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The $\mathcal{W}^*$-curvature Tensor on Relativistic Space-times

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Abstract. This paper aims to study the $\mathcal{W}^*-$curvature tensor on relativistic space-times. The energy-momentum tensor T of a space-time having a semi-symmetric $\mathcal{W}^\star-$curvature tensor is semi-symmetric, whereas the whereas the energy-momentum tensor T of a space-time having a divergence free $\mathcal{W}^\star-$curvature tensor is of Codazzi type. A space-time having a traceless $\mathcal{W}^*-$curvature tensor is Einstein. A $\mathcal{W}^*-$curvature flat space-time is Einstein. Perfect fluid space-times which admits $\mathcal{W}^\star-$curvature tensor are considered.

1. Introduction

In [12, 13, 14, 15, 16], the authors introduced some curvature tensors similar to the projective curvature tensor of [9]. They investigated their geometrical properties and physical significance. These tensors have been recently studied in different ambient spaces [1, 4, 5, 18, 17, 20, 11]. However, we have noticed that little attention has been paid to the $\mathcal{W}_3^*-$curvature tensor. This tensor is a $(0,4)$ tensor defined as

$$\mathcal{W}_3^\star(U,V,Z,T) = R(U,V,Z,T) - \frac{1}{n-1}\left[g(V,Z)\operatorname{Ric}(U,T) - g(V,T)\operatorname{Ric}(U,Z)\right],$$

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where $R(U,V,Z,T) = g(R((U,V)Z,T)$, $R(U,V)Z = \nabla_U \nabla_V - \nabla_U \nabla_V - \nabla_{[U,V]} Z$ is the Riemann curvature tensor, ∇ is the Levi-Civita connection, and $\mathrm{Ric}(U,V)$ is the Ricci tensor. For simplicity, we will denote $\mathcal{W}_3^\star$ by $\mathcal{W}^\star$; in local coordinates, it is

$$\mathcal{W}^\star_{ijkl} = R_{ijkl} - \frac{1}{n-1}\left[g_{jk}R_{il} - g_{jl}R_{ik}\right]. \tag{1.1}$$

The $\mathcal{W}^*-$curvature tensor has neither symmetry nor cyclic properties.

A semi-Riemannian manifold M is *semi-symmetric* [19] if

$$R(\zeta,\xi)\cdot R = 0,$$

where $R(\zeta,\xi)$ acts as a derivation on R. M is *Ricci semi-symmetric* [8] if

$$R(\zeta,\xi)\cdot \mathrm{Ric} = 0,$$

where $R(\zeta,\xi)$ acts as a derivation on Ric. A semi-symmetric manifold is known to be Ricci semi-symmetric as well. The converse does not generally hold. Along the same line of the above definitions we say that M has a semi-symmetric $\mathcal{W}^\star-$curvature tensor if

$$R(\zeta,\xi)\cdot \mathcal{W}^\star = 0,$$

where $R(\zeta,\xi)$ acts as a derivation on $\mathcal{W}^\star$.

This study was designed to fill the above mentioned gap. The relativistic significance of the $\mathcal{W}^\star-$curvature tensor is investigated. First, it is shown that space-times with semi-symmetric $\mathcal{W}^\star_{jk} = g^{il}\mathcal{W}^\star_{ijkl}$ tensor have Ricci semi-symmetric tensor and consequently the energy-momentum tensor is semi-symmetric. The divergence of the $\mathcal{W}^\star-$curvature tensor is considered and it is proved that the energy-momentum tensor T of a space-time M is of Codazzi type if M has a divergence free $\mathcal{W}^\star-$curvature tensor. If M admits a parallel $\mathcal{W}^\star-$curvature tensor, then T is a parallel. Finally, a $\mathcal{W}^\star-$flat perfect fluid space-time performs as a cosmological constant. A dust fluid $\mathcal{W}^\star-$flat space-time satisfies Einstein's field equation is a vacuum space.

