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On Some Fractional Quadratic Integral Inequalities

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Abstract. Integral inequalities provide a very useful and handy tool for the study of qualitative as well as quantitative properties of solutions of differential and integral equations. The main object of this work is to generalize some integral inequalities of quadratic type not only for integer order but also for arbitrary (fractional) order. We also study some inequalities of Pachpatte type.

1. Introduction and Preliminaries

In the theory of differential and integral equations, Gronwall's lemma has been widely used in various applications since its first appearance in the article by Bellman in 1943. In this article the author gave a fundamental lemma, known as Gronwall-Bellman Lemma, which is used to study the stability and asymptotic behavior of solutions of differential equations. Gronwall's lemma has seen several generalizations to various forms [3, 11, 28].

The literature on these inequalities and their applications is vast; see [1, 2, 4] and the references given therein [19, 20, 23]. In addition, as the theory of calculus on time scales has developed over the last few years, some Gronwall-type integral inequalities on time scales have been established by many authors [17, 21, 24].

In this work, we shall study some fractional integral inequalities which can be used to prove the uniqueness of solutions for differential and integral equations of

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fractional-order. These inequalities are similar to Bellman-Gronwall type inequalities which play a fundamental role in the qualitative as well as quantitative study of differential equations.

Let $L_1 = L_1[a, b]$ be the class of Lebesgue integrable functions on $[a, b]$ with the standard norm.

Now, we shall introduce the definitions of the fractional-order integral operators (see [18, 25, 26, 27]). Let β be a positive real number.

Definition 1.1. The *left-sided fractional integral* of order β of the function f is defined on $[a, b]$ by

$$I_{a+}^{\beta} f(t) \; = \; \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds, \quad t > a \tag{1.1}$$

and when $a = 0$, we have $I_{0+}^{\beta} f(t),\ t > 0$.

Definition 1.2. The *right-sided fractional integral* of order β of the function f is defined on $[a, b]$ by

$$I_{b-}^{\beta} f(t) \; = \; \int_t^b \frac{(s-t)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds, \quad t < b \tag{1.2}$$

and when $b = 0$, we have $I_{0-}^{\beta} f(t),\ t < 0$.

For further properties of fractional calculus see [18, 25, 26, 27].

2. Quadratic Integral Inequalities

The existence of solutions of nonlinear quadratic integral equations and some properties of their solutions have been well studied recently (see [5, 6, 7, 8, 9, 10] and [12, 13, 14, 15, 16]).

Let $\alpha, \beta > 0$. The existence of solutions of the quadratic integral equation of arbitrary (fractional) orders α and β

$$x(t) \; = \; h(t) \; + \; I_{a+}^{\alpha} \; f(t, x(t)) \; I_{a+}^{\beta} \; g(t, x(t)), \quad t > a \tag{2.1}$$

have been studied in [16] and [14] under the following assumptions.

(i) The functions $f, g : [a, b] \times R_+ \; \to R_+$ satisfy the Caratheòdory condition (i.e. are measurable in t for all $x \in R_+$ and continuous in x for all $t \in [a, b]$). Moreover, there exist two functions $m_1, m_2 \in L_1$, and a positive constant k such that

$$|f(t, x)| \; \leq \; m_1(t) \; + \; k \; x(t), \qquad |g(t, x)| \; \leq \; m_2(t), \;\; \forall \; (t, x) \; \in \; [a, b] \; \times \; R_+.$$

(ii) There exists a positive constant M_2 such that $I_{a+}^{\gamma} m_2(t) \leq M_2,\ \gamma < \beta$.

(iii) There exists a positive constant M_1 such that $I_{a+}^{\sigma} m_1(t) \leq M_1,\ \sigma < \alpha$.

(iv) $h \in L_1$.

