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A q-ANALOGUE OF QI FORMULA FOR r-DOWLING NUMBERS

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Abstract. In this paper, we establish an explicit formula for r-Dowling numbers in terms of r-Whitney Lah and r-Whitney numbers of the second kind. This is a generalization of the Qi formula for Bell numbers in terms of Lah and Stirling numbers of the second kind. Moreover, we define the q,r-Dowling numbers, q,r-Whitney Lah numbers and q,r-Whitney numbers of the first kind and obtain several fundamental properties of these numbers such as orthogonality and inverse relations, recurrence relations, and generating functions. Hence, we derive an analogous Qi formula for q,r-Dowling numbers expressed in terms of q,r-Whitney Lah numbers and q,r-Whitney numbers of the second kind.

1. Introduction

Cheon and Jung [3] defined the r-Dowling numbers, denoted by $D_{m,r}(n,x)$, as a sum of the r-Whitney numbers of the second kind

$$D_{m,r}(n) = \sum_{k=0}^{n} W_{m,r}(n,k), \tag{1}$$

which is a generalization of the Dowling numbers introduced by Benoumhani [1]. Fundamental properties of the r-Dowling numbers were derived such as the exponential generating function

$$\sum_{n\geq 0} D_{m,r}(n)\frac{z^n}{n!} = \exp\left(rz + \frac{e^{mz}-1}{m}\right), \tag{2}$$

and the recurrence relation

$$D_{m,r}(n+1) = rD_{m,r}(n) + \sum_{j=0}^{n}\binom{n}{j} m^{n-j} D_{m,r}(j), \quad n \geq 0. \tag{3}$$

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In the same paper, Cheon and Jung also defined the r-Whitney Lah numbers, denoted by $L_{m,r}(n,k)$, in terms of the r-Whitney numbers of the first kind $w_{m,r}(n,k)$ and the second kind $W_{m,r}(n,k)$

$$L_{m,r}(n,k)=\sum_{j=k}^{n}w_{m,r}(n,j)W_{m,r}(j,k), \tag{4}$$

which generalizes the identity for the Lah numbers in terms of the Stirling numbers of the first kind $s(n,k)$ and the second kind $S(n,k)$

$$(-1)^nL(n,k)=\sum_{j=k}^{n}(-1)^js(n,j)S(j,k). \tag{5}$$

Other properties for $L_{m,r}(n,k)$ were established which include its horizontal generating function

$$\langle x+2r|m\rangle_n=\sum_{k=0}^{n}L_{m,r}(n,k)(x|m)_k, \tag{6}$$

where

$$\langle x+2r|m\rangle_n=\begin{cases}(x+2r)\cdots(x+2r+(n-1)m), & n\geq 1,\\ 0, & n=0,\end{cases}$$

and the triangular recurrence relation

$$L_{m,r}(n,k)=L_{m,r}(n-1,k-1)+(2r+(n+k-1)m)L_{m,r}(n-1,k), \tag{7}$$

with $L_{m,r}(n,n)=1$ for $n\geq 0$ and $L_{m,r}(n,n)=0$ for $n<k$, or $n,k<0$. We can use (7) to generate the first values of $L_{m,r}(n,k)$.

Qi [10] established a new way of computing Bell numbers by applying the nth derivative of the exponential function $e^{\pm 1/t}$

$$\frac{d^n}{dt^n}e^{\pm 1/t}=(-1)^ne^{\pm 1/t}\sum_{k=0}^{n}(\pm 1)^kL(n,k)\frac{1}{t^{n+k}},$$

to the famous Faá di Bruno formula [4]

$$\frac{d^n}{dt^n}f\circ h(t)=\sum_{k=0}^{n}f^{(k)}(h(t))B_{n,k}(h'(t),h''(t),\ldots,h^{(n-k+1)}(t)),$$

where $B_{n,k}(x_1,x_2,\ldots,x_{n-k+1})$ is the Bell polynomial of the second kind with $n-k+1$ variables defined by

$$B_{n,k}(x_1,x_2,\ldots,x_{n-k+1})=\sum_{\substack{1\leq i\leq n,\, l_i\in\mathbb{N}\\ \sum_{i=0}^{n} il_i=n\\ \sum_{i=0}^{n} l_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}l_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{l_i}$$

such that $B_k(1, 1, \ldots, 1) = S(n, k)$. The said explicit formula which we refer as the Qi Formula for the Bell numbers is given by

$$B_n = \sum_{k=0}^{n} (-1)^{n-k} \left\{ \sum_{j=0}^{k} L(k, j) \right\} S(n, k). \tag{8}$$

However, particularly interesting is the second proof provided in the same paper which utilized already existing fundamental properties of the Lah numbers and the Stirling numbers such as the well-known identity for the Lah numbers [4]

$$(-1)^n L(n, k) = \sum_{j=k}^{n} (-1)^j s(n, j) S(j, k),$$

and the inverse relation of the Stirling numbers

$$f_n = \sum_{k=0}^{n} s(n, j) g_j \Longleftrightarrow g_n = \sum_{k=0}^{n} S(n, j) f_j. \tag{9}$$

By letting $f_n = (-1)^n L(n, k)$ and $g_j = (-1)^j S(j, k)$, (9) yields an expression for the Stirling numbers of the second kind, which is given by

$$S(n, k) = (-1)^n \sum_{j=0}^{n} S(n, j)(-1)^j L(j, k).$$

Summing up both sides over k gives (8). Moreover, the r-Dowling numbers $D_{m,r}(n)$, which are defined in [3] as the sum of the r-Whitney numbers of the second kind

$$D_{m,r}(n) = \sum_{k=0}^{n} W_{m,r}(n, k),$$

generalize the Bell-type numbers. A comprehensive study of r-Dowling numbers has been done by Gyimesi and Nyul in [7].

In the case when $m = 1$ and $r = 0$, the r-Whitney numbers, the r-Dowling numbers, and the r-Whitney Lah numbers reduce to the Stirling numbers, the Bell numbers, and the Lah numbers, respectively.

Corcino et al. [5] obtained an explicit formula parallel to (8) expressing r-Dowling numbers in terms of r-Whitney Lah numbers and r-Whitney numbers of the second kind, which is discussed in Section 6. From this result, it is interesting to establish a corresponding q-analogue of the Qi formula for r-Dowling numbers.

