Non Existence of $\mathcal{PR}$-semi-slant Warped Product Submanifolds in a Para-Kähler Manifold

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Abstract. In this paper, we prove that there are no non-trivial $\mathcal{PR}$-semi-slant warped product submanifolds with proper slant coefficients in para-Kähler manifolds $\overline{M}$. We also present a numerical example that illustrates the existence of a $\mathcal{PR}$-warped product submanifold in $\overline{M}$.

1. Introduction

Warped product manifolds are arguably the most natural and significant generalization of Riemannian product manifolds. Bishop and O'Neill in [3] introduced the concept of warped product manifolds to construct manifolds of nonpositive curvature. Later O'Neill in [20] extended the theory of Riemannian warped product manifolds for pseudo-Riemannian warped product manifolds. More precisely, let (B, g_B) and (F, g_F) be two pseudo-Riemannian manifolds and let $f : B \to (0, \infty)$ be a positive differentiable function. Consider the product manifold $B \times F$ with projections

$$\pi : B \times F \to B \quad \text{and} \quad \sigma : B \times F \to F.$$

The warped product $B \times_f F$ is the manifold $B \times F$ endowed with the pseudo-Riemannian structure such that

$$g(X, Y) = g_B \left(\pi_*(X), \pi_*(Y) \right) + (f \circ \pi)^2 g_F \left(\sigma_*(X), \sigma_*(Y) \right),$$

(or equivalently, $g = g_B + (f)^2 g_F$,) for any $X, Y \in \Gamma(TM)$, where '$*$' stands for derivation map. The function f is called *the warping function* of the warped product.

The theory of warped products attained attention when Chen, in [7], introduced a new class of CR-submanifolds [5] called CR-warped product Kaehlerian manifolds and investigated some existence and non-existence results. Hasegawa-Mihai [15] and Munteanu [19] continued this line of research for Sasakian manifold, which can be viewed as an odd-dimensional counterpart of a Kähler manifold. Since then, several differential geometer's have studied the existence (or non-existence) of warped product submanifolds in Riemannian metric manifolds [9, 10, 25].

In addition to Riemannian metric manifolds, manifolds with neutral metric structures (i.e., para-Hermitian manifolds) have been explored in various geometric and physical problems, and the theory successfully applied to supersymmetric field theory, string theory and black holes [13, 16, 17, 21, 22]. The study of neutral metric manifolds was less prominant until Davidov et al. in [12] presented some analogies and differences between the structures admitting the neutral metric and the Riemannian metric. Thereafter, Chen and others initiated the study of the geometry of pseudo-Riemannian warped product submanifolds in para complex (contact) manifolds [1, 11, 24]. Motivated by the above-mentioned works, in this paper we investigate the existence (or non-existence) of $\mathcal{PR}$-semi-slant warped product submanifolds in para-Kähler manifolds by assuming proper and improper slant factor aspects.

The organization of the article is as follows. In Section 2 we recall some fundamental concepts about para-Kähler manifolds and their submanifolds, warped product manifolds, along with a few relevant results required for the present study. Section 3 deals with the definitions of slant, $\mathcal{PR}$-semi-slant submanifolds, and derivations of the necessary and sufficient conditions for integrability and totally geodesic foliation of the distributions associated with the definition of $\mathcal{PR}$-semi-slant submanifolds. In Section 4, we derive non-existence results for non-trivial $\mathcal{PR}$-semi-slant warped product submanifolds of the forms $M_T \times_f M_\lambda$ and $M_\lambda \times_f M_T$, where M_T is an invariant submanifold and M_λ is a proper slant submanifold in $\overline{M}$. Finally in Section 5, we present an example that illustrates the existence of a $\mathcal{PR}$-semi-slant warped product submanifold with an improper slant coefficient (called $\mathcal{PR}$-warped product) in $\overline{M}$.

2. Preliminaries

Let $\overline{M}$ be an even-dimensional smooth manifold. A smooth manifold $\overline{M}$ is said to have an almost product structure if

$$(2.1) \qquad\qquad\qquad\qquad \mathcal{P}^2 = I,$$

where $\mathcal{P}$ is a tensor field of type $(1, 1)$ and I is the identity transformation on $\overline{M}$. For this, the pair $(\overline{M}, \mathcal{P})$ is called an almost product manifold. An almost para-Hermitian manifold $(\overline{M}, \mathcal{P}, \overline{g})$ [18] is a smooth manifold associated with an almost product structure $\mathcal{P}$ and a pseudo-Riemannian metric $\overline{g}$ satisfying

$$(2.2) \qquad\qquad\qquad \overline{g}(\mathcal{P}X, \mathcal{P}Y) + \overline{g}(X, Y) = 0,$$

