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IDEALIZATION OF EM-HERMITE RINGS

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Abstract. A commutative ring R with unityis called EM-Hermite if for each $a, b \in R$ there exist $c, d, f \in R$ such that $a = cd, b = cf$ and the ideal (d, f) is regular in R. We showed in this article that R is a PP-ring if and only if the idealization $R(+)R$ is an EM-Hermite ring if and only if $R[x]/(x^{n+1})$ is an EM-Hermite ring for each $n \in \mathbb{N}$. We generalize some results, and answer some questions in the literature.

1. Introduction

Let R be a commutative ring with unity. Let $Z(R)$ be the set of zero-divisors in R, and $Reg(R) = R\backslash Z(R)$ be the set of regular elements. An ideal I of R is called a regular ideal if I contains a regular element.

A ring R is called EM-Hermite if for each $a, b \in R$, there exist $a_1, b_1, d \in R$ such that $a = a_1d$, $b = b_1d$ and the ideal (a_1, b_1) is regular.

If for each $f(x) \in Z(R[x])$ we can write $f(x) = c_f f_1(x)$, where $c_f \in R$ and $f_1(x) \in reg(R[x])$, then R is called an EM-ring, see [1]. It is clear that any EM-Hermite is EM-ring, but the converse is not in general true, see [4].

A ring R is called a morphic ring if for each $a \in R$ there exists $b \in R$ such that $Ann(a) = bR$ and $Ann(b) = aR$. It is called generalized morphic if for each $a \in R$ there exists $b \in R$ such that $Ann(a) = bR$, see [7].

A ring R is called a PP-ring if every principal ideal of R is a projective R-module. It is well known that R is a PP-ring if and only if for each $a \in R, Ann(a)$ is generated by an idempotent if and only if for each $a \in R$ there exist an idempotent e and a regular element r such that $a = er$, see [2].

A ring R is called von Neumann regular if for each $a \in R$ there exists $b \in R$ such that $a = a^2b$. It is well known that R is von Neumann regular if and only if for each $a \in R$ there exist an idempotent e and a unit u such that $a = eu$.

A ring R is said to have property A, if a finitely generated ideal I is contained in $Z(R)$ if and only if it has a non-zero annihilator.

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Recall that if R is a ring, and M is an R-module, then the idealization $R(+)M$ is the set of all ordered pairs $(r, m) \in R \times M$, equipped with addition defined by $(r, m) + (s, n) = (r + s, m + n)$ and multiplication defined by $(r, m)(s, n) = (rs, rn + sm)$. It is well-known that $R(+)R \cong R[x] \big/ (x^2)$. For the general case, we consider the ring $R[x] \big/ (x^{n+1})$, where $n \in \mathbb{N}$. In this case we set $R[x] \big/ (x^{n+1}) = \{\sum_{i=0}^{n} a_i X^i : a_i \in R, X = x + (x^{n+1})\}$.

The following proposition was proved in [5], and [6].

Proposition 1.1. *Let R be a ring. Then the following are equivalent*:
(1) *The ring R is von Neumann regular.*
(2) *The ring $R[x]/(x^{n+1})$ is morphic for each $n \in \mathbb{N}$.*
(3) *The ring $R(+)R$ is morphic.*

Motivated by these results, the authors in [3] proved the following:

Proposition 1.2. *The following statements are equivalent for a ring R*:
(1) *R is a PP-ring.*
(2) *$R[x]/(x^{n+1})$ is a generalized morphic ring for each $n \in \mathbb{N}$.*
(3) *$R(+)R$ is a generalized morphic ring.*
(4) *$R(+)R$ is an EM-ring.*

But the authors could not prove whether this is also equivalent to

$$R[x]/(x^{n+1})$$

being EM-ring for each $n \in \mathbb{N}$.

In this article we answer the question raised in [3] positively. We also show that it is equivalent to $R(+)R$ is EM-Hermite, and it is also equivalent to $R[x]/(x^{n+1})$ is EM-Hermite for each $n \in \mathbb{N}$.

2. Idealization of EM-Hermite rings

Lemma 2.1. *If $R(+)R$ is EM-Hermite, then R is EM-Hermite.*

Proof. Let $a, b \in R$. Then $(a, 0)$, $(b, 0) \in R(+)R$, and so, there exist (c, d), (x, y), $(z, w) \in R(+)R$ such that

$$(a, 0) = (c, d)(x, y),$$
$$(b, 0) = (c, d)(z, w),$$

with the ideal $((x, y), (z, w))$ is regular in $R(+)R$. So there exist (x_1, y_1), (x_2, y_2), $(r_1, r_2) \in R(+)R$ such that $(x_1, y_1)(x, y) + (x_2, y_2)(z, w) = (r_1, r_2) \in Reg(R(+)R)$.