2. The r-Whitney numbers

Mezo [9] introduced the r-Whitney numbers as a consequence for deriving a new formula for the Bernoulli polynomials. As a further study, Cheon and

Jung [3] defined the (signless) r-Whitney numbers of the first kind $w_{m,r}(n,k)$ and the second kind $W_{m,r}(n,k)$ as coefficients of following relations

$$m^n(x)_n = \sum_{k=0}^{n}(-1)^{n-k}w_{m,r}(n,k)(mx+r)^k, \tag{10}$$

$$(mx+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k)(x)_k, \tag{11}$$

where $(x)_n$ is the well-known Pochammer symbol for the falling factorial defined by

$$(x)_n = \begin{cases} x(x-1)\cdots(x-n+1), & n \geq 1, \\ 1, & n = 0. \end{cases}$$

These numbers generalize the Whitney numbers of the first kind and the second kind which were developed by Benoumhani [1].

The Stirling numbers satisfy the following orthogonality relations

$$\sum_{k=m}^{n} S(n,k)s(k,m) = \sum_{k=m}^{n} s(n,k)S(k,m) = \delta_{nm}. \tag{12}$$

By applying (11) into (10), Cheon and Jung [3] showed that the r-Whitney numbers satisfy the orthogonality relation

$$\sum_{k=0}^{n}(-1)^{n-k}w_{m,r}(n,k)W_{m,r}(k,p) = \delta_{np}, \tag{13}$$

where δ_{np} denotes the Kronecker delta function.

By making use of the orthogonality relation in (13), one may easily establish the following inverse relations between r-Whitney numbers of the first kind and the second kind.

Theorem 2.1. *The r-Whitney numbers satisfy the following inverse relations*:

$$\text{i.} \quad f_n = \sum_{k=0}^{n}(-1)^{n-k}w_{m,r}(n,k)g_k \Longleftrightarrow g_n = \sum_{k=0}^{n} W_{m,r}(n,k)f_k; \tag{14}$$

$$\text{ii.} \quad f_k = \sum_{n=k}^{\infty}(-1)^{n-k}w_{m,r}(n,k)g_n \Longleftrightarrow g_k = \sum_{n=k}^{\infty} W_{m,r}(n,k)f_n. \tag{15}$$

Other properties of the r-Whitney numbers established in [3] include the triangular recurrence relations

$$w_{m,r}(n,k) = w_{m,r}(n-1,k-1) + (r+(n-1)m)w_{m,r}(n-1,k), \tag{16}$$

$$W_{m,r}(n,k) = W_{m,r}(n-1,k-1) + (r+km)W_{m,r}(n-1,k), \tag{17}$$

with $w_{m,r}(n,n) = W_{m,r}(n,n) = 1$ for $n \geq 0$ and $w_{m,r}(n,k) = W_{m,r}(n,k) = 0$ for $n < k$, or $n,k < 0$. Equations (16) and (17) can be used to generate the first values of $w_{m,r}(n,k)$ and $W_{m,r}(n,k)$. The r-Whitney numbers also satisfy the following exponential generating functions

$$\sum_{n\geq k}(-1)^{n-k}w_{m,r}(n,k)\frac{z^n}{n!}=(1+mz)^{-\frac{r}{m}}\frac{\ln^k(1+mz)}{m^k k!}, \tag{18}$$

$$\sum_{n\geq k}W_{m,r}(n,k)\frac{z^n}{n!}=\frac{e^{rz}}{k!}\left(\frac{e^{mz}-1}{m}\right)^k. \tag{19}$$

Using the inverse relation (15), the generating functions (18) and (19) can be transformed, respectively, into the following identities

$$\sum_{n=k}^{\infty}W_{m,r}(n,k)(1+mz)^{-\frac{r}{m}}\frac{\ln^n(1+mz)}{m^n n!}\frac{k!}{z^k}=1, \tag{20}$$

$$\sum_{n=k}^{\infty}(-1)^{n-k}w_{m,r}(n,k)\frac{e^{rz}}{n!}\left(\frac{e^{mz}-1}{m}\right)^n\frac{k!}{z^k}=1. \tag{21}$$

Recently, Gyimesi and Nyul [8] gave new combinatorial interpretation for r-Whitney numbers as well as for r-Whitney-Lah numbers

3. A q-analogue of r-Whitney numbers

A given polynomial $a_k(q)$ is said to be a q-analogue of an integer a_k if

$$\lim_{q\to 1}a_k(q)=a_k.$$

An example is the q-analogue $[x]_q$ of an integer x

$$[x]_q=\frac{1-q^x}{1-q}=1+q+q^2+\cdots+q^{x-1},$$

which is a polynomial in q with degree $x-1$.

Corcino and Montero [6] noted that the r-Whitney numbers of the second kind are exactly the same numbers with the Rucinski-Voigt numbers, introduced by Rucinski and Voigt in [12], and defined a q-analogue of these numbers, denoted by $\sigma[n,k]_q^{\beta,r}$, in a form of triangular recurrence relation as follows

$$\sigma[n,k]_q^{\beta,r}=\sigma[n-1,k-1]_q^{\beta,r}+\big([k\beta]_q+[r]_q\big)\sigma[n-1,k]_q^{\beta,r}, \tag{22}$$

with $\sigma[n,0]_q^{\beta,r}=[r]_q^n$ for $n\geq 0$, $\sigma[n,n]_q^{\beta,r}=1$ for $n\geq 0$, and $\sigma[n,k]_q^{\beta,r}=0$ for $n<k$ or $n,k<0$. We shall consider $\sigma[n,k]_q^{\beta,r}$ as a q-analogue of r-Whitney numbers of the second kind $W_{m,r}(n,k)$ with $\beta=m$. Equation (22) can be used to compute the first values of $\sigma[n,k]_q^{\beta,r}$.

When $q\to 1$,

$$[k\beta]_q+[r]_q\to k\beta+r$$

and (22) gives back the recurrence relation (17) for $W_{m,r}(n,k)$ with $\beta=m$. For brevity, we use the term q,r-Whitney numbers of the second kind to refer to $\sigma[n,k]_q^{\beta,r}$ as a q-analogue of the r-Whitney numbers of the second kind. Other properties of $\sigma[n,k]_q^{\beta,r}$ included in the same paper are its horizontal generating function

$$(t+[r]_q)^n = \sum_{k=0}^{n} \sigma[n,k]_q^{\beta,r} (t|[\beta]_q)_k, \tag{23}$$

where

$$(t|[\beta]_q)_k = \begin{cases} \prod_{i=0}^{k-1}(t-[i\beta]_q), & n \geq 1, \\ 1, & n=0, \end{cases} \tag{24}$$

and exponential generating function

$$\sum_{n\geq 0} \sigma[n,k]_q^{\beta,r} \frac{t^n}{n!} = \left[\Delta_{q^\beta}^k \left(\frac{e^{([x\beta]_q+[r]_q)t}}{\left([k\beta]_q|[\beta]_q\right)_k}\right)\right]_{x=0}, \tag{25}$$

where

$$\left([k\beta]_q|[\beta]_q\right)_k = q^{\beta\binom{k}{2}}[k\beta]_q[(k-1)\beta]_q\cdots[\beta]_q = q^{\beta\binom{k}{2}}[k]_{q^\beta}![\beta]_q^k.$$

Moreover, the q,r-Whitney numbers of the second kind were proven to satisfy the identity

$$\sigma[n,k]_q^{\beta,r} = \sum_{j=k}^{n} \binom{n}{j} q^{(n-j)r_2}[r_1]_q^{n-j}\sigma[j,k]_q^{\beta,r_2}, \tag{26}$$

which we can use to obtain an equivalent recurrence relation for the q,r-Dowling numbers to be introduced in the next section.