for any X, Y tangent to $\overline{M}$. Clearly, from equations (2.1) and (2.2), we conclude that the signature of $\overline{g}$ is necessarily (m, m), and

$$(2.3) \qquad \overline{g}(\mathcal{P}X, Y) + \overline{g}(X, \mathcal{P}Y) = 0,$$

for any $X, Y \in \Gamma(T\overline{M})$; where $\Gamma(T\overline{M})$ is the Lie algebra of vector fields in $\overline{M}$. The fundamental 2-form ω of $\overline{M}$ is defined by

$$(2.4) \qquad \omega(X, Y) = \overline{g}(X, \mathcal{P}Y), \quad \forall X, Y \in \Gamma(T\overline{M}).$$

Definition 2.1. An almost para-Hermitian manifold $\overline{M}$ is called a *para-Kähler manifold* [8] if $\mathcal{P}$ is parallel with respect to $\overline{\nabla}$ i.e.,

$$(2.5) \qquad (\overline{\nabla}_X \mathcal{P})Y = 0, \quad \forall X, Y \in \Gamma(T\overline{M})$$

where $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}$ with respect to $\overline{g}$.

Let us consider that M be an isometrically immersed submanifold of a para-Kähler manifold $\overline{M}$ in the sense of O'Neill [20]. Let g be the induced metric on M such that $g = \overline{g}|_M$ [14]. We denote the set of vector fields normal to M by $\Gamma(TM^\perp)$ and the sections of tangent bundle TM of M by $\Gamma(TM)$. Then the Gauss-Weingarten formulas are given by, respectively,

$$(2.6) \qquad \overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$
$$(2.7) \qquad \overline{\nabla}_X \zeta = -A_\zeta X + \nabla_X^\perp \zeta,$$

for any $X, Y \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^\perp)$, where ∇ is the induced connection, $\nabla^\perp$ is the normal connection on the normal bundle $\Gamma(TM^\perp)$, h is the second fundamental form, and the shape operator A_ζ associated with the normal section ζ is given by [4]

$$(2.8) \qquad g(A_\zeta X, Y) = \overline{g}(h(X, Y), \zeta).$$

If we write, for all $\tau \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^\perp)$ that

$$(2.9) \qquad \mathcal{P}\tau = t\tau + n\tau,$$
$$(2.10) \qquad \mathcal{P}\zeta = t'\zeta + n'\zeta,$$

where $t\tau$ (resp., $n\tau$) is tangential (resp., normal) part of $\mathcal{P}\tau$ and $t'\zeta$ (resp., $n'\zeta$) is tangential (resp., normal) part of $\mathcal{P}\zeta$, then the submanifold M is said to be *invariant* if n is identically zero and *anti-invariant* if t is identically zero. Now, from equations (2.3) and (2.9), we obtain for any $X, Y \in \Gamma(TM)$ that

$$(2.11) \qquad g(X, tY) = -g(tX, Y).$$

Here, we state some important results on warped product manifolds;

Proposition 2.2.([3]) *For $X, Y \in \Gamma(TB)$ and $Z, W \in \Gamma(TF)$, we obtain for the warped product manifold $M = B \times_f F$ that*

(i) $\nabla_X Y \in \Gamma(TB)$,

(ii) $\nabla_X Z = \nabla_Z X = \left(\frac{Xf}{f}\right) Z$,

(iii) $\nabla_Z W = \frac{-g(Z,W)}{f} \nabla f$,

where ∇ denotes the Levi-Civita connection on M and ∇f is the gradient of f defined by $g(\nabla f, X) = Xf$.

Remark 2.3.([3]) It is also important to note that for a warped product $M = B \times_f F$,

(1) M is said to be *trivial* if f is constant,

(2) B is totally geodesic and F is totally umbilical in M,

(3) To reduce the complexity, we represent a vector field X on B with its lift $\overline{X}$ and a vector field Z on F with its lift $\overline{Z}$.

Now, we present a lemma for later use:

Lemma 2.4. *Let $M = B \times_f F$ be a non-trivial warped product submanifold of a para-Kähler manifold $\overline{M}$, then*

$$(2.12) \qquad g(A_{nZ}X, W) = g(A_{nW}X, Z) = -tX(\ln f)g(Z,W),$$

for any $X, Y \in \Gamma(TB)$ and $Z, W \in \Gamma(TF)$.

Proof. The proof of this lemma follows easily by using equations (2.5), (2.9), (2.11) and Gauss-Weingarten formulas. $\square$

3. $\mathcal{PR}$-semi-slant Submanifolds

In this section, motivated by [1, 24], we first state the definition of a slant submanifold in a para-Kähler manifold $\overline{M}$, and then continue the study by introducing a generalized class of a $\mathcal{PR}$-submanifold [11] called $\mathcal{PR}$-semi-slant submanifold in $\overline{M}$.