Now, consider the nonlinear quadratic integral inequality of fractional orders

$$x(t) \ \leq \ h(t) \ + \ I_{a+}^{\alpha} \ f(t, x(t)) \ I_{a+}^{\beta} \ g(t, x(t)), \quad t > a, \quad \alpha, \ \beta > 0 \tag{2.2}$$

Theorem 2.1. *Let assumptions* (i)–(iv) *be satisfied. Let* $x(t) \in L_1$ *and satisfies inequality* (2.2) *for almost all* $t \in [a, b]$. *If* $\frac{kM_2 b^{\alpha+\beta-\gamma}}{\Gamma(\beta-\gamma+1)\Gamma(\alpha+1)} < 1$, *then*

$$x(t) \ \leq \ \sum_{n \ = \ 0}^{\infty} \ (\frac{k \ M_2 \ b^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} \ I_{a+}^{\alpha})^n \left[h(t) \ + \ \frac{M_1 \ M_2 \ b^{\beta-\gamma+\alpha-\sigma}}{\Gamma(\beta-\gamma+1)\Gamma(\alpha-\sigma+1)}\right].$$

Proof. Using assumptions (i)-(iv) and inequality (2.2), we have

$$x(t) \ \leq \ h(t) + I_{a+}^{\alpha} \ (m_1(t) + k \ x(t)).I_{a+}^{\beta} \ m_2(t), \ t > a, \alpha, \beta > 0$$

$$x(t) \ \leq \ h(t) + I_{a+}^{\alpha-\sigma} I_{a+}^{\sigma} m_1(t).I_{a+}^{\beta-\gamma} I_{a+}^{\gamma} m_2(t) + k \ I_{a+}^{\alpha} x(t).I_{a+}^{\beta-\gamma} I_{a+}^{\gamma} m_2(t),$$

$$x(t) \ \leq \ h(t) + \frac{M_1 M_2 b^{\beta-\gamma+\alpha-\sigma}}{\Gamma(\beta-\gamma+1)\Gamma(\alpha-\sigma+1)} + \frac{kM_2 b^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} I_{a+}^{\alpha} \ x(t), \ t > a, \alpha > 0$$

$$\Rightarrow \qquad (I \ - \ \frac{k \ M_2 \ b^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} I_{a+}^{\alpha}) \ x(t) \ \leq \ h(t) \ + \ \frac{M_1 \ M_2 \ b^{\beta-\gamma+\alpha-\sigma}}{\Gamma(\beta-\gamma+1)\Gamma(\alpha-\sigma+1)}$$

Since

$$\begin{aligned}
\| \frac{k \ M_2 \ b^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} \ I_{a+}^{\alpha} \ x(t) \ \| &= \int_a^b \ | \ \frac{k \ M_2 \ b^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} \ I_{a+}^{\alpha} \ x(t) \ | \ dt \\
&\leq \int_a^b \ \frac{k \ M_2 \ b^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} \ \int_a^t \ \frac{(t \ - \ s)^{\alpha \ - \ 1}}{\Gamma(\alpha)} \ |x(s)| \ ds \ dt \\
&\leq \frac{k \ M_2 \ b^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} \int_a^b \ \int_s^b \ \frac{(t \ - \ s)^{\alpha \ - \ 1}}{\Gamma(\alpha)} \ dt \ |x(s)| \ ds \\
&\leq \frac{k \ M_2 \ b^{\alpha+\beta-\gamma}}{\Gamma(\beta-\gamma+1)\Gamma(\alpha+1)} \int_a^b \ |x(s)| \ ds
\end{aligned}$$

and $\frac{k \ M_2 \ b^{\alpha+\beta-\gamma}}{\Gamma(\beta-\gamma+1)\Gamma(\alpha+1)} < 1$, then

$$\| \ \frac{k \ M_2 \ b^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} \ I_{a+}^{\alpha} \ x(t) \ \| < \|x(t)\|$$

and $(I - \frac{k \ M_2 \ b^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} I_{a+}^{\alpha})^{-1}$ exists.