Following the same approach used by Corcino and Montero to define a q-analogue of r-Whitney numbers of the second kind which was originally introduced by Carlitz [2] to define the q-Stirling numbers of the second kind, we also define a q-analogue of r-Whitney numbers of the first kind in a form of a triangular recurrence relation as follows.

Definition 3.1. For non-negative integers n and k and real numbers β and r, a q-analogue $\phi_{\beta,r}[n,k]_q$ of $w_{\beta,r}(n,k)$ is defined by

$$\phi_{\beta,r}[n,k]_q = \phi_{\beta,r}[n-1,k-1]_q + ([r]_q+[(n-1)\beta]_q)\,\phi_{\beta,r}[n-1,k]_q, \tag{27}$$

where $\phi_{\beta,r}[0,0]_q = 1$ and $\phi_{\beta,r}[n,k]_q = 0$ for $n<k$ and $n,k<0$. For brevity, the term *q,r-Whitney numbers of the first kind* is used to refer to $\phi_{\beta,r}[n,k]_q$.

Note that when $q\to 1$,

$$[r]_q + [(n-1)\beta]_q \to r+(n-1)\beta.$$

Hence, the numbers $\phi_{\beta,r}[n,k]_q$ maybe considered as a q-analogue of $w_{\beta,r}(n,k)$. Moreover, the recurrence relation in (27) will give back the recurrence relation in (16) with $\beta = m$.

Thus, by Definition 3.1,

$$\phi_{\beta,r}[n,n]_q = 1,\ n\geq 0; \tag{28}$$

$$\phi_{\beta,r}[n,0]_q = \prod_{i=0}^{n-1} \left([r]_q + [i\beta]_q\right), \ n \geq 1. \tag{29}$$

The first few values of $\phi_{\beta,r}[n,k]_q$ can be generated using (27), (28), and (29). Now, note that by replacing t with $t-[r]_q$, (24) can be rewritten as

$$(t-[r]_q|[\beta]_q)_n = \begin{cases} \prod\limits_{i=0}^{n-1}(t-[r]_q-[i\beta]_q), & n \geq 1, \\ 1, & n = 0. \end{cases} \tag{30}$$

Using (30), we can now introduce the theorem below, which is analogous to (10). This is necessary to obtain the orthogonality and the inverse relations of q,r-Whitney numbers and to establish a q-analogue of the r-Whitney Lah numbers.

Theorem 3.2. *For non-negative integers n and k and real numbers β and r, the q,r-Whitney numbers of the first kind $\phi_{\beta,r}[n,k]_q$ satisfy the relation*

$$\sum_{k=0}^{n}(-1)^{n-k}\phi_{\beta,r}[n,k]_q t^k = (t-[r]_q|[\beta]_q)_n, \tag{31}$$

where

$$(t-[r]_q|[\beta]_q)_n = \begin{cases} \prod\limits_{i=0}^{n-1}(t-[r]_q-[i\beta]_q), & n \geq 1, \\ 1, & n = 0. \end{cases}$$

Proof. (By induction on n) Note that (31) is true for $n=0$. Assume that (31) is true for $n>0$. We want to show that it also holds for $n+1$. By (27),

$$\phi_{\beta,r}[n+1,k]_q = \phi_{\beta,r}[n,k-1]_q + \left([r]_q+[n\beta]_q\right)\phi_{\beta,r}[n,k]_q.$$

Note that $\phi_{\beta,r}[n,k]_q = 0$ for $n<k$, then

$$\begin{aligned}
&\sum_{k=0}^{n+1}(-1)^{n+1-k}\phi_{\beta,r}[n+1,k]_q t^k \\
&= \sum_{k=0}^{n+1}(-1)^{n+1-k}\phi_{\beta,r}[n,k-1]_q t^k + \sum_{k=0}^{n+1}(-1)^{n+1-k}\left([r]_q+[n\beta]_q\right)\phi_{\beta,r}[n,k]_q t^k \\
&= \sum_{k=0}^{n}(-1)^{n-k}\phi_{\beta,r}[n,k]_q t^{k+1} - \sum_{k=0}^{n}(-1)^{n-k}\left([r]_q+[n\beta]_q\right)\phi_{\beta,r}[n,k]_q t^k \\
&= (t-[r]_q-[n\beta]_q)\sum_{k=0}^{n}(-1)^{n-k}\phi_{\beta,r}[n,k]_q t^k \\
&= (t-[r]_q-[n\beta]_q)(t-[r]_q|[\beta]_q)_n = (t-[r]_q|[\beta]_q)_{n+1}.
\end{aligned}$$

□

Now, we are ready to establish the orthogonality and the inverse relations of the q,r-Whitney numbers of the first kind $\phi_{\beta,r}[n,k]_q$ and the second kind $\sigma[n,k]_q^{\beta,r}$ parallel to (13), (14), and (15).

Theorem 3.3. *For non-negative integers n and k and real numbers β and r, the q,r-Whitney numbers of the first kind $\phi_{\beta,r}[n,k]_q$ and the second kind $\sigma[n,k]_q^{\beta,r}$ satisfy the following orthogonality relations*

$$\sum_{k=p}^{n}(-1)^{n-k}\phi_{\beta,r}[n,k]_q\sigma[k,p]_q^{\beta,r} = \sum_{k=p}^{n}\sigma[n,k]_q^{\beta,r}(-1)^{k-p}\phi_{\beta,r}[k,p]_q$$

$$= \delta_{np} = \begin{cases} 0, & n \neq p, \\ 1, & n = p, \end{cases} \tag{32}$$

where δ_{np} is the Kronecker delta.