2. $\mathcal{W}^\star$-semi-symmetric Space-times

A 4$-$dimensional relativistic space-time M is said to *have a semi-symmetric $\mathcal{W}^\star$-curvature tensor* if

$$R(\zeta,\xi)\cdot \mathcal{W}^\star = 0,$$

where $R(\zeta,\xi)$ acts as a derivation on the tensor $\mathcal{W}^\star$. In local coordinates, one gets

$$\begin{aligned}(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\mathcal{W}^\star_{ijkl} = {} & (\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)R_{ijkl} - \frac{1}{3}[g_{jk}(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)R_{il} \\ & - g_{jl}(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)R_{ik}.\end{aligned} \tag{2.1}$$

Contracting both sides with g^{il} yields

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \mathcal{W}^\star_{jk} = \frac{4}{3} (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) R_{jk}, \tag{2.2}$$

where $\mathcal{W}^\star_{jk} = g^{il}\mathcal{W}^\star_{ijkl}$. Thus we have the following theorem.

Theorem 2.1. *M is Ricci semi-symmetric if and only if $\mathcal{W}^\star_{jk} = g^{il}\mathcal{W}^\star_{ijkl}$ is semi-symmetric.*

The following result is a direct consequence of this theorem.

Corollary 2.2. *M is Ricci semi-symmetric if the $\mathcal{W}^*$-curvature is semi-symmetric.*

A space-time manifold is *conformally semi-symmetric* if the conformal curvature tensor $\mathcal{C}$ is semi-symmetric.

Theorem 2.3. *Assume that M is a space-time admitting a semi-symmetric $\mathcal{W}^\star_{jk} = g^{il}\mathcal{W}^\star_{ijkl}$. Then, M is conformally semi-symmetric if and only if it is semi-symmetric i.e. $\nabla_{[\mu}\nabla_{\nu]}R_{ijkl} = 0 \Leftrightarrow \nabla_{[\mu}\nabla_{\nu]}\mathcal{C}_{ijkl} = 0$.*

The Einstein's field equation is

$$R_{ij} - \frac{1}{2} g_{ij} R + g_{ij}\Lambda = kT_{ij}, \tag{2.3}$$

where Λ, R, k are the cosmological constant, the scalar curvature, and the gravitational constant. Then

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) R_{ij} = k (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) T_{ij}, \tag{2.4}$$

i.e., M is Ricci semi-symmetric if and only if the energy-momentum tensor is semi-symmetric.

Theorem 2.4. *The energy-momentum tensor of a space-time M is semi-symmetric if and only if $\mathcal{W}^\star_{jk} = g^{il}\mathcal{W}^\star_{ijkl}$ is semi-symmetric.*

Remark 2.5. A space-time M with semi-symmetric energy-momentum tensor has been studied by De and Velimirovic in [2].

It is clear that $\nabla_\mu \mathcal{W}^\star_{ijkl} = 0$ implies $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \mathcal{W}^\star_{ijkl} = 0$. Thus the following result rises.

Corollary 2.6. *Let M be a space-time having a covariantly constant $\mathcal{W}^*-$curvature tensor. Then M is conformally semi-symmetric and the energy-momentum tensor is semi-symmetric.*

A space-time is called *Ricci recurrent* if the Ricci curvature tensor satisfies

$$\nabla_\mu R_{ij} = b_\mu R_{ij}, \tag{2.5}$$

where b is called the associated recurrence $1-$form. Assume that the Ricci tensor is recurrent, then

$$
\begin{aligned}
(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) R_{ij} &= \nabla_\mu (\nabla_\nu R_{ij}) - \nabla_\nu (\nabla_\mu R_{ij}) \\
&= \nabla_\mu (b_\nu R_{ij}) - \nabla_\nu (b_\mu R_{ij}) \\
&= (\nabla_\mu b_\nu) R_{ij} + b_\nu \nabla_\mu R_{ij} - (\nabla_\nu b_\mu) R_{ij} - b_\mu \nabla_\nu R_{ij} \\
&= [\nabla_\mu b_\nu - \nabla_\nu b_\mu] R_{ij}. \qquad (2.6)
\end{aligned}
$$

Corollary 2.7. *The following conditions on a space-time M are equivalent*

(1) *The Ricci tensor is recurrent with closed recurrence one form,*

(2) *T is semi-symmetric, and*

(3) *$\mathcal{W}^\star_{jk} = g^{il}\mathcal{W}^\star_{ijkl}$ is semi-symmetric.*