Let M be a non-degenerate submanifold of a para-Kähler manifold $\overline{M}$ such that $t^2 X = \lambda X = \lambda \mathcal{P}^2 X$, $g(tX, Y) = -g(X, tY)$ for any $X, Y \in \Gamma(TM)$, where λ is a real coefficient. Then with the help of equation (2.11), we have

$$(3.1) \quad \frac{g(\mathcal{P}X, tY)}{|\mathcal{P}X||tY|} = -\frac{g(X, \mathcal{P}tY)}{|\mathcal{P}X||tY|} = -\frac{g(X, t^2 Y)}{|\mathcal{P}X||tY|} = -\lambda\frac{g(X, \mathcal{P}^2 Y)}{|\mathcal{P}X||tY|} = \lambda\frac{g(\mathcal{P}X, \mathcal{P}Y)}{|\mathcal{P}X||tY|}.$$

On the other hand,

$$(3.2) \qquad\qquad \frac{g(\mathcal{P}X, tY)}{|\mathcal{P}X||tY|} = \frac{g(tX, tY)}{|\mathcal{P}X||tY|}.$$

In particular, from equations (3.1) and (3.2), we obtain for $X = Y$ that $\frac{g(\mathcal{P}X,tX)}{|\mathcal{P}X||tX|} = \sqrt{\lambda}$. Here we call λ a slant coefficient and consequently M a slant submanifold. Conversely, assume that M is a slant submanifold then $\lambda\frac{|\mathcal{P}X|}{|tX|} = \frac{|tX|}{|\mathcal{P}X|}$, where X is a non-lightlike vector field. We obtain by the consequence of previous equation for any $X, Y \in \Gamma(TM)$ that $-\lambda\frac{g(X,\mathcal{P}^2Y)}{|\mathcal{P}X||tY|} = \frac{g(\mathcal{P}X,tY)}{|\mathcal{P}X||tY|}$, which yields $g(X, t^2Y) = \lambda g(X, \mathcal{P}^2Y)$, $g(tX, Y) = -g(X, tY)$. Hence, $t^2 = \lambda I$, $g(tX, Y) = -g(X, tY)$ by virtue of the fact that structure is para-Kähler and X is any non-lightlike vector field.

Remark 3.1. The slant coefficient λ is sometimes chosen as $\cos^2\theta$ or $\cosh^2\theta$ or $-\sinh^2\theta$ for vector fields tangent to M, where θ is a real constant, called slant angle. The nature of the λ equals to $\cos^2\theta$ or $\cosh^2\theta$ or $-\sinh^2\theta$ depending on the consideration of vector fields (that is, angle between spacelike–spacelike or timelike–timelike or timelike–spacelike vector fields). The authors in [1, 2] distinguished different cases for λ, depending on the behaviour of vector fields.

As a consequence of the above characterization, we define slant submanifold in an almost para-Hermitian manifold;

Definition 3.2. Let M be an isometrically immersed submanifold of an almost para-Hermitian manifold $\overline{M}$ and let $\mathfrak{D}_\lambda$ be the distribution on M. Then $\mathfrak{D}_\lambda$ is said to be *slant distribution* on M, accordingly M *slant submanifold*, if there exist a real valued constant λ such that

$$t^2 = \lambda I, \quad g(tX, Y) = -g(X, tY),$$

for any non lightlike tangent vector fields $X, Y \in \mathfrak{D}_\lambda$ on M. Here, λ will be called *slant coefficient* that is globally constant and independent of the choice of point on M in $\overline{M}$.

Remark 3.3. Since the manifold M is non-degenerate (*i.e.*, M includes either spacelike vector fields or timelike vector fields), thus our definition of slant submanifold can be considered as the generalization of definitions given in [6, 23] that covers only the spacelike vector fields which implies $\lambda = \cos^2\theta$, where θ is a slant angle.

Now, we have the following definition.

Definition 3.4. Let M be an isometrically immersed submanifold of an almost para-Hermitian metric manifold $\overline{M}(\mathcal{P}, \overline{g})$. Then the submanifold M is said to be a $\mathcal{PR}$-*semi-slant* if it is furnished with a pair of non-degenerate orthogonal distribution $(\mathfrak{D}_T, \mathfrak{D}_\lambda)$ satisfying the following conditions:

(i) $TM = \mathfrak{D}_T \oplus \mathfrak{D}_\lambda$,

(ii) the distribution $\mathfrak{D}_T$ is invariant under $\mathcal{P}$, i.e., $\mathcal{P}(\mathfrak{D}_T) \subset \mathfrak{D}_T$ and

(iii) the distribution $\mathfrak{D}_\lambda$ is slant distribution with slant coefficient λ.