Proof. Note that (23) and (31) can be rewritten as

$$(t+[r]_q)^k = \sum_{p=0}^{k}\sigma[k,p]_q^{\beta,r}(t|[\beta]_q])_p, \tag{33}$$

$$(t|[\beta]_q)_n = \sum_{k=0}^{n}(-1)^{n-k}\phi_{\beta,r}[n,k]_q(t+[r]_q)^k, \tag{34}$$

respectively. Substituting (33) into (34) gives

$$\begin{aligned}(t|[\beta]_q)_n &= \sum_{k=0}^{n}(-1)^{n-k}\phi_{\beta,r}[n,k]_q\sum_{p=0}^{k}\sigma[k,p]_q^{\beta,r}(t|[\beta]_q)_p \\ &= \sum_{p=0}^{n}\left\{\sum_{k=p}^{n}(-1)^{n-k}\phi_{\beta,r}[n,k]_q\sigma[k,p]_q^{\beta,r}\right\}(t|[\beta]_q)_p.\end{aligned}$$

Hence,

$$\sum_{k=p}^{n}(-1)^{n-k}\phi_{\beta,r}[n,k]_q\sigma[k,p]_q^{\beta,r} = \begin{cases} 0, & n \neq p, \\ 1, & n = p. \end{cases}$$

Similarly, by replacing t with $t+[r]_q$, (31) can be rewritten as

$$(t|[\beta]_q)_k = \sum_{p=0}^{k}(-1)^{k-p}\phi_{\beta,r}[k,p]_q(t+[r]_q)^p.$$

Substitute this to

$$(t+[r]_q)^n = \sum_{k=0}^{n}\sigma[n,k]_q^{\beta,r}(t|[\beta]_q)_k,$$

it follows that

$$(t+[r]_q)^n = \sum_{k=0}^{n}\sigma[n,k]_q^{\beta,r}\sum_{p=0}^{k}(-1)^{k-p}\phi_{\beta,r}[k,p]_q(t+[r]_q)^p$$

$$= \sum_{p=0}^{n} \left\{ \sum_{k=p}^{n} \sigma[n,k]_q^{\beta,r} (-1)^{k-p} \phi_{\beta,r}[k,p]_q \right\} (t+[r]_q)^p.$$

Therefore,

$$\sum_{k=p}^{n} \sigma[n,k]_q^{\beta,r} (-1)^{k-p} \phi_{\beta,r}[k,p]_q = \delta_{np} = \begin{cases} 0, & n \neq p, \\ 1, & n = p. \end{cases} \qquad \square$$

Consequently using the orthogonality relation (3.3), we can easily derive the following inverse relations of q,r-Whitney numbers.

Theorem 3.4. *For non-negative integers n and k and real numbers β and r, the q,r-Whitney numbers of the first kind $\phi_{\beta,r}[n,k]_q$ and the second kind $\sigma[n,k]_q^{\beta,r}$ satisfy the following inverse relations*

$$\text{(35)} \qquad \text{i.} \qquad f_n = \sum_{k=0}^{n} (-1)^{n-k} \phi_{\beta,r}[n,k]_q g_k \Longleftrightarrow g_n = \sum_{k=0}^{n} \sigma[n,k]_q^{\beta,r} f_k,$$

$$\text{(36)} \qquad \text{ii.} \qquad f_k = \sum_{n=k}^{\infty} (-1)^{n-k} \phi_{\beta,r}[n,k]_q g_n \Longleftrightarrow g_k = \sum_{n=k}^{\infty} \sigma[n,k]_q^{\beta,r} f_n.$$

By applying the inverse relation (36), the generating function (25) can be transformed into the following identity

$$\text{(37)} \qquad \sum_{n=0}^{\infty} (-1)^{n-k} \phi_{\beta,r}[n,k]_q \left[\Delta_{q^\beta}^{n} \left(\frac{e^{([x\beta]_q + [r]_q)t}}{([n\beta]_q | [\beta]_q)_n} \right) \right]_{x=0} \frac{k!}{t^k} = 1.$$

Equation (37) can be considered as an identity for the q,r-Whitney numbers of the first kind.

4. A q-analogue of r-Whitney Lah numbers

Cheon and Jung [3] defined the r-Whitney Lah numbers $L_{m,r}(n,k)$ in terms of the r-Whitney numbers of the first kind $w_{m,r}(n,k)$ and the second kind $W_{m,r}(n,k)$

$$\text{(38)} \qquad L_{m,r}(n,k) = \sum_{j=k}^{n} w_{m,r}(n,j) W_{m,r}(j,k),$$

which generalizes the identity for the Lah numbers in terms of the Stirling numbers of the first kind $s(n,k)$ and the second kind $S(n,k)$

$$(-1)^n L(n,k) = \sum_{j=k}^{n} (-1)^j s(n,j) S(j,k).$$

Ramirez and Shattuck established a p,q-analogue of r-Whitney Lah numbers in [11], which is a p,q-generalization of the r-Whitney Lah numbers defined by Cheon and Jung. Parallel to (38), a q-analogue of the r-Whitney Lah numbers is defined as follows.

Definition 4.1. For non-negative integers n and k and real numbers β and r, a q-analogue $L_{\beta,r}[n,k]_q$ of $L_{\beta,r}(n,k)$ is defined by

$$L_{\beta,r}[n,k]_q = \sum_{j=k}^{n} \phi_{\beta,r}[n,j]_q \sigma[j,k]_q^{\beta,r}. \tag{39}$$

For brevity, the term q,r-Whitney Lah numbers is used to refer to $L_{\beta,r}[n,k]_q$.

Note that when $q \to 1$,

$$\phi_{\beta,r}[n,k]_q \to w_{\beta,r}(n,k) \quad \text{and} \quad \sigma[n,k]_q^{\beta,r} \to W_{\beta,r}(n,k).$$

Furthermore, (39) reduces to

$$L_{m,r}(n,k) = \sum_{j=k}^{n} w_{m,r}(n,j) W_{m,r}(j,k)$$

as $q \to 1$. Hence, $L_{\beta,r}[n,k]_q$ reduces to $L_{m,r}(n,k)$ with $\beta = m$.

Recall that the q,r-Whitney numbers of the first kind $\phi_{\beta,r}[n,k]_q$ and the second kind $\sigma[n,k]_q^{\beta,r}$ are connection constants in the relations

$$(t - [r]_q | [\beta]_q)_n = \sum_{k=0}^{n} (-1)^{n-k} \phi_{\beta,r}[n,k]_q t^k, \tag{40}$$

$$(t + [r]_q)^n = \sum_{k=0}^{n} \sigma[n,k]_q^{\beta,r} (t | [\beta]_q)_k. \tag{41}$$

The following theorem contains an equivalent horizontal generating function which is needed to obtain the recurrence relation of $L_{\beta,r}[n,k]_q$.