3. Space-times admitting Divergence Free $\mathcal{W}^\star$-curvature Tensor

The tensor $\mathcal{W}^{\star h}_{jkl}$ of type $(1,3)$ is given by

$$
\begin{aligned}
\mathcal{W}^{\star h}_{jkl} &= g^{hi}\mathcal{W}^\star_{ijkl} \\
&= R^h_{jkl} - \frac{1}{3}[g_{jk}R^h_l - g_{jl}R^h_k].
\end{aligned}
$$

Consequently, one defines its divergence as

$$
\begin{aligned}
\nabla_h \mathcal{W}^{\star h}_{jkl} &= \nabla_h R^h_{jkl} - \frac{1}{3}[g_{jk}\nabla_h R^h_l - g_{jl}\nabla_h R^h_k] \\
&= \nabla_h R^h_{jkl} - \frac{1}{3}[g_{jk}\nabla_l R - g_{jl}\nabla_k R]. \qquad (3.1)
\end{aligned}
$$

It is well known that the contraction of the second Bianchi identity gives

$$\nabla_h R^h_{jkl} = \nabla_l R_{jk} - \nabla_k R_{jl}.$$

Thus, equation (3.1) becomes

$$\nabla_h \mathcal{W}^{\star h}_{jkl} = \nabla_l R_{jk} - \nabla_k R_{jl} - \frac{1}{3}[g_{jk}\nabla_l R - g_{jl}\nabla_k R]. \qquad (3.2)$$

If the $\mathcal{W}^\star-$curvature tensor is divergence free, then equation (3.2) turns into

$$0 = \nabla_l R_{jk} - \nabla_k R_{jl} - \frac{1}{3}[g_{jk}\nabla_l R - g_{jl}\nabla_k R].$$

Multiplying by g^{jk} we have

$$\nabla_l R = 0. \qquad (3.3)$$

Thus, the tensor R_{ij} is a Codazzi tensor and R is constant. Conversely, assume that the Ricci tensor is a Codazzi tensor. Then

$$\nabla_h \mathcal{W}^{\star h}_{jkl} = -\frac{1}{3}[g_{jk}\nabla_l R - g_{jl}\nabla_k R]$$
$$0 = \nabla_l R_{jk} - \nabla_k R_{jl}$$

However, the last equation implies that $\nabla_l R = 0$. Consequently, the $\mathcal{W}^\star-$curvature tensor has zero divergence.

Theorem 3.1. *The $\mathcal{W}^\star-$curvature tensor has zero divergence if and only if the Ricci tensor is a Codazzi tensor. In both cases, the scalar curvature is constant.*

The divergence of the Weyl curvature $\mathcal{C}$ tensor is given by

$$\nabla_h \mathcal{C}^h_{ijk} = \frac{n-3}{n-2}\left[\nabla_k R_{ij} - \nabla_j R_{ik}\right] + \frac{1}{2\,(n-1)}[g_{ij}\nabla_k R - g_{ik}\nabla_j R].$$

Remark 3.2. Since divergence free of $\mathcal{W}^\star-$curvature tensor implies that R_{ij} is a Codazzi tensor, the conformal curvature tensor has zero divergence.

Equation (2.3) yields

$$\nabla_l R_{ij} - \frac{1}{2} g_{ij}\nabla_l R = k\nabla_l T_{ij}.$$

The above theorem now implies the following result.

Corollary 3.3. *The energy-momentum tensor is a Codazzi tensor if and only if the $\mathcal{W}^\star-$curvature tensor has zero divergence. In both cases, the scalar curvature is constant.*

Einstein's field equation infers

$$\begin{aligned} k\left(\nabla_l T_{ij} - \nabla_i T_{jl}\right) &= \nabla_l\left(R_{ij} - \frac{1}{2}g_{ij}R\right) - \nabla_i\left(R_{lj} - \frac{1}{2}g_{lj}R\right) \\ &= \nabla_l R_{ij} - \nabla_i R_{lj} - \frac{1}{2}\left(g_{ij}\nabla_l R - g_{lj}\nabla_i R\right) \\ &= \nabla_h \mathcal{W}^{\star h}_{jil} - \frac{1}{6}\left(g_{ij}\nabla_l R - g_{lj}\nabla_i R\right). \end{aligned} \tag{3.4}$$

Now, it is noted that the above theorem may be proved using this identity.