Then

$$x(t) \leq \sum_{n \ = \ 0}^{\infty} \ (\frac{k \ M_2 \ b^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} \ I_{a+}^{\alpha})^n \left[h(t) + \frac{M_1 \ M_2 \ b^{\beta-\gamma+\alpha-\sigma}}{\Gamma(\beta-\gamma+1)\Gamma(\alpha-\sigma+1)}\right]. \qquad \square$$

Remark 2.2. If we replace assumptions (ii) and (iii) by (ii*) There exists a positive constant M_2 such that $I^{\beta}_{a+} m_2(t) \leq M_2$;

(iii*) There exists a positive constant M_1 such that $I^{\alpha}_{a+} m_1(t) \leq M_1$,

then we can obtain the following result.

Theorem 2.3. *Let assumptions* (i), (ii*), (iii*) *and*(iv) *be satisfied. Let* $x(t) \in L_1$ *satisfies the inequality* (2.2) *for almost all* $t \in [a, b]$. *If* $\frac{kM_2 b^{\alpha+\beta}}{\Gamma(\beta+1)\Gamma(\alpha+1)} < 1$, *then*

$$x(t) \;\leq\; \sum_{n\,=\,0}^{\infty} \Big(\frac{k\; M_2\; b^{\beta}}{\Gamma(\beta+1)}\; I^{\alpha}_{a+}\Big)^n \left[h(t) \;+\; \frac{M_1\; M_2\; b^{\beta+\alpha}}{\Gamma(\beta+1)\Gamma(\alpha+1)}\right].$$

Inequality (2.2) involves many integral inequalities of fractional order. So, some particular cases can be obtained as follows.

Corollary 2.4. *Let* $f, g : [a, b] \times R_+ \to R_+$ *are continuous functions, and* $h \in L_1$. *If* $x(t) \in L_1$ *and satisfies the inequality (2.2) for almost all* $t \in [a, b]$. *Then*

$$x(t) \;\leq\; h(t) \;+\; \frac{M_1\; M_2\; b^{\beta+\alpha}}{\Gamma(\beta+1)\Gamma(\alpha+1)},$$

where $M_1 = \sup\limits_{\forall t \in [a,b]} |g(t, x)|, M_2 = \sup\limits_{\forall t \in [a,b]} |f(t, x)|$.

Corollary 2.5. *Let assumptions* (i)–(iv) (*with* $\alpha = \beta, f = g$) *be satisfied. Let* $x(t) \in L_1$ *and satisfies the inequality*

$$x(t) \;\leq\; h(t) \;+\; \left(I^{\alpha}_{a+}\; f(t, x(t))\right)^2, \quad t > a, \qquad \alpha \;>\; 0$$

for almost all $t \in [a, b]$. *If* $\frac{kM_2 b^{2\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)\Gamma(\alpha+1)} < 1$, *then*

$$x(t) \;\leq\; \sum_{n\,=\,0}^{\infty} \Big(\frac{k\; M_2\; b^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\; I^{\alpha}_{a+}\Big)^n \left[h(t) \;+\; \frac{M_2^2\; b^{2\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)\Gamma(\alpha+1)}\right].$$

Corollary 2.6. *Let assumptions* (i)-(iv) (*with* $\alpha = \beta, f = g$ *and* $h = 0$) *be satisfied. Let* $x(t) \in L_1$ *and satisfies the inequality*

$$\sqrt{x(t)} \;\leq I^{\alpha}_{a+}\; f(t, x(t)), \quad t > a, \qquad \alpha \;>\; 0$$

for almost all $t \in [a, b]$. *If* $\frac{kM_2 b^{2\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)\Gamma(\alpha+1)} < 1$, *then*

$$x(t) \;\leq\; \sum_{n\,=\,0}^{\infty} \Big(\frac{k\; M_2\; b^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\; I^{\alpha}_{a+}\Big)^n \left[\frac{M_2^2\; b^{2\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)\Gamma(\alpha+1)}\right].$$