Thus we have

$$a = cx,$$
$$b = cz,$$
$$r_1 = x_1x + x_2z \in Reg(R).$$

Hence, R is EM-Hermite as required. $\square$

The converse of the above lemma is not in general true, since $\mathbb{Z}_4$ is EM-Hermite, but $\mathbb{Z}_4(+)\mathbb{Z}_4$ is not. The following lemma is easily proved.

Lemma 2.2. *Let R be a ring, $n \in \mathbb{N}$, and $\sum_{i=0}^{n} \beta_i X^i$, $\sum_{i=0}^{n} \gamma_i X^i \in R[x]/(x^{n+1})$. Then*:

(1) *$\sum_{i=0}^{n} \beta_i X^i$ is zero-divisor in $R[x]/(x^{n+1})$ if and only if β_0 is a zero-divisor in R.*

(2) *$Ann(\sum_{i=0}^{n} \beta_i X^i, \sum_{i=0}^{n} \gamma_i X^i) \neq \{0\}$ if and only if $Ann(\beta_0, \gamma_0) \neq \{0\}$.*

(3) *If R has property A, then the ideal $(\sum_{i=0}^{n} \beta_i X^i, \sum_{i=0}^{n} \gamma_i X^i)$ is regular in $R[x]/(x^{n+1})$ if and only if the ideal (β_0, γ_0) is regular in R.*

Theorem 2.3. *The following are equivalent for a ring R*:

(1) *The ring R is a PP-ring.*

(2) *The ring $R[x]/(x^{n+1})$ is EM-Hermite for each $n \in \mathbb{N}$.*

(3) *The idealization $S = R(+)R$ is EM-Hermite.*

Proof. (1) $\Rightarrow$ (2) Let $a(X), b(X) \in S$ such that $a(X) = \sum_{i=0}^{n} a_i X^i$, $b(X) = \sum_{i=0}^{n} b_i X^i$. Since R is a PP-ring, then for each $i = 0, \ldots, n$, we can write $a_i = u_i r_i, b_i = v_i s_i$ such that $u_i^2 = u_i$, $v_i^2 = v_i$, and r_i, s_i are regular elements in R.

Define

$$1 - e_j = \prod_{i=0}^{j} (1 - u_i)(1 - v_i) \quad \text{for } j = 0, 1, 2, \ldots, n.$$

Then clearly we have $e_j^2 = e_j$ for each j. Moreover, for each i and j, $e_j e_i = e_i$, whenever $j \geq i$, and so $e_j a_i = a_i$, and $e_j b_i = b_i$ whenever $j \geq i$. Thus, $(e_j - e_{j-1})a_i = 0 = (e_j - e_{j-1})b_i$, whenever $j > i$. Also, it is clear that

$$(e_{i+1} - e_i)(e_{k+1} - e_k) = \begin{cases} (e_{i+1} - e_i) & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

Now, let $d_n(X) \in S$ such that

$$d_n(X) = \sum_{i=0}^{n} \alpha_i X^i,$$

with

$$\alpha_0 = e_0 \text{ and } \alpha_i = \sum_{j|i} (-1)^{\frac{i}{j}+1} (e_j - e_{j-1}),$$

and let $p_n(X), q_n(X) \in S$ such that

$$p_n(X) = \sum_{i=0}^{n} \beta_i X^i,$$

$$q_n(X) = \sum_{i=0}^{n} \gamma_i X^i,$$

such that for $i = 0, 1, \ldots, n-1$,

$$\beta_i = (a_i + \sum_{j=1}^{n-i}(e_j - e_{j-1})a_{i+j} + (1 - e_n)),$$
$$\gamma_i = (b_i + \sum_{j=1}^{n-i}(e_j - e_{j-1})b_{i+j} + (1 - e_n)), \text{ and}$$
$$\beta_n = a_n, \ \gamma_n = b_n.$$

We claim that $d_n(X)p_n(X) = a(X)$ and $d_n(X)q_n(X) = b(X)$.