A $\mathcal{PR}$-semi-slant submanifold is

(1) $\mathcal{PR}$-*submanifold* if $\mathfrak{D}_T \neq \{0\}$ and $\mathfrak{D}_\lambda \neq \{0\}$ with $\lambda = 0$ [11],

(2) *proper* if $\mathfrak{D}_T \neq \{0\}$, $\mathfrak{D}_\lambda \neq \{0\}$ and $\lambda \neq 0, 1$.

Here, we give an example for demonstrating the existence of proper $\mathcal{PR}$-semi-slant submanifold M in a para-Kähler manifold $\overline{M}$.

Example 3.5. Let $\overline{M} = \mathbb{R}^8$ be a 8-dimensional manifold with the standard Cartesian coordinates

$$(\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4, \overline{x}_5, \overline{x}_6, \overline{x}_7, \overline{x}_8).$$

Define the para-Kähler pseudo-Riemannian metric structure $(\mathcal{P}, \overline{g})$ on $\overline{M}$ by

$$\mathcal{P}e_1 = e_5, \ \mathcal{P}e_2 = e_6, \ \mathcal{P}e_3 = e_7, \ \mathcal{P}e_4 = e_8, \ \mathcal{P}e_8 = e_4, \ \mathcal{P}e_7 = e_3, \ \mathcal{P}e_6 = e_2$$

$$(3.3) \qquad \mathcal{P}e_5 = e_1, \ \overline{g} = \sum_{i=1}^{4}(d\overline{x}_i)^2 - \sum_{j=5}^{8}(d\overline{x}_j)^2.$$

Here, $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ is a local orthonormal frame for $\Gamma(T\overline{M})$, given by $e_i = \dfrac{\partial}{\partial \overline{x}_i}, \ e_j = \dfrac{\partial}{\partial \overline{x}_j}$.

Let M be an isometrically immersed pseudo-Riemannian submanifold in a para-Kähler manifold $\overline{M}$ defined by

$$\Omega(x_1, x_2, x_3, x_4) = \left(\begin{array}{l} x_1 \cos(\alpha), x_3 \cosh(\alpha), x_4 \sinh(\alpha), x_1 \sin(\alpha), \\ \quad x_2, x_4 \cosh(\alpha), x_3 \sinh(\alpha), k \end{array} \right),$$

where $\alpha \in (0, \pi/2)$ and $k \neq 0$. Then the tangent bundle $\Gamma(TM)$ of M is spanned by the vector fields

$$(3.4) \qquad \begin{aligned} X_1 &= \cos(\alpha)e_1 + \sin(\alpha)e_4, \quad X_2 = e_5, \\ X_3 &= \cosh(\alpha)e_2 + \sinh(\alpha)e_7, \\ X_4 &= \sinh(\alpha)e_3 + \cosh(\alpha)e_6, \end{aligned}$$

where X_1, X_3 are spacelike and X_2, X_4 are timelike vector fields tangent to M. Now, the space $\mathcal{P}(TM)$ with respect to the paracosymplectic pseudo-Riemannian metric structure $(\mathcal{P}, \overline{g})$ of $\overline{M}$ becomes

$$(3.5) \qquad \begin{aligned} \mathcal{P}(X_1) &= \cos(\alpha)e_5 + \sin(\alpha)e_8, \quad \mathcal{P}(X_2) = e_1, \\ \mathcal{P}(X_3) &= \cosh(\alpha)e_6 + \sinh(\alpha)e_3, \\ \mathcal{P}(X_4) &= \sinh(\alpha)e_7 + \cosh(\alpha)e_2. \end{aligned}$$

Therefore from equations (5.2), (5.3) and (5.1), we obtain that $\mathfrak{D}_T$ and $\mathfrak{D}_\lambda$, are the distributions defined by $span\{X_3, X_4\}$, and $span\{X_1, X_2\}$ respectively, where $\mathfrak{D}_T$ is an invariant distribution and $\mathfrak{D}_\lambda$ is a slant distribution with slant coefficient $\lambda = |\cos(\alpha)|$. Thus M becomes a proper $\mathcal{PR}$-semi-slant submanifold M of a para-Kähler manifold $\overline{M}$.

Next, we prove the following result as a proposition for later use.