Letting $\alpha, \beta \to 1$, then we have

Corollary 2.7. *Let $h(t), f(t,x)$ and $g(t,x)$ satisfy the assumptions of Theorem 2.3. Let $x(t) \in L_1$ and satisfies the quadratic integral inequality*

$$x(t) \ \leq \ h(t) \ + \ \int_a^t \ f(t, x(t)) \ dt. \int_a^t \ g(t, x(t)) \ dt, \quad t > a$$

for almost all $t \in [a, b]$. If $kM_2 \ b^2 < 1$, then

$$x(t) \ \leq \ \sum_{n \ = \ 0}^{\infty} \ (k \ M_2 \ b \ I_{a+}^{\alpha})^n \left[h(t) \ + \ M_1 \ M_2 \ b^2\right].$$

Letting $\beta \to 0$, then we have

Corollary 2.8. *Let $h(t), f(t,x)$ and $g(t,x)$ satisfy the assumptions of Theorem 2.3. Let $x(t) \in L_1$ and satisfies the quadratic integral inequality*

$$x(t) \ \leq \ h(t) \ + \ g(t, x(t)) I_{a+}^{\alpha} \ f(t, x(t)), \quad t > a, \qquad \alpha \ > \ 0 \tag{2.3}$$

for almost all $t \ \in \ [a, b]$. If $\frac{k.M_2 b^{\alpha}}{\Gamma(\alpha+1)} < 1 \ \forall t \in [a, b]$, then

$$x(t) \ \leq \ \sum_{n \ = \ 0}^{\infty} \ (k \ m_2(t) \ I_{a+}^{\alpha})^n \left[h(t) \ + \ M_1 \ m_2(t)\right].$$

Proof. Using assumptions of Theorem 2.3, then inequality (2.3) becomes

$$x(t) \ \leq \ h(t) \ + \ m_2(t).I_{a+}^{\alpha} \ (m_1(t) + k \ x(t)), \quad t > a, \quad \alpha, \ \beta > 0$$

$$x(t) \ \leq \ h(t) \ + \ m_2(t) I_{a+}^{\alpha} m_1(t) + k \ m_2(t) I_{a+}^{\alpha} x(t),$$

$$x(t) \ \leq \ h(t) \ + \ M_1 \ m_2(t) + \ k \ m_2(t).I_{a+}^{\alpha} \ x(t), \quad t > a, \qquad \alpha \ > \ 0$$

$$\Rightarrow \qquad (I \ - \ k \ m_2(t) \ I_{a+}^{\alpha}) \ x(t) \ \leq \ h(t) \ + \ M_1.m_2(t).$$

Similarly

$$\| \ k \ m_2(t) \ I_{a+}^{\alpha} \ x(t) \| \ \leq \ \frac{k.M_2 \ b^{\alpha}}{\Gamma(\alpha+1)} \ \| \ x(t) \ \|$$

Then

$$x(t) \ \leq \ \sum_{n \ = \ 0}^{\infty} \ (k \ m_2(t) \ I_{a+}^{\alpha})^n \left[h(t) \ + \ M_1 \ m_2(t)\right]. \qquad \square$$

In the same fashion, we can prove the following corollary.

Corollary 2.9. *Let $h(t), f(t,x)$ satisfy assumptions of Corollary 2.8 and $g(t,x) = 1$. Let $x(t) \in L_1$ and satisfies the inequality*

$$x(t) \;\leq\; h(t) \;+\; I_{a+}^{\alpha}\; f(t,x(t)), \quad t > a, \qquad \alpha \;>\; 0 \tag{2.4}$$

for almost all $t \in [a,b]$. If $\frac{kb^{\alpha}}{\Gamma(\alpha+1)} < 1$, then

$$x(t) \;\leq\; \sum_{n\,=\,0}^{\infty}\; (k\; I_{a+}^{\alpha})^{n}\,[h(t) \;+\; M_1]\,. \qquad \square$$

Letting $f(t,x(t)) = m(t)x(t), \beta \to 0$ and $g(t,x(t)) = 1$, then we have the following result.