Theorem 4.2. *The q,r-Whitney Lah numbers $L_{\beta,r}[n,k]_q$ are connection constants in the identity*

$$\big\langle t + 2[r]_q | [\beta]_q \big\rangle_n = \sum_{k=0}^{n} L_{\beta,r}[n,k]_q \big(t | [\beta]_q\big)_k, \tag{42}$$

where

$$\big\langle t + 2[r]_q | [\beta]_q \big\rangle_n = \begin{cases} \prod_{i=0}^{n-1} (t + 2[r]_q + [i\beta]_q), & n > 0, \\ 1, & n = 0. \end{cases} \tag{43}$$

Proof. Replacing t with $-(t + [r]_q)$, (40) gives

$$\big(-(t + [r]_q) - [r]_q | [\beta]_q \big)_n = \sum_{j=0}^{n} (-1)^{n-j} \phi_{\beta,r}[n,j]_q (-t - [r]_q)^j,$$

$$\big(-t - 2[r]_q | [\beta]_q \big)_n = \sum_{j=0}^{n} (-1)^n \phi_{\beta,r}[n,j]_q (t + [r]_q)^j. \tag{44}$$

Note that the left-hand side of (44) can be expanded as

$$\begin{aligned}\big(-t-2[r]_q|[\beta]_q\big)_n &= \prod_{i=0}^{n-1}(-t-2[r]_q-[i\beta]_q)\\ &= (-1)^n\big\langle t+2[r]_q|[\beta]_q\big\rangle_n. \qquad (45)\end{aligned}$$

Combining (44) and (45) gives

$$\big\langle t+2[r]_q|[\beta]_q\big\rangle_n = \sum_{j=0}^{n}\phi_{\beta,r}[n,j]_q(t+[r]_q)^j. \qquad (46)$$

Now (41) can be rewritten as

$$(t+[r]_q)^j = \sum_{k=0}^{j}\sigma[j,k]_q^{\beta,r}\big(t|[\beta]_q\big)_k.$$

Applying this to (46) results to

$$\begin{aligned}\big\langle t+2[r]_q|[\beta]_q\big\rangle_n &= \sum_{j=0}^{n}\phi_{\beta,r}[n,j]_q\sum_{k=0}^{j}\sigma[j,k]_q^{\beta,r}\big(t|[\beta]_q\big)_k\\ &= \sum_{k=0}^{n}\left\{\sum_{j=k}^{n}\phi_{\beta,r}[n,j]_q\sigma[j,k]_q^{\beta,r}\right\}\big(t|[\beta]_q\big)_k\\ &= \sum_{k=0}^{n}L_{\beta,r}[n,k]_q\big(t|[\beta]_q\big)_k.\end{aligned}$$

□

To generate the first values of $L_{\beta,r}[n,k]_q$, the following recurrence relation is established using the horizontal generating function in the previous theorem.

Theorem 4.3. *The q,r-Whitney Lah numbers $L_{\beta,r}[n,k]_q$ satisfy the recurrence relation*

$$\begin{aligned}L_{\beta,r}[n,k]_q &= L_{\beta,r}[n-1,k-1]_q\\ &\quad+(2[r]_q+[k\beta]_q+[(n-1)\beta]_q)\,L_{\beta,r}[n-1,k]_q, \qquad (47)\end{aligned}$$

with $L_{\beta,r}[0,0]_q=1$ and $L_{\beta,r}[n,k]_q=0$ for $n<k$ or $n,k<0$.

Proof. By Theorem 4.2, we have

$$\sum_{k=0}^{n}L_{\beta,r}[n,k]_q\big(t|[\beta]_q\big)_k = \big\langle t+2[r]_q|[\beta]_q\big\rangle_n.$$

Evaluate the right-hand side of the equation,

$$\sum_{k=0}^{n}L_{\beta,r}[n,k]_q\big(t|[\beta]_q\big)_k$$

$$
\begin{aligned}
&= \big(t + 2[r]_q + [(n-1)\beta]_q\big) \sum_{k=0}^{n-1} L_{\beta,r}[n-1,k]_q \big(t|[\beta]_q\big)_k \\
&= \big(t - [k\beta]_q + [k\beta]_q + 2[r]_q + [(n-1)\beta]_q\big) \sum_{k=0}^{n-1} L_{\beta,r}[n-1,k]_q \big(t|[\beta]_q\big)_k \\
&= \sum_{k=0}^{n-1} \big(t - [k\beta]_q\big) L_{\beta,r}[n-1,k]_q \big(t|[\beta]_q\big)_k \\
&\quad + \sum_{k=0}^{n-1} \big(2[r]_q + [k\beta]_q + [(n-1)\beta]_q\big) L_{\beta,r}[n-1,k]_q \big(t|[\beta]_q\big)_k \\
&= \sum_{k=0}^{n} L_{\beta,r}[n-1,k-1]_q \big(t|[\beta]_q\big)_k \\
&\quad + \sum_{k=0}^{n} \big(2[r]_q + [k\beta]_q + [(n-1)\beta]_q\big) L_{\beta,r}[n-1,k]_q \big(t|[\beta]_q\big)_k \\
&= \sum_{k=0}^{n} \big\{L_{\beta,r}[n-1,k-1]_q + \big(2[r]_q + [k\beta]_q + [(n-1)\beta]_q\big) L_{\beta,r}[n-1,k]_q\big\} \big(t|[\beta]_q\big)_k.
\end{aligned}
$$

By comparing the coefficients of $\big(t|[\beta]_q\big)_k$, we prove the theorem. □

As $q \to 1$, (47) reduces to

$$L_{\beta,r}(n,k) = L_{\beta,r}(n-1,k-1) + (2r + (k+n-1)\beta)\, L_{\beta,r}(n-1,k),$$

which is equivalent to

$$L_{m,r}(n,k) = L_{m,r}(n-1,k-1) + (2r + (k+n-1)m)\, L_{m,r}(n-1,k),$$

with $\beta = m$.

By Theorem 4.3, we can deduce the following

$$L_{\beta,r}[n,n]_q = 1, \qquad n \geq 0, \tag{48}$$

$$L_{\beta,r}[n,0]_q = \prod_{i=0}^{n-1} (2[r]_q + [i\beta]_q), \qquad n \geq 1. \tag{49}$$

The following are the first few values of $L_{\beta,r}[n,k]_q$ with $\beta = r = 2$ generated using (47), (48), and (49):

$$
\begin{aligned}
L_{\beta,r}[0,0]_q &= 1, \\
L_{\beta,r}[1,0]_q &= 2[2]_q = 2q + 2, \\
L_{\beta,r}[1,1]_q &= 1, \\
L_{\beta,r}[2,0]_q &= (2[2]_q + [2]_q)(2[2]_q) = (3q+3)(2q+2) = 6q^2 + 12q + 6, \\
L_{\beta,r}[2,1]_q &= L_{\beta,r}[1,0]_q + (2[2]_q + [2]_q + [2]_q) L_{\beta,r}[1,1]_q, \\
&= 2q + 2 + (4q+4)(1) = 6q + 6,
\end{aligned}
$$

$L_{\beta,r}[2,2]_q = 1.$

Hence, $L_{\beta,r}[n,k]_q$ is a polynomial in q.