Proposition 3.6. *Let M be a slant submanifold of an almost para-Hermitian manifold $\overline{M}(\mathcal{P}, \overline{g})$. Then*

$$(3.6) \qquad g(tX, tY) = \lambda\{-g(X, Y)\} = \lambda g(\mathcal{P}X, \mathcal{P}Y),$$

$$(3.7) \qquad g(nX, nY) = (1 - \lambda)\{-g(X, Y)\} = (1 - \lambda)g(\mathcal{P}X, \mathcal{P}Y),$$

for any $X, Y \in \Gamma(TM)$.

Proof. We have from equations (2.1), (2.3) and (2.11), that $g(tX, tY) = -g(X, t^2 Y) = -\lambda g(X, \mathcal{P}^2 Y) = \lambda g(\mathcal{P}X, \mathcal{P}Y)$. Therefore, by the use of equation (2.2) and the Definition 3.2(a), we get equation (3.6). Equation (3.7) follows from equations (2.9) and (3.6). This completes the proof of the proposition. $\qquad\square$

Here, we obtain some necessary and sufficient conditions for the foliation determined by distributions allied with the definition of $\mathcal{PR}$-semi-slant submanifolds in para-Kähler manifolds $\overline{M}$ to be involutive and totally geodesic.

Theorem 3.7. *Let M be a proper $\mathcal{PR}$-semi-slant submanifold M of a para-Kähler manifold $\overline{M}$. Then the distribution $(\mathfrak{D}_T)$*

(1) *is involutive if and only if $h(X, tY) = h(tX, Y)$;*

(2) *defines totally geodesic foliation if and only if $A_{nZ}tY = A_{ntZ}Y$;*

for any $X, Y \in \Gamma(\mathfrak{D}_T)$ and $Z \in \Gamma(\mathfrak{D}_\lambda)$.

Proof. For M to be a proper $\mathcal{PR}$-semi-slant submanifold M of a para-Kähler manifold, we have

$$(3.8) \qquad g([X, Y], Z) = g(\nabla_X Y, Z) - g(\nabla_Y X, Z).$$

Now, using equations (2.2) and (2.6), we achieve that

$$(3.9) \qquad g([X, Y], Z) = -\overline{g}(\mathcal{P}\overline{\nabla}_X Y, \mathcal{P}Z) + \overline{g}(\mathcal{P}\overline{\nabla}_Y X, \mathcal{P}Z).$$

Employing equations (2.5), (2.8), (2.9), (3,2) and Gauss-Weingarten formula in equation (3.9), we obtain that

$$(3.10) \qquad g([X, Y], Z) = \lambda g([X, Y], Z) + \overline{g}(h(X, \mathcal{P}Y), nZ) - \overline{g}(h(Y, \mathcal{P}X), nZ).$$

We can rewrite above equation as follows;

$$(3.11) \qquad (1 - \lambda)g([X, Y], Z) = \overline{g}(h(X, \mathcal{P}Y), nZ) - \overline{g}(h(Y, \mathcal{P}X), nZ).$$

Since $\lambda = 1$ is impossible and X, Y, Z are all non-degenerate vector fields, hence from equation (3.11), we can easily deduce that the distribution $(\mathfrak{D}_T)$ is involutive if and only if $h(X, tY) = h(tX, Y)$. Furthermore, for totally geodesic M in $\overline{M}$, we obtain that $g(\nabla_X Y, Z) = \overline{g}(\overline{\nabla}_X Y, Z)$. Now in light of equations (2.2), (2.5), (2.9),

(3.2) and the fact that the distributions $(\mathfrak{D}_T, \mathfrak{D}_\lambda)$ are orthogonal, we derive the required condition for $(\mathfrak{D}_T)$ to be totally geodesic. This completes the proof of the theorem. $\qquad\square$

Theorem 3.8. *Let M be a proper $\mathcal{PR}$-semi-slant submanifold M of a para-Kähler manifold $\overline{M}$. Then the distribution $(\mathfrak{D}_\lambda)$*

(1) *is involutive if and only if $g(A_{nW}Z - A_{nZ}W, tX) = g(A_{ntZ}W - A_{ntW}Z, X)$;*

(2) *defines totally geodesic if and only if $A_{ntW}X = A_{nW}tX$;*

for any $Z, W \in \Gamma(\mathfrak{D}_\lambda)$ and $X, Y \in \Gamma(\mathfrak{D}_T)$.

Proof. The proof of this theorem can be obtained by following the steps of the Theorem 3.7. $\qquad\square$

4. $\mathcal{PR}$-semi-slant Warped Product Submanifolds

In this section, we investigate the existence or non-existence of non-trivial $\mathcal{PR}$-semi-slant warped product submanifolds of the forms $M_T \times_f M_\lambda$, $M_\lambda \times_f M_T$ in para-Kähler manifolds $\overline{M}$, where M_T and M_λ are invariant and slant submanifolds of $\overline{M}$, respectively.