Corollary 2.10. *Let $h(t), m(t) \in L_1$ and $m(t) > 0$. Let $x(t) \in L_1$ and satisfies the inequality*

$$x(t) \;\leq\; h(t) \;+\; I_{a+}^{\alpha}\; m(t)\; x(t), \quad t > a, \qquad \alpha \;>\; 0 \tag{2.5}$$

for almost all $t \in [a,b]$. If $Mb^{\alpha} < \Gamma(\alpha+1)$, where M is a positive constant such that $\sup\limits_{\forall t \in [a,b]} |m(t)| = M$, then

$$x(t) \;\leq\; \frac{1}{m(t)}\; \sum_{j\,=\,0}^{\infty}\; (m(t)\; I_{a+}^{\alpha})^{j}\; m(t)\; h(t),\; t > a.$$

When $m(t)$ is continuous, then we get the following corollary.

Corollary 2.11. *Let $h(t) \in L_1$, $m(t) > 0$ and $m(t)$ is continuous on $[a,b]$. Let $x(t) \in L_1$ and satisfies (2.5) for almost all $t \in [a,b]$. If $b^{\alpha}M < \Gamma(\alpha+1)$, then*

$$x(t) \;\leq\; \sum_{i\,=\,0}^{\infty}\; M^{i}\; I_{a+}^{\alpha i}\; h(t),\; t > a$$

where $\sup\limits_{t \in [a,b]} |m(t)| = M$.

Some special cases will be considered, when $m(t) = K, K \neq 0$.

Corollary 2.12. *Let $h(t) \in L_1$ and $x(t) \in L_1$ satisfying the inequality*

$$x(t) \;\leq\; h(t) \;+\; K\; I_{a+}^{\alpha}\; x(t),\; t > a, \qquad \alpha \;>\; 0$$

for almost all $t \in [a,b]$. If $b^{\alpha}K < \Gamma(\alpha+1)$, then

$$x(t) \;\leq\; \sum_{i\,=\,0}^{\infty}\; K^{i}\; I_{a+}^{\alpha i}\; h(t),\; t > a.$$

Corollary 2.13. *Let $0 \leq \beta \leq \alpha, A \geq 0$ and let $x(t) \in L_1$ satisfying the inequality*

$$x(t) \ \leq \ A \, t^{- \ \beta} \ + \ K \ I^{\alpha}_{0+} \ x(t), \ t > 0, \qquad \alpha \ > \ 0$$

for almost all $t \in [0, b]$. If $b^{\alpha} K < \Gamma(\alpha + 1)$, then

$$x(t) \ \leq \ C \ A \ t^{- \ \beta}, \ t > 0$$

where C depends only on K, α and b.

For $h(t) = 0$, we have the following corollary.

Corollary 2.14. *Let $x(t) \in L_1$ and $x(t) > 0$ satisfying*

$$x(t) \ \leq \ K \ I^{\alpha}_{a+} \ x(t), \ t > a, \qquad \alpha \ > \ 0$$

for almost all $t \in [a, b]$. If $b^{\alpha} K < \Gamma(\alpha + 1)$, then

$$x(t) \ = \ 0, \ t > a.$$

Now, if we reverse the inequality (2.5) we shall obtain the next lemma.

Lemma 2.15. *Let $h(t), k(t) \in L_1$ and $k(t) > 0$. Let $x(t) \in L_1$ and satisfies the inequality*

$$x(t) \ \geq \ h(t) \ - \ I^{\alpha}_{a+} \ k(t) \ x(t), \ t > a, \qquad \alpha \ > \ 0 \tag{2.6}$$

for almost all $t \in [a, b]$. If there exist a function $m \in L_1$ and a positive constant M such that $I^{\beta}_{b-} m(t) \leq M, \beta < \alpha$. Moreover, $M b^{\alpha - \beta} < \Gamma(\alpha - \beta + 1)$. Then

$$x(t) \ \geq \ \frac{1}{k(t)} \ \sum_{i \ = \ 0}^{\infty} \ (- \ k(t) \ I^{\alpha}_{a+})^{i} \ k(t) \ h(t), \ t > a.$$