5. A q-analogue of r-Dowling numbers

The r-Dowling polynomials introduced by Cheon and Jung [3] give the r-Dowling numbers when $x = 1$. Specifically, the r-Dowling numbers $D_{m,r}(n)$ are given as the sum of the r-Whitney numbers of the second kind

$$D_{m,r}(n) = \sum_{k=0}^{n} W_{m,r}(n,k). \tag{50}$$

Parallel to (50), a q-analogue of r-Dowling numbers is defined as follows.

Definition 5.1. For non-negative integers n and k and real numbers β and r, a q-analogue of r-Dowling numbers, denoted by $D_{\beta,r}[n]$, is defined by

$$D_{\beta,r}[n]_q = \sum_{k=0}^{n} \sigma[n,k]_q^{\beta,r}. \tag{51}$$

For brevity, we use the term q, r-*Dowling numbers* to refer to $D_{\beta,r}[n]_q$.

Note that when $q \to 1$, $\sigma[n,k]_q^{\beta,r} \to W_{\beta,r}(n,k)$. Consequently,

$$D_{\beta,r}[n]_q \to D_{\beta,r}(n).$$

Hence the numbers $D_{\beta,r}[n]_q$ may be considered as a q-analogue of $D_{m,r}(n)$ with $\beta = m$.

By (51), the values of $D_{\beta,r}[n]_q$ can be taken from the sum of $\sigma[n,k]_q^{\beta,r}$. Using (22), the following are the first few values of $\sigma[n,k]_q^{\beta,r}$ with $\beta = r = 2$:

$$\begin{aligned}
\sigma[0,0]_q^{2,2} &= 1,\\
\sigma[1,0]_q^{2,2} &= [2]_q = q+1,\\
\sigma[1,1]_q^{2,2} &= 1,\\
\sigma[2,0]_q^{2,2} &= [2]_q^2 = (q+1)^2 = q^2+2q+1,\\
\sigma[2,1]_q^{2,2} &= \sigma[1,0]_q^{2,2} + ([2]_q + [2]_q)\sigma[1,1]_q^{2,2}\\
&= q+1+(2q+2)(1) = 3q+3,\\
\sigma[2,2]_q^{2,2} &= 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
D_{2,2}[0]_q &= 1,\\
D_{2,2}[1]_q &= q+2,\\
D_{2,2}[2]_q &= q^2+5q+5.
\end{aligned}$$

Moreover, $\sigma[n,k]_q^{\beta,r}$ and $D_{\beta,r}[n]_q$ are shown to be polynomials in q.

The following theorem contains the exponential generating function of q, r-Dowling numbers $D_{\beta,r}[n]_q$ which is analogous to (2).

Theorem 5.2. *For non-negative integers n and k and real numbers β and r, the q, r-Dowling numbers $D_{\beta,r}[n]_q$ satisfy the exponential generating function*

$$\sum_{n=0}^{\infty} D_{\beta,r}[n]_q \frac{t^n}{n!} = \sum_{k=0}^{\infty} \left\{ \Delta_{q^\beta}^k \left(\frac{e^{([x\beta]_q+[r]_q)t}}{\left([k\beta]_q|[\beta]_q\right)_k} \right) \right\}_{x=0}. \tag{52}$$

Proof. Multiplying both sides of (51) with $\frac{t^n}{n!}$ and summing over n gives

$$\begin{aligned}\sum_{n=0}^{\infty} D_{\beta,r}[n]_q \frac{t^n}{n!} &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n} \sigma[n,k]_q^{\beta,r}\right\}\frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty}\left\{\sum_{n=0}^{\infty} \sigma[n,k]_q^{\beta,r}\frac{t^n}{n!}\right\}.\end{aligned}$$

Applying the exponential generating function (25) of q, r-Whitney numbers of the second kind $\sigma[n,k]_q^{\beta,r}$ yields

$$\sum_{n=0}^{\infty} D_{\beta,r}[n]_q \frac{t^n}{n!} = \sum_{k=0}^{\infty} \left\{ \Delta_{q^\beta}^k \left(\frac{e^{([x\beta]_q+[r]_q)t}}{\left([k\beta]_q|[\beta]_q\right)_k} \right) \right\}_{x=0}. \qquad \square$$

The following theorem contains certain recurrence relation for $D_{\beta,r}[n]_q$ with respect to r.

Theorem 5.3. *The q, r-Dowling numbers $D_{\beta,r}[n]_q$ satisfy the following relation:*

$$D_{\beta,r}[n]_q = \sum_{j=0}^{n} \binom{n}{j} q^{(n-j)(r-1)} D_{\beta,r-1}[j]_q. \tag{53}$$

Proof. By making use of (26)

$$\sigma[n,k]_q^{\beta,r} = \sum_{j=k}^{n} \binom{n}{j} q^{(n-j)r_2} [r_1]_q^{n-j} \sigma[j,k]_q^{\beta,r_2},$$

with $r_1 = 1$ and $r_2 = r-1$, then

$$\sigma[n,k]_q^{\beta,r} = \sum_{j=k}^{n} \binom{n}{j} q^{(n-j)(r-1)} [1]_q^{n-j} \sigma[j,k]_q^{\beta,r-1}.$$

Summing up both sides of the equation over k such that $0 \le k \le n$,

$$\begin{aligned}\sum_{k=0}^{n} \sigma[n,k]_q^{\beta,r} &= \sum_{k=0}^{n}\sum_{j=k}^{n} \binom{n}{j} q^{(n-j)(r-1)} \sigma[j,k]_q^{\beta,r-1} \\ &= \sum_{j=0}^{n} \binom{n}{j} q^{(n-j)(r-1)} \left\{\sum_{k=0}^{j} \sigma[j,k]_q^{\beta,r-1}\right\}.\end{aligned}$$

Hence,

$$D_{\beta,r}[n]_q = \sum_{j=0}^{n} \binom{n}{j} q^{(n-j)(r-1)} D_{\beta,r-1}[j]_q. \qquad \square$$

The next corollary is a direct consequence of Theorem 5.3.