We will proceed by induction on n to show that $d_n(X)p_n(X) = a(X)$.

n	$d_n(X)$	$p_n(X)$	$d_n(X)p_n(X)$
0	e_0	a_0	a_0
1	$e_0 + (e_1 - e_0)X$	$[a_0 + (e_1 - e_0)a_1 + (1 - e_1)] + a_1X$	$a_0 + a_1X$
2	$e_0 + (e_1 - e_0)X$ $+[(e_0 - e_1)$ $+(e_2 - e_1)]X^2$	$[a_0 + (e_1 - e_0)a_1$ $+(e_2 - e_1)a_2 + (1 - e_2)]$ $+[a_1 + (e_1 - e_0)a_2 + (1 - e_2)]X$ $+a_2X^2$	$a_0+a_1X+a_2X^2$

Assume now that the result is true for $n = k-1$, i.e., $d_{k-1}(X)p_{k-1}(X) = \sum_{i=0}^{k-1}\alpha_iX^i \sum_{i=0}^{k-1}\beta_iX^i = \sum_{i=0}^{k-1}a_iX^i$.

Define $d_k(X) = d_{k-1}(X) + [\sum_{j|k}(-1)^{\frac{k}{j}+1}(e_j - e_{j-1})]X^k$, $p_k(X) = \sum_{i=0}^{k}\delta_iX^i$, where $\delta_i = \beta_i + (e_{k-i} - e_{k-i-1})a_k$ for $i = 0, 1, \ldots, k-1$, and $\delta_k = a_k$.

Then for $i + j < k$, $\alpha_i\delta_j = \alpha_i\beta_j + \alpha_i(e_{k-j} - e_{k-j-1})a_k = \alpha_i\beta_j$, since if α_i contains the term $(e_{k-j} - e_{k-j-1})$, then $k - j$ would divide i, which is not the case, since $k - j > i$.

Hence, $d_k(X)p_k(X) = d_{k-1}(X)p_{k-1}(X) + [\sum_{i+j=k}\alpha_i\delta_j]X^k$.

We are done if we show that $\sum_{i+j=k}\alpha_i\delta_j = a_k$.

Assume that $(e_m - e_{m-1})a_s$ is a term in $\sum_{i+j=k}\alpha_i\delta_j$. Then we have 3 cases:

Case 1: $m = s < k$. This term occurs when multiplying $(e_m - e_{m-1})$ from α_{k-m} with a_m in δ_m, and this implies that $k - m = ml_1$, and so, $m|k$ which implies that the term will also occur when multiplying $(e_m - e_{m-1})$ from α_k with $(e_m - e_{m-1})a_m$ in δ_0, and therefore $k = ml_2$, and so $l_2 = l_1 + 1$. Thus, we have the terms $(-1)^{l_1+1}(e_m - e_{m-1})a_m + (-1)^{l_2+1}(e_m - e_{m-1})a_m = 0$ in $\sum_{i+j=k}\alpha_i\delta_j$.

A similar argument will be obtained if the term occurs when multiplying $(e_m - e_{m-1})$ from α_k with $(e_m - e_{m-1})a_m$ in δ_0.

Case 2: $m < s < k$. This term occurs when multiplying $(e_m - e_{m-1})$ from α_{k-s} with a_s in δ_s, and this implies that $k - s = ml_1$, and so $m|(k - s + m)$ which means that the term will also occur when multiplying $(e_m - e_{m-1})$ from α_{k-s+m} with $(e_m - e_{m-1})a_s$ in δ_{s-m}, and this implies that $k - s + m = ml_2$, and so $l_2 = l_1 + 1$. Thus, we have the terms $(-1)^{l_1+1}(e_m - e_{m-1})a_s + (-1)^{l_2+1}(e_m - e_{m-1})a_s = 0$ in $\sum_{i+j=k}\alpha_i\delta_j$.

A similar argument will be obtained if the term occurs when multiplying $(e_m - e_{m-1})$ from α_{k-s+m} with $(e_m - e_{m-1})a_m$ in δ_{s-m}.

Case 3: $m \leq s = k$. Then we will have the terms: $(e_k - e_{k-1})a_k + (e_{k-1} - e_{k-2})a_k + \cdots + (e_2 - e_1)a_k + (e_1 - e_0)a_k + e_0 a_k = e_k a_k = a_k$.

Thus $d_k(X)p_k(X) = \sum_{i=0}^{k} a_i X^i$, and so it follows by induction that

$$d_n(X)p_n(X) = a(X).$$

Similarly, one can show that $d_n(X)q_n(X) = b(X)$.

To show that $(p_n(X), q_n(X))$ is a regular ideal in $R[x]/(x^{n+1})$, it suffices using Lemma 2.2 to show that the ideal $I = (\beta_0, \gamma_0)$ is regular in R.