Now, we derive the following major results.

Theorem 4.1. *Let $M \to \overline{M}$ be an isometric immersion of a proper $\mathcal{PR}$-semi-slant submanifold M into a para-Kähler manifold $\overline{M}$. Then there does not exist any non-trivial $\mathcal{PR}$-semi-slant warped product submanifold $M = M_T \times_f M_\lambda$ in $\overline{M}$.*

Proof. Let us assume that $M = M_T \times_f M_\lambda$ is a non-trivial $\mathcal{PR}$-semi-slant warped product submanifold in a para-Kähler manifold $\overline{M}$. Then by using equations (2.5) and (2.6), we obtain that

$$(4.1) \qquad\qquad \overline{\nabla}_Z \mathcal{P}X = \mathcal{P}\overline{\nabla}_Z X$$

for any non-degenerate vector fields $X \in \Gamma(M_T)$ and $Z \in \Gamma(M_\lambda)$. By putting together equations (2.9), (2.10) and Proposition 2.2 in (4.1), we get that

$$(4.2) \quad \nabla_Z tX + h(Z, tX) = X(\ln f)tZ + X(\ln f)nZ + t'h(X, Z) + n'h(X, Z).$$

By equating the tangential and normal parts of equation (4.2), we get

$$(4.3) \qquad\qquad tX(\ln f)Z = X(\ln f)tZ + t'h(X, Z),$$
$$(4.4) \qquad\qquad h(Z, tX) = X(\ln f)nZ + n'h(X, Z).$$

On the other hand, by virtue of Lemma 2.4 and the fact that $\Gamma(M_T)$ is a invariant submanifold, we have

$$(4.5) \qquad\qquad \overline{g}(h(X, Z), nZ) = \overline{g}(h(Z, Z), nX) = 0,$$

for all $X \in \Gamma(M_T)$ and $Z \in \Gamma(M_\lambda)$. Moreover, we can write from equations (2.3), (2.10) and (4.5) that

$$(4.6) \qquad g(h(X,Z), \mathcal{P}Z) = g(t'h(X,Z), Z) + g(n'h(X,Z), Z) = 0.$$

We attain from equation (4.6), that

$$g(t'h(X,Z), Z) = 0 \text{ implies } t'h(X,Z) = 0.$$

Now, from above expression and the fact that the distributions are orthogonal, we conclude that

$$(4.7) \qquad t'h(X,Z) = 0 \implies t'h(X,Z) \in \Gamma(M_T).$$

Next, by taking an inner product of tZ with equation(4.3), we compute that

$$(4.8) \qquad tX(\ln f)g(Z,tZ) = X(\ln f)g(tZ,tZ) + g(t'h(X,Z), tZ).$$

Then using equations (3.6) and (4.7) in (4.8), we achieve that

$$(4.9) \qquad tX(\ln f)g(Z,tZ) = -\lambda X(\ln f)g(Z,Z).$$

Furthermore, employing equations (2.2), (2.6) and (2.7), we get

$$(4.10) \qquad X(\ln f)g(Z,Z) = (tX, \nabla_Z tZ) - g(tX, A_{nZ}Z).$$

The above equation with the help of equations (2.8), (4.5) and Proposition 2.2(iii), becomes

$$(4.11) \qquad X(\ln f)g(Z,Z) = -g(tX, g(Z,tZ)grad(\ln f)) = -tX(\ln f)g(Z,tZ).$$

Now, from equations (4.9) and (4.11), we derive that

$$(4.12) \qquad X(\ln f)g(Z,Z) = \lambda X(\ln f)g(Z,Z),$$

for all non-degenerate vector fields $X \in \Gamma(M_T)$ and $Z \in \Gamma(M_\lambda)$. Therefore from equation (4.12), we conclude that either $\lambda = 1$ or f is a constant function on M_T. But $\lambda = 1$ is impossible since M is a proper $\mathcal{PR}$-semi-slant submanifold in $\overline{M}$. Hence f must be constant on M_T. Which is contradiction to our assumption. This completes the proof of the theorem. $\qquad \square$

Theorem 4.2. *Let M be an isometrically immersed $\mathcal{PR}$-semi-slant submanifold of a para-Kähler manifold $\overline{M}$. Then there doesn't exist a non-trivial $\mathcal{PR}$-semi-slant warped product submanifold $M = M_\lambda \times_f M_T$ in $\overline{M}$.*