Corollary 5.4. *The q, r-Dowling numbers satisfy the following relation*:

$$D_{\beta,r-1}[n]_q = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} q^{(n-j)(r-1)} D_{\beta,r}[j]_q. \tag{54}$$

Proof. Equation (53) can be rewritten as

$$q^{-n(r-1)} D_{\beta,r}[n]_q = \sum_{j=0}^{n} \binom{n}{j} q^{(-j)(r-1)} D_{\beta,r-1}[j]_q. \tag{55}$$

Applying the binomial inversion formula

$$f_n = \sum_{j=0}^{n} \binom{n}{j} g_j \Longleftrightarrow g_n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f_j,$$

with

$$f_n = q^{-n(r-1)} D_{\beta,r}[n]_q$$

and

$$g_j = q^{(-j)(r-1)} D_{\beta,r-1}[j]_q,$$

Equation (55) yields

$$q^{(-n)(r-1)} D_{\beta,r-1}[n]_q = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} q^{-j(r-1)} D_{\beta,r}[j]_q.$$

Hence,

$$D_{\beta,r-1}[n]_q = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} q^{(n-j)(r-1)} D_{\beta,r}[j]_q. \tag{56}$$

$\square$

6. Explicit formula for q, r-Dowling numbers

Recently, Corcino et al. [5] obtained a new explicit formula for r-Dowling numbers which is expressed in terms of r-Whitney numbers of the second kind and the r-Whitney Lah numbers. This formula is stated in the following theorem.

Theorem 6.1 ([5])**.** *The explicit formula for r-Dowling numbers is given by*

$$D_{m,r}(n) = \sum_{j=0}^{n} (-1)^{n-j} \left\{ \sum_{k=0}^{j} L_{m,r}(j,k) \right\} W_{m,r}(n,j). \tag{57}$$

To verify this formula for a specific value of n, m, and r, we have the following table of values:

TABLE 1. A triangular array of values for $W_{m,r}(n,k)$ with $m = r = 2$

$D_{2,2}(n)$	n/k	0	1	2	3	4
1	0	1				
3	1	2	1			
11	2	4	6	1		
49	3	8	28	12	1	
257	4	16	120	100	20	1

TABLE 2. A triangular array of values for $L_{m,r}(n,k)$ with $m = r = 2$

$\sum_{k=0}^{n} L_{m,r}(n,k)$	n/k	0	1	2	3	4
1	0	1				
5	1	4	1			
37	2	24	12	1		
361	3	192	144	24	1	
4361	4	1920	1920	480	40	1

Using the explicit formula in Theorem 6.1, we obtain

$$\begin{aligned}
D_{2,2}(n=2) &= \sum_{j=0}^{2}(-1)^{2-j}\left\{\sum_{k=0}^{j} L_{2,2}(j,k)\right\} W_{2,2}(2,j) \\
&= (1)(4) - (5)(6) + (37)(1) \\
&= 11, \\
D_{2,2}(n=3) &= \sum_{j=0}^{3}(-1)^{3-j}\left\{\sum_{k=0}^{j} L_{2,2}(j,k)\right\} W_{2,2}(3,j) \\
&= -(1)(8) + (5)(28) - (37)(12) + (361)(1) \\
&= 49, \\
D_{2,2}(n=4) &= \sum_{j=0}^{4}(-1)^{4-j}\left\{\sum_{k=0}^{j} L_{2,2}(j,k)\right\} W_{2,2}(4,j) \\
&= (1)(16) - (5)(120) + (37)(100) - (361)(20) + (4361)(1) \\
&= 257.
\end{aligned}$$

Note that the values obtained for $D_{2,2}(2), D_{2,2}(3)$, and $D_{2,2}(4)$ are the exact values that appeared in Table 2.

Remark 6.2. When $m = 1$, and $r = 0$, (57) reduces to

$$D_{1,0}(n) = \sum_{j=0}^{n} (-1)^{n-j} \left\{ \sum_{k=0}^{j} L_{1,0}(j,k) \right\} W_{1,0}(n,j),$$

which is equivalent to the Qi formula for Bell numbers

$$B_n = \sum_{k=0}^{n} (-1)^{n-k} \left\{ \sum_{j=0}^{k} L(k,j) \right\} S(n,k),$$

with

$$B_n = D_{1,0}(n), L(j,k) = L_{1,0}(j,k) \text{ and } S(n,j) = W_{1,0}(n,j).$$

Remark 6.3. The r-Dowling numbers $D_{\beta,r}[n]_q$ equal to the sum of the entries of the *ith* row of the product of two matrices

$$\left[(-1)^{i-j} W_{m,r}(i,j)\right]_{n\times n} \left[L_{m,r}(i,j)\right]_{n\times n}, \tag{58}$$

whose entries are respectively the r-Whitney numbers of the second kind and the r-Whitney Lah numbers.

The following theorem contains the desired explicit formula for q, r-Dowling numbers, which is a q-analogue of the explicit formula in Theorem 6.1.

Theorem 6.4. *The explicit formula for q, r-Dowling numbers is given by*

$$D_{\beta,r}[n]_q = \sum_{j=0}^{n} (-1)^{n-j} \left\{ \sum_{k=0}^{j} L_{\beta,r}[j,k]_q \right\} \sigma[n,j]_q^{\beta,r}. \tag{59}$$

Proof. Equation (39) can be rewritten as

$$(-1)^n L_{\beta,r}[n,k]_q = \sum_{j=0}^{n} (-1)^{n-j} \phi_{\beta,r}[n,j]_q (-1)^j \sigma[j,k]_q^{\beta,r}. \tag{60}$$

Applying the inverse relation in (36)

$$f_n = \sum_{j=0}^{n} (-1)^{n-j} \phi_{\beta,r}[n,j]_q g_j \Longleftrightarrow g_n = \sum_{j=0}^{n} \sigma[n,j]_q^{\beta,r} f_j, \tag{61}$$

with

$$f_n = (-1)^n L_{\beta,r}[n,k]_q \text{ and } g_j = (-1)^j \sigma[j,k]_q^{\beta,r},$$

implies that equation (60) is equivalent to

$$(-1)^n \sigma[n,k]_q^{\beta,r} = \sum_{j=0}^{n} \sigma[n,j]_q^{\beta,r} (-1)^j L_{\beta,r}[j,k]_q,$$

$$\sigma[n,k]_q^{\beta,r} = \sum_{j=0}^{n} (-1)^{n-j} \sigma[n,j]_q^{\beta,r} L_{\beta,r}[j,k]_q.$$

Taking the sum on both sides over k such that $0 \le k \le n$ yields

$$\begin{aligned}\sum_{k=0}^{n} \sigma[n,k]_q^{\beta,r} &= \sum_{k=0}^{n}\sum_{j=0}^{n}(-1)^{n-j}\sigma[n,j]_q^{\beta,r} L_{\beta,r}[j,k]_q \\ &= \sum_{j=0}^{n}(-1)^{n-j}\sum_{k=0}^{n} L_{\beta,r}[j,k]_q \sigma[n,j]_q^{\beta,r}.\end{aligned}$$