Proof. Multiply both sides of (2.6) by $k(t)$, then

$$k(t) \ x(t) \ \geq \ k(t) \ h(t) \ - \ k(t) \ I^{\alpha}_{a+} \ k(t) \ x(t)$$

and setting $y(t) \ = \ k(t) \ x(t)$, we get

$$y(t) \ \geq \ k(t) \ h(t) \ - \ k(t) \ I^{\alpha}_{a+} \ y(t)$$

$$\Rightarrow \qquad (I \ - \ (-k(t) \ I^{\alpha}_{a+})) \ y(t) \ \geq \ k(t) \ h(t)$$

Since

$$\begin{aligned}
\| (-k(t)\ I_{a+}^{\alpha}\ y(t)) \| &= \int_a^b \mid -k(t)\ I_{a+}^{\alpha}\ y(t) \mid dt \\
&\le \int_a^b |-k(t)| \int_a^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} |y(s)|\ ds\ dt \\
&\le \int_a^b \int_s^b \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} k(t)\ dt \mid y(s) \mid ds \\
&\le \int_a^b I_{b-}^{\alpha-\beta} I_{b-}^{\beta} m(t) dt \mid y(s) \mid ds \\
&\le M \int_a^b \int_s^b \frac{(t - s)^{\alpha - \beta - 1}}{\Gamma(\alpha-\beta)} dt \mid y(s) \mid ds \\
&\le M \frac{b^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \int_a^b \mid y(s) \mid ds.
\end{aligned}$$

Then

$$\begin{aligned}
&\| (-k(t))\ I_{a+}^{\alpha}\ y(t) \| < \| y(t) \| \\
\Rightarrow\quad &\| (-k(t))\ I_{a+}^{\alpha} \| < 1 \quad \text{and} \quad (I - (-k(t))\ I_{a+}^{\alpha})^{-1} \quad \text{exists.}
\end{aligned}$$

Then

$$y(t) \ \ge \ \sum_{i = 0}^{\infty} (-k(t))\ I_{a+}^{\alpha})^{i}\ k(t)\ h(t),$$

and therefore $x(t)$ is estimated by

$$x(t) \ \ge \ \frac{1}{k(t)} \sum_{i = 0}^{\infty} (-k(t))\ I_{a+}^{\alpha})^{i}\ k(t)\ h(t). \qquad \square$$

The next corollaries give particular cases for inequality (2.6).

Corollary 2.16. *Let $h(t) \in L_1$, $k(t) > 0$ and $k(t)$ is continuous on $[a, b]$. Let $x(t) \in L_1$ and satisfies (2.6) for almost all $t \in [a, b]$. If $b^{\alpha} M < \Gamma(\alpha + 1)$, then*

$$x(t) \ \ge \ \sum_{i = 0}^{\infty} M^{i}\ I_{a+}^{\alpha i}\ h(t),\ \ t > a,$$

where $\sup_{t\in[a,b]} |k(t)| = M.$

Corollary 2.17. *Let $h(t) \in L_1$ and $x(t) \in L_1$ satisfying the inequality*

$$x(t) \ \ge \ h(t) \ - \ k\ I_{a+}^{\alpha}\ x(t),\ \ t > a, \qquad \alpha \ > \ 0$$

for almost all $t \in [a,b]$. *If* $b^\alpha k < \Gamma(\alpha+1)$, *then*

$$x(t) \ \geq \ \sum_{i\,=\,0}^{\infty} \ (-\,k)^i \ I_{a+}^{\alpha i} \ h(t), \ t > a, .$$

Corollary 2.18. *Let* $0 \leq \beta \leq \alpha, A \geq 0$ *and let* $x(t) \in L_1$ *satisfying the inequality*

$$x(t) \ \geq \ A \ t^{-\ \beta} \ - \ k \ I_{0+}^{\alpha} \ x(t), \ t > a, \qquad \alpha \ > \ 0$$

for almost all $t \in [0,b]$. *If* $b^\alpha k < \Gamma(\alpha+1)$, *then*

$$x(t) \ \geq \ C \ A \ t^{-\ \beta}, \quad t > 0.$$

where C *depends only on* k, α *and* b.