Proof. Let us assume that $M = M_\lambda \times_f M_T$ be a non-trivial $\mathcal{PR}$-semi-slant warped product submanifold in a para-Kähler manifold $\overline{M}$. Then we can write from Proposition 2.2, Gauss formula and the connection property for $\overline{\nabla}$, that

$$(4.13) \qquad \overline{g}(\overline{\nabla}_X X, Z) = -g(X, \nabla_X Z) = -Z(\ln f)g(X,X),$$

for any $X \in \Gamma(M_T)$ and $Z \in \Gamma(M_\lambda)$. Also from equations (2.2), (2.6), (2.9) and the fact that structure is para-Kähler, we have that

$$(4.14) \qquad \overline{g}(\overline{\nabla}_X X, Z) = -g(\nabla_X \mathcal{P}X, tZ) - \overline{g}(h(X, \mathcal{P}X), nZ).$$

The above equation by the use of connection property for ∇ and Proposition 2.2, reduces to

$$(4.15) \qquad \overline{g}(\overline{\nabla}_X X, Z) = tZ(\ln f)g(\mathcal{P}X, X) - \overline{g}(h(X, \mathcal{P}X), nZ).$$

Now by the virtue of equations (4.13) and (4.15), we obtain for all $X \in \Gamma(M_T)$ and $Z \in \Gamma(M_\lambda)$ that

$$(4.16) \qquad -Z(\ln f)g(X, X) = tZ(\ln f)g(tX, X) - \overline{g}(h(X, tX), nZ).$$

On the other hand, using equations (2.5), (2.9), (2.10) and the fact that M_T is an invariant submanifold, we derive that

$$(4.17) \qquad h(Z, tX) = t'h(Z, X) + n'h(Z, X).$$

By comparing the tangential and normal parts from the above expression, we get

$$(4.18) \qquad h(Z, tX) = n'h(Z, X) \quad and \quad t'h(Z, X) = 0,$$

for all $X \in \Gamma(M_T)$ and $Z \in \Gamma(M_\lambda)$. Analogous to equation (4.17), we obtain by simplifying equation $\overline{\nabla}_X \mathcal{P}Z = \mathcal{P}\overline{\nabla}_X Z$ and using equation (4.18) that

$$(4.19) \qquad tZ(\ln f)X + h(X, tZ) - A_{nZ}X + \nabla^\perp_X nZ = Z(\ln f)tX + n'h(X, Z).$$

Now, equating the tangential parts of equation (4.19) and interchanging X by tX, we achieve that

$$(4.20) \qquad tZ(\ln f)tX - A_{nZ}tX = Z(\ln f)t^2 X$$

Taking inner product of equation (4.20) with X and then using equations (2.8), (2.11), we deduce that

$$(4.21) \qquad tZ(\ln f)g(X, tX) - \overline{g}(h(X, tX), nZ) = -Z(\ln f)g(tX, tX).$$

Employing equation (4.16) in equation (4.21), we find that

$$(4.22) \qquad Z(\ln f)g(X, X) = Z(\ln f)g(tX, tX).$$

Furthermore by applying Proposition 3.6 and the fact that M_T is an invariant submanifold in the above equation, we get

$$(4.23) \qquad Z(\ln f)g(X, X) = 0$$

for any non-degenerate vector fields $X \in \Gamma(M_T)$ and $Z \in \Gamma(M_\lambda)$. Now, from equation (4.23), we deduce that f must be constant on M_λ. Which is contradiction to our assumption that $M = M_\lambda \times_f M_T$ is a non-trivial $\mathcal{PR}$-semi-slant warped product submanifold in a para-Kähler manifold $\overline{M}$. Hence the proof of the theorem completed.   □

5. Example

In contrast to the results in Section 4, we hereby give a numerical example to illustrates the existence of a $\mathcal{PR}$-warped product submanifold M of the form $M_T \times_f M_\lambda$ with vanishing slant coefficient (i.e. $M_T \times_f M_\perp$) in a para-Kähler manifold.

Let $\overline{M} = \mathbb{R}^8$ be a 8-dimensional manifold with the standard Cartesian coordinates

$$(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4).$$

Define the para-Kähler pseudo-Riemannian metric structure $(\mathcal{P}, \overline{g})$ on $\overline{M}$ by

$$\mathcal{P}e_1 = e_5, \ \mathcal{P}e_2 = e_6, \ \mathcal{P}e_3 = e_7, \ \mathcal{P}e_4 = e_8, \ \mathcal{P}e_8 = e_4, \ \mathcal{P}e_7 = e_3, \ \mathcal{P}e_6 = e_2$$

$$(5.1) \qquad \mathcal{P}e_5 = e_1, \ \overline{g} = \sum_{i=1}^{4}(dx_i)^2 - \sum_{j=5}^{8}(dy_j)^2,$$

here $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ is a local orthonormal frame for $\Gamma(T\overline{M})$, given by $e_i = \dfrac{\partial}{\partial x_i}, \ e_j = \dfrac{\partial}{\partial y_j}$.