Note that

$$\begin{aligned}
(1 - e_n)\beta_0 &= (1 - e_n) \in I, \\
e_0\beta_0 &= a_0 \in I, \\
e_1(\beta_0 - a_0) &= (e_1 - e_0)a_1 \in I, \\
e_2(\beta_0 - a_0 - (e_1 - e_0)a_1) &= (e_2 - e_1)a_2 \in I, \\
&\vdots \\
(e_n - e_{n-1})a_n &\in I.
\end{aligned}$$

Similarly, one can show that $b_0, (e_i - e_{i-1})b_i \in I$ for $i = 1, 2, \ldots, n$.

Let $\alpha \in Ann(I)$. Then $\alpha a_0 = 0 = \alpha b_0$, and so, $\alpha u_0 = 0 = \alpha v_0$, hence $\alpha e_0 = 0$.

Also, $\alpha a_1 = \alpha(e_1 - e_0)a_1 = 0 = \alpha(e_1 - e_0)b_1 = \alpha b_1$, and so, $\alpha u_1 = 0 = \alpha v_1$, which implies that $\alpha e_1 = 0$.

Continue to get $\alpha e_i = 0$ for $i = 0, 1, \ldots, n$. But we have also $(1 - e_n) \in I$, and so $0 = \alpha(1 - e_n) = \alpha - \alpha e_n = \alpha$.

Thus, $Ann(\beta_0, \gamma_0) = \{0\}$, and since R has property A, we must have $I = (\beta_0, \gamma_0)$ a regular ideal in R.

Using Lemma 2.2, we get that $(p_n(X), q_n(X))$ is a regular ideal in $R[x]/(x^{n+1})$, and so, $R[x]/(x^{n+1})$ is an EM-Hermite ring.

(2) $\Rightarrow$ (3) The result is clear since $R(+)R$ is isomorphic to $R[x]/(x^2)$.

(3) $\Rightarrow$ (1) Assume $S = R(+)R$ is an EM-Hermite ring. Let $b \in Z(R)\backslash\{0\}$. Then it suffices to show that $Ann_R(b)$ is generated by an idempotent and hence R is a PP-ring.

Now, let $(0, 1), (b, 0) \in S$. Since S is an EM-Hermite ring, there exist elements $(n_1, m_1), (n_2, m_2), (\alpha, \beta) \in S$ such that

$$(0, 1) = (n_1, m_1)(\alpha, \beta),$$

$$(b, 0) = (n_2, m_2)(\alpha, \beta),$$

where the ideal $(n_1, m_1)S + (n_2, m_2)S \subsetneq Z(S)$. This implies that

$$\begin{aligned}
n_1\alpha &= 0, \\
n_1\beta + m_1\alpha &= 1,
\end{aligned}$$

$$n_2\alpha = b,$$
$$n_2\beta + m_2\alpha = 0.$$

Therefore,

$$b = b1 = b(n_1\beta + m_1\alpha) = bn_1\beta + bm_1\alpha = (n_2\alpha)n_1\beta + bm_1\alpha = 0 + bm_1\alpha.$$

Thus $Ann_R(m_1\alpha) \subseteq Ann_R(b)$. Also note that

$$\begin{aligned}(m_1\alpha)^2 &= (m_1\alpha)^2 + \beta m_1(0) = (m_1\alpha)^2 + \beta m_1(n_1\alpha)\\ &= m_1\alpha(m_1\alpha + n_1\beta) = m_1\alpha(1) = m_1\alpha.\end{aligned}$$

Now, let $d \in Ann_R(b)$. Then

$$\begin{aligned}(d\alpha m_1)n_1 &= dm_1(\alpha n_1) = dm_1(0) = 0,\\ (d\alpha m_1)n_2 &= dm_1(\alpha n_2) = dm_1 b = (db)m_1 = 0,\end{aligned}$$

Thus $d\alpha m_1 \in Ann_R(n_1) \cap Ann_R(n_2)$, and so we have

$$(0, d\alpha m_1) \in Ann_S(n_1, m_1) \cap Ann_S(n_2, m_2) = \{(0,0)\}.$$

Hence, $d \in Ann_R(\alpha m_1)$, and so $Ann_R(b) = Ann_R(\alpha m_1) = (1 - \alpha m_1)R$, is generated by an idempotent. □

To clarify the above proof, we give the following example.

Example 2.4. The ring $\mathbb{Z}_{10}[x]/(x^4)$ is an EM-Hermite ring, since $\mathbb{Z}_{10}$ is a PP-ring.

Let $a(X) = 6 + 4X + 5X^2 + 8X^3, b(X) = 4 + 6X + 7X^2 + 9X^3 \in \mathbb{Z}_{10}[x]/(x^4)$.