Since

$$D_{\beta,r}[n]_q = \sum_{k=0}^{n} \sigma[n,k]_q^{\beta,r},$$

hence

$$D_{\beta,r}[n]_q = \sum_{j=0}^{n}(-1)^{n-j}\left\{\sum_{k=0}^{j} L_{\beta,r}[j,k]_q\right\}\sigma[n,j]_q^{\beta,r},$$

which is the desired result. □

TABLE 3. This table contains the first values of $\sigma[n,k]_q^{\beta,r}$ and $D_{\beta,r}[n]_q$ with $\beta = r = 2$

$D_{2,2}[n]_q$	n/k	0	1
1	0	1	0
$q+2$	1	$q+1$	1
q^2+5q+5	2	q^2+2q+1	$3q+3$
$2q^3+11q^2+22q+14$	3	q^3+3q^2+3q+1	$7q^2+14q+7$
$q^6+3q^5+10q^4+35q^3$ $+77q^2+90q+41$	4	$q^4+4q^3+6q^2+4q+1$	$15q^3+45q^2+45q+15$

TABLE 4. This table contains the first values of $L_{\beta,r}(n,k)$ with $\beta = r = 2$

$\sum_{k=0}^{n} L_{2,2}[n,k]_q$	n/k	0
1	0	1
$2q+3$	1	$2q+2$
$6q^2+18q+13$	2	$6q^2+12q+6$
$6q^5+24q^4+50q^3+98q^2+124q+59$	3	$6q^5+18q^4+36q^3+60q^2+54q+18$
$6q^{10}+30q^9+80q^8+178q^7+316q^6+438q^5+560q^4$ $+750q^3+942q^2+784q+277$	4	$6q^{10}+24q^9+60q^8+120q^7+186q^6+240q^5+294q^4$ $+360q^3+360q^2+216q+54$

n/k	1	2	3	4
0	0	0	0	0
1	1	0	0	0
2	$6q+6$	1	0	0
3	$6q^4+12q^3+36q^2+60q+30$	$2q^3+2q^2+10q+10$	1	0
4	$6q^9+18q^8+54q^7+114q^6+168q^5+216q^4$ $+318q^3+474q^2+414q+138$	$2q^8+4q^7+16q^6+28q^5+48q^4+68q^3$ $+104q^2+140q+70$	$2q^5+2q^4+4q^3+4q^2+14q+14$	1

Using the explicit formula in Theorem 6.4, we get

$$\begin{aligned}
D_{2,2}[n=1]_q &= \sum_{j=0}^{1}(-1)^{1-j}\left\{\sum_{k=0}^{j} L_{2,2}[j,k]_q\right\}\sigma[1,j]_q^{2,2}\\
&= -(q+1)+2q+3\\
&= q+2,\\
D_{2,2}[n=2]_q &= \sum_{j=0}^{2}(-1)^{2-j}\left\{\sum_{k=0}^{j} L_{2,2}[j,k]_q\right\}\sigma[2,j]_q^{2,2}\\
&= (q^2+2q+1)-(2q+3)(3q+3)+(6q^2+18q+13)\\
&= q^2+5q+5,\\
D_{2,2}[n=3]_q &= \sum_{j=0}^{3}(-1)^{3-j}\left\{\sum_{k=0}^{j} L_{2,2}[j,k]_q\right\}\sigma[3,j]_q^{2,2}\\
&= -(q^3+3q^2+3q+1)+(2q+3)(7q^2+14q+7)\\
&\quad -(6q^2+18q+13)(q^3+q^2+5q+5)\\
&\quad +6q^5+24q^4+50q^3+98q^2+124q+59\\
&= 2q^3+11q^2+22q+14,\\
D_{2,2}[n=4]_q &= \sum_{j=0}^{4}(-1)^{4-j}\left\{\sum_{k=0}^{j} L_{2,2}[j,k]_q\right\}\sigma[4,j]_q^{2,2}\\
&= (q^4+4q^3+6q^2+4q+1)-(2q+3)(15q^3+45q^2+45q+15)\\
&\quad +(6q^2+18q+13)(q^6+2q^5+8q^4+14q^3+24q^2+34q+17)\\
&\quad -(6q^5+24q^4+50q^3+98q^2+124q+59)(q^5+q^4+2q^3+2q^2\\
&\quad +7q+7)+6q^{10}+30q^9+80q^8+178q^7+316q^6+438q^5\\
&\quad +560q^4+750q^3+942q^2+784q+277\\
&= q^6+3q^5+10q^4+35q^3+77q^2+90q+41.
\end{aligned}$$

The values obtained for $D_{2,2}[1]_q, D_{2,2}[2]_q, D_{2,2}[3]_q$, and $D_{2,2}[4]_q$ are the exact values that appeared in Table 3.

Remark 6.5. Note that as $q \to 1$, (59) reduces to

$$D_{\beta,r}(n) = \sum_{j=0}^{n}(-1)^{n-j}\left\{\sum_{k=0}^{j} L_{\beta,r}(j,k)\right\} W_{\beta,r}(n,k),$$

which is equivalent to the explicit formula for r-Dowling numbers

$$D_{m,r}(n) = \sum_{j=0}^{n}(-1)^{n-j}\left\{\sum_{k=0}^{j} L_{m,r}(j,k)\right\} W_{m,r}(n,j),$$

with $\beta = m$.

Remark 6.6. The q, r-Dowling numbers $D_{\beta,r}[n]_q$ equal to the sum of the entries of the ith row of the product of two matrices

$$\left[(-1)^{i-j}\sigma[i,j]_q^{\beta,r}\right]_{n\times n}\left[L_{\beta,r}[i,i]_q\right]_{n\times n}, \tag{62}$$

whose entries are respectively the q, r-Whitney numbers of the second kind and the q, r-Whitney Lah numbers.

Remark 6.6 is equivalent to the following matrix relation

$$\left[\sigma[i,j]_q^{\beta,r}\right]_{n\times n} = \left[(-1)^{i-j}\sigma[i,j]_q^{\beta,r}\right]_{n\times n}\left[L_{\beta,r}[i,i]_q\right]_{n\times n}. \tag{63}$$

Now, the orthogonality relation in Theorem 3.3 implies the following matrix relation

$$\left[(-1)^{i-j}\sigma[i,j]_q^{\beta,r}\right]_{n\times n}\left[\phi_{\beta,r}[i,j]_q\right]_{n\times n} = I_n, \tag{64}$$

where I_n is the identity matrix of order n. That is,

$$\left[(-1)^{i-j}\sigma[i,j]_q^{\beta,r}\right]_{n\times n}^{-1} = \left[\phi_{\beta,r}[i,j]_q\right]_{n\times n},$$

which implies

$$\left[\phi_{\beta,r}[i,j]_q\right]_{n\times n}\left[\sigma[i,j]_q^{\beta,r}\right]_{n\times n} = \left[L_{\beta,r}[i,i]_q\right]_{n\times n}. \tag{65}$$

The matrix equation (65) is equivalent to equation (39).

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