [9, 10], $u = \frac{\Gamma^4(1/4)}{8\pi}$. It is obvious that the results of Tsumura [11–13] can be established by using the methods of the present paper.

We finally remark that using the method of this paper, it is possible to evaluate other hyperbolic series. For example, in recent paper [15], Tsumura studied the double series $\tilde{H}_{2k}(\tau)$ (τ belongs to the upper half-plane) and proved that $\tilde{H}_{2k}(i) \in \mathbb{Q}[\pi, \Gamma^2(1/4)/(2\sqrt{2\pi})]$, where $\tilde{H}_{2k}(\tau)$ is defined by the double series (Though we treats only the case $\tau = i$ in this paper, it is desirable and possible to treat the general case of τ)

$$\tilde{H}_{2k}(\tau) = \sum_{\substack{m,n \in \mathbb{Z}, \\ (m,n) \neq (0,0)}} \frac{(-1)^m}{\cosh(m\pi i/\tau)(m+n\tau)^{2k}} \quad (k \in \mathbb{N}).$$

If letting $g(m) = (-1)^m / \cosh(m\pi)$ and $a = 1$ in (3.7), we find that $\tilde{H}_{2k}(i)$ can be expressed in terms of hyperbolic series

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{\cosh(m\pi) \sinh^{2l}(m\pi)} \quad (l \leq k).$$

For instance,

$$\begin{aligned} &\sum_{\substack{m,n \in \mathbb{Z}, \\ m \neq 0}} \frac{(-1)^m}{\cosh(m\pi)(m+ni)^4} \\ &= \frac{4\pi^4}{3} \sum_{m=1}^{\infty} \frac{(-1)^m}{\cosh(m\pi) \sinh^2(m\pi)} + 2\pi^4 \sum_{m=1}^{\infty} \frac{(-1)^m}{\cosh(m\pi) \sinh^4(m\pi)} \\ &= -\frac{83}{360}\pi^4 + \frac{\Gamma^2(1/4)}{6\sqrt{2}}\pi^{5/2} - \frac{\Gamma^8(1/4)}{640\pi^2}. \end{aligned}$$

Conversely, we obtain the conclusion

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{\cosh(m\pi) \sinh^{2l}(m\pi)} \in \mathbb{Q}[\pi, \Gamma^2(1/4)/(2\sqrt{2\pi})].$$

With the help of results in [15], we have

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{(-1)^m}{\cosh(m\pi) \sinh^2(m\pi)} &= \frac{5}{12} - \frac{\Gamma^2(1/4)}{4\sqrt{2}\pi^{3/2}}, \\ \sum_{m=1}^{\infty} \frac{(-1)^m}{\cosh(m\pi) \sinh^4(m\pi)} &= -\frac{283}{720} + \frac{\Gamma^2(1/4)}{4\sqrt{2}\pi^{3/2}} - \frac{\Gamma^8(1/4)}{1280\pi^6}. \end{aligned}$$

References

- [1] R. Ayoub, *Euler and the zeta function*, Amer. Math. Monthly **81** (1974), 1067–1086. <https://doi.org/10.2307/2319041>
- [2] B. C. Berndt, *Modular transformations and generalizations of several formulae of Ramanujan*, Rocky Mountain J. Math. **7** (1977), no. 1, 147–189. <https://doi.org/10.1216/RMJ-1977-7-1-147>
- [3] ———, *Analytic Eisenstein series, theta-functions, and series relations in the spirit of Ramanujan*, J. Reine Angew. Math. **303/304** (1978), 332–365. <https://doi.org/10.1515/crll.1978.303-304.332>
- [4] ———, *Ramanujan's Notebooks. Part II*, Springer-Verlag, New York, 1989. <https://doi.org/10.1007/978-1-4612-4530-8>
- [5] ———, *Ramanujan's Notebooks. Part III*, Springer-Verlag, New York, 1991. <https://doi.org/10.1007/978-1-4612-0965-2>
- [6] G. H. Hardy, P. V. Seshu Aiyar, and B. M. Wilson, *Collected papers of Srinivasa Ramanujan*, Cambridge University Press, 1927,
- [7] A. Hurwitz, *Mathematische Werke. Bd. I*, Herausgegeben von der Abteilung für Mathematik und Physik der Eidgenössischen Technischen Hochschule in Zürich, Birkhäuser Verlag, Basel, 1962.
- [8] Y. Komori, K. Matsumoto, and H. Tsumura, *Infinite series involving hyperbolic functions*, Lith. Math. J. **55** (2015), no. 1, 102–118. <https://doi.org/10.1007/s10986-015-9268-x>
- [9] C. B. Ling, *On summation of series of hyperbolic functions*, SIAM J. Math. Anal. **5** (1974), 551–562. <https://doi.org/10.1137/0505055>
- [10] ———, *On summation of series of hyperbolic functions. II*, SIAM J. Math. Anal. **6** (1975), 129–139. <https://doi.org/10.1137/0506013>
- [11] H. Tsumura, *On certain analogues of Eisenstein series and their evaluation formulas of Hurwitz type*, Bull. Lond. Math. Soc. **40** (2008), no. 2, 289–297. <https://doi.org/10.1112/blms/bdn014>
- [12] ———, *Evaluation of certain classes of Eisenstein-type series*, Bull. Aust. Math. Soc. **79** (2009), no. 2, 239–247. <https://doi.org/10.1017/S0004972708001159>
- [13] ———, *Analogues of the Hurwitz formulas for level 2 Eisenstein series*, Results Math. **58** (2010), no. 3–4, 365–378. <https://doi.org/10.1007/s00025-010-0058-9>
- [14] ———, *Analogues of level-N Eisenstein series*, Pacific J. Math. **255** (2012), no. 2, 489–510. <https://doi.org/10.2140/pjm.2012.255.489>
- [15] ———, *Double series identities arising from Jacobi's identity of the theta function*, Results Math. **73** (2018), no. 1, Art. 10, 12 pp. <https://doi.org/10.1007/s00025-018-0770-4>
- [16] C. Xu, *Some evaluation of infinite series involving trigonometric and hyperbolic functions*, Results Math. **73** (2018), no. 4, Art. 128, 18 pp. <https://doi.org/10.1007/s00025-018-0891-9>

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