Writing $a_i = u_i r_i, b_i = v_i s_i$, with u_i and v_i are idempotents, r_i and s_i are regular, and $1 - e_i = \prod_{j=0}^{i}(1 - u_j)(1 - v_j)$. Thus we have:

i	a_i	u_i	b_i	v_i	e_i
0	6	6	4	6	6
1	4	6	6	6	6
2	5	5	7	1	1
3	8	6	9	1	1

Now let

$$\begin{aligned}p_3(X) = {} & [a_0 + (e_1 - e_0)a_1 + (e_2 - e_1)a_2 + (e_3 - e_2)a_3 + (1 - e_3)]\\ & + [a_1 + (e_1 - e_0)a_2 + (e_2 - e_1)a_3 + (1 - e_3)]X\\ & + [a_2 + (e_1 - e_0)a_3 + (1 - e_3)]X^2 + a_3X^3.\end{aligned}$$

Then $p_3(X) = 1 + 4X + 5X^2 + 8X^3$.

$$\begin{aligned}q_3(X) = {} & [b_0 + (e_1 - e_0)b_1 + (e_2 - e_1)b_2 + (e_3 - e_2)b_3 + 1 - e_3]\\ & + [b_1 + (e_1 - e_0)b_2 + (e_2 - e_1)b_3 + (1 - e_3)]X\\ & + [b_2 + (e_1 - e_0)b_3 + (1 - e_3)]X^2 + b_3X^3.\end{aligned}$$

Then $q_3(X) = 9 + X + 7X^2 + 9X^3$.

$$d_3(X) = e_0 + (e_1 - e_0)X + [(e_0 - e_1) + (e_2 - e_1)]X^2 \\ + [(e_1 - e_0) + (e_3 - e_2)]X^3.$$

Then $d_3(X) = 6 + 5X^2$.

Simple computations yield $a(X) = p_3(X)d_3(X), b(X) = q_3(X)d_3(X)$.

Now, the ideal $(\alpha_0, \beta_0)\mathbb{Z}_{10} = (1, 9)\mathbb{Z}_{10} = \mathbb{Z}_{10}$. Hence, the ideal

$$(p_3(X), q_3(X))\mathbb{Z}_{10}[x]/(x^4) = \mathbb{Z}_{10}[x]/(x^4)$$

is regular as required.

It was shown in [4] that if R is Noetherian, then R is generalized morphic if and only if it is an EM-ring if and only if it is an EM-Hermite ring. While if R was not Noetherian, then the result is false. Now we answer the question raised in [3] concerning the case at which $R[x]/(x^{n+1})$ is an EM-ring, and we give more equivalent conditions to Proposition 1.2 above.

Theorem 2.5. *The following statements are equivalent for a ring R:*
(1) $R[x]/(x^{n+1})$ *is a generalized morphic ring for each* $n \in \mathbb{N}$.
(2) $R[x]/(x^{n+1})$ *is an EM-Hermite ring for each* $n \in \mathbb{N}$.
(3) $R[x]/(x^{n+1})$ *is an EM-ring for each* $n \in \mathbb{N}$.
(4) $R(+)R$ *is a generalized morphic ring.*
(5) $R(+)R$ *is an EM-Hermite ring.*
(6) $R(+)R$ *is an EM-ring.*
(7) R *is a PP-ring.*

Proof. For the equivalence of (1), (4), (6) and (7), see [3].

The equivalence of (2), (5) and (7) follows from Theorem 2.3.

(2) $\Rightarrow$ (3) $\Rightarrow$ (6) are clear. □

References

[1] E. Abuosba and M. Ghanem, *Annihilating content in polynomial and power series rings*, To appear in J. Korean Math. Soc.

[2] S. Endo, *Note on p.p. rings.* (*A supplement to Hattori's paper*), Nagoya Math. J. **17** (1960), 167–170. http://projecteuclid.org/euclid.nmj/1118800460

[3] M. Ghanem and E. Abu Osba, *Some extensions of generalized morphic rings and EM-rings*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. **26** (2018), no. 1, 111–123. https://doi.org/10.2478/auom-2018-0007

[4] ———, *EM-Hermite rings*, Submitted.

[5] T.-K. Lee and Y. Zhou, *Morphic rings and unit regular rings*, J. Pure Appl. Algebra **210** (2007), no. 2, 501–510. https://doi.org/10.1016/j.jpaa.2006.10.005

[6] ———, *Regularity and morphic property of rings*, J. Algebra **322** (2009), no. 4, 1072–1085. https://doi.org/10.1016/j.jalgebra.2009.04.046

[7] H. Zhu and N. Ding, *Generalized morphic rings and their applications*, Comm. Algebra **35** (2007), no. 9, 2820–2837. https://doi.org/10.1080/00927870701354017

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