Let M be an isometrically immersed pseudo-Riemannian submanifold in a para-Kähler manifold $\overline{M}$ defined by

$$\Omega(u, v, \gamma, \beta) = \begin{pmatrix} u\cosh(\gamma), u\cosh(\beta), v\sinh(\gamma), v\sinh(\beta), \\ v\cosh(\gamma), v\cosh(\beta), u\sinh(\gamma), u\sinh(\beta) \end{pmatrix},$$

where $\gamma, \beta \in (0, \pi/2)$ and either $u < v \in \mathbb{R}_+$ or $v < u \in \mathbb{R}_-$. Then the tangent bundle $\Gamma(TM)$ of M is spanned by the vectors

$$\begin{aligned} X_u &= \cosh(\gamma)e_1 + \cosh(\beta)e_2 + \sinh(\gamma)e_7 + \sinh(\beta)e_8, \\ X_v &= \sinh(\gamma)e_3 + \sinh(\beta)e_4 + \cosh(\gamma)e_5 + \cosh(\beta)e_6, \\ (5.2) \qquad X_\gamma &= u\sinh(\gamma)e_1 + v\cosh(\gamma)e_3 + v\sinh(\gamma)e_5 + u\cosh(\gamma)e_7, \\ X_\beta &= u\sinh(\beta)e_2 + v\cosh(\beta)e_4 + v\sinh(\beta)e_6 + u\cosh(\beta)e_8, \end{aligned}$$

where X_u, X_γ, X_β are spacelike vector fields and X_v is a timelike vector field tangent to M. Now the space $\mathcal{P}(TM)$ with respect to the para-Kähler pseudo-Riemannian

metric structure $(\mathcal{P}, \overline{g})$ of $\overline{M}$ becomes

$$\mathcal{P}(X_u) = \sinh(\gamma)e_3 + \sinh(\beta)e_4 + \cosh(\gamma)e_5 + \cosh(\beta)e_6,$$
$$\mathcal{P}(X_v) = \cosh(\gamma)e_1 + \cosh(\beta)e_2 + \sinh(\gamma)e_7 + \sinh(\beta)e_8,$$
$$(5.3) \qquad \mathcal{P}(X_\gamma) = v\sinh(\gamma)e_1 + u\cosh(\gamma)e_3 + u\sinh(\gamma)e_5 + v\cosh(\gamma)e_7,$$
$$\mathcal{P}(X_\beta) = v\sinh(\beta)e_2 + u\cosh(\beta)e_4 + u\sinh(\beta)e_6 + v\cosh(\beta)e_8.$$

From equations (5.2) and (5.3), we obtain that $\mathcal{P}(X_\gamma)$ and $\mathcal{P}(X_\beta)$ are orthogonal to $\Gamma(TM)$, $\mathcal{P}(X_u)$ and $\mathcal{P}(X_v)$ are tangent to M. This means $\mathfrak{D}_T$ and $\mathfrak{D}_\lambda$ can be taken as a subspace span$\{X_u, X_v\}$ and a subspace span$\{X_\gamma, X_\beta\}$, respectively. Consequently, M becomes a $\mathcal{PR}$-semi-slant submanifold. Furthermore, we can say from Theorem 3.7 and Theorem 3.8 that $\mathfrak{D}_T$ and $\mathfrak{D}_\lambda$ are integrable. We denote the integral manifolds of $\mathfrak{D}_T$ and $\mathfrak{D}_\lambda$ by M_T and M_λ, respectively, then the induced pseudo-Riemannian metric tensor g of M is given by

$$[g(e_i, e_j)] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & v^2 - u^2 & 0 \\ 0 & 0 & 0 & v^2 - u^2 \end{bmatrix},$$

that is,

$$g = 2du^2 - 2dv^2 + (v^2 - u^2)\{d\gamma^2 + d\beta^2\} = g_{M_T} +_{(v^2-u^2)} g_{M_\lambda}.$$

Hence, M is a 4-dimensional $\mathcal{PR}$-semi-slant warped product submanifold in $\overline{M}$ with warping function $f = \sqrt{(v^2 - u^2)}$ and slant coefficient λ for $\mathfrak{D}_\lambda$ is 0. Thus, M is a non-trivial $\mathcal{PR}$-semi-slant warped product submanifold with improper slant coefficient in M (in particular, a $\mathcal{PR}$-warped product submanifold in a para-Kähler manifold [11]).

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