Corollary 2.19. *Let* $x(t) \in L_1$ *and satisfies the inequality*

$$x(t) \ \geq \ k \ I_{a+}^{\alpha} \ x(t), \qquad \alpha \ > \ 0$$

for almost all $t \in [a,b]$. *If* $b^\alpha k < \Gamma(\alpha+1)$, *then*

$$x(t) \ \geq \ 0, \quad t > 0.$$

Remark 2.20 Clearly, we can obtain similar results if we replace the *left-sided fractional-order integral* I_{a+}^{α} by the *right-sided fractional-order integral* I_{b-}^{α} in inequality (2.2).

3. Inequality of Pachpatte Type

Over the years integral inequalities have become an important tool in the analysis of various differential and integral equations. These inequalities are useful in investigating the asymptotic behavior and the stability on the solutions of integral equations. Pachpatte [22] gave a new integral inequality and studied the boundedness, asymptotic behavior and growth of the solutions of an integral equation using the inequality. We introduce this inequality as follows.

Theorem 3.1.([22]) *Let* u, f, g *be real-valued nonnegative continuous functions defined on* R^+, *and* c_1, c_2 *be nonnegative constants. If*

$$u(t) \leq (c_1 + \int_0^t f(s)u(s)ds)(c_2 + \int_0^t g(s)u(s)ds)$$

and $c_1c_2 \int_0^t R(s)Q(s)ds < 1$ *for all* $t \in R^+$, *then*

$$u(t) \leq \frac{c_1c_2 \ Q(t)}{1 - c_1c_2 \int_0^t R(s)Q(s)ds}, t \in R^+,$$

where

$$R(t) = \int_0^t [f(t)g(s) + f(s)g(t)]ds, \qquad Q(t) = \exp\,(\int_0^t [c_1 g(s) + c_2 f(s)]ds).$$

Now consider the quadratic inequality of fractional order

$$x(t) \;\le\; (c_1 \;+\; I_{a+}^{\alpha}\; f(t,x(t)))(c_2 + \; I_{a+}^{\beta}\; g(t,x(t))), \quad t > a, \quad \alpha,\; \beta > 0 \tag{3.1}$$

Theorem 3.2. *Let assumptions* (i), (ii*), (iii*) *and* (iv) *be satisfied. Let* $x(t) \in L_1$ *satisfies the inequality* (3.1) *for almost all* $t \in [a, b]$. *If* $\frac{k(c_2+M_1)b^{\alpha}}{\Gamma(\alpha+1)} < 1$, *then*

$$x(t) \;\le\; \sum_{n\,=\,0}^{\infty} \;\left(k(c_2 \;+M_1)\; I_{a+}^{\alpha}\right)^n (c_1\; c_2 + c_1\; M_1 + M_2 c s_2 + M_1\; M_2)\,.$$

$$x(t) \;\le\; \sum_{n\,=\,0}^{\infty} \;\frac{(k(c_2 \;+M_1))^n\; t^{n\alpha}\,(c_1\; c_2 + c_1\; M_1 + M_2 c s_2 + M_1\; M_2)}{\Gamma(\alpha\; n+1)}.$$

Proof. The proof can straight forward as in Theorem 2.1. □

When $f(t, x) = m(t)x(t)$ and $g(t, x) = k(t)x(t)$, then we obtain Pachpatte inequality which is studied in [22].

Competing Interests. The authors declare that they have no competing interests.

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