4. $\mathcal{W}^\star$-symmetric Space-times

A space-time M is called $\mathcal{W}^\star$-*symmetric* if

$$\nabla_m \mathcal{W}^\star_{ijkl} = 0.$$

Applying the covariant derivative on the both sides of equation (1.1), one gets

$$\nabla_m \mathcal{W}^{\star}_{ijkl} = \nabla_m R_{ijkl} - \frac{1}{n-1}\left[g_{jk}\nabla_m R_{il} - g_{jl}\nabla_m R_{ik}\right]. \tag{4.1}$$

If M is a $\mathcal{W}^{\star}-$symmetric space-time, then

$$\nabla_m R_{ijkl} = \frac{1}{3}[g_{jk}\nabla_m R_{il} - g_{jl}\nabla_m R_{ik}].$$

Multiplying the both sides by g^{il}, we get

$$\nabla_m R_{jk} = \frac{1}{3}[g_{jk}\nabla_m R - \nabla_m R_{jk}],$$

and hence

$$\nabla_m R_{jk} = \frac{1}{4} g_{jk}\nabla_m R. \tag{4.2}$$

Now, the following theorem rises.

Theorem 4.1. *Assume that M is a $\mathcal{W}^{\star}-$symmetric space-time, then M is a Ricci symmetric if the scalar curvature is constant.*

The second Bianchi identity for $\mathcal{W}^{\star}-$curvature tensor is

$$\begin{aligned} &\nabla_m \mathcal{W}^{\star}_{ijkl} + \nabla_k \mathcal{W}^{\star}_{ijlm} + \nabla_l \mathcal{W}^{\star}_{ijmk} \\ &= -\frac{1}{3}\left[g_{jk}(\nabla_m R_{il} - \nabla_l R_{im}) + g_{jl}(\nabla_k R_{im} - \nabla_m R_{ik})\right] \\ &\quad - \frac{1}{3} g_{jm}(\nabla_l R_{ik} - \nabla_k R_{il}). \end{aligned} \tag{4.3}$$

If the Ricci tensor satisfies $\nabla_m R_{il} = \nabla_l R_{im}$, then

$$\nabla_m \mathcal{W}^{\star}_{ijkl} + \nabla_k \mathcal{W}^{\star}_{ijlm} + \nabla_l \mathcal{W}^{\star}_{ijmk} = 0. \tag{4.4}$$

Conversely, if the above equation holds, then equation (4.3) implies

$$g_{jk}(\nabla_m R_{il} - \nabla_l R_{im}) + g_{jl}(\nabla_k R_{im} - \nabla_m R_{ik}) + g_{jm}(\nabla_l R_{ik} - \nabla_k R_{il}) = 0. \tag{4.5}$$

Multiplying the both sides with g^{ik}, then we have

$$\nabla_m R_{jl} = \nabla_l R_{jm}, \tag{4.6}$$

which means that the Ricci tensor is of Codazzi type.

Theorem 4.2. *The Ricci tensor satisfies $\nabla_m R_{il} = \nabla_l R_{im}$ if and only if the $\mathcal{W}^{\star}-$curvature tensor satisfies equation (4.4).*

For a purely electro-magnetic distribution, Eequation (2.3) reduces to

$$R_{ij} = kT_{ij}. \tag{4.7}$$

Its contraction with g^{ij} gives

$$R = -kT. \tag{4.8}$$

In this case, it is $T = R = 0$. Thus equation (4.2) yields $\nabla_m T_{jk} = 0$.

Theorem 4.3. *The energy-momentum tensor of a $\mathcal{W}^\star-$symmetric space-time obeying Einstein's field equation for a purely electro-magnetic distribution is locally symmetric.*

5. $\mathcal{W}^\star$-flat Space-times

Now, we consider $\mathcal{W}^\star-$flat space-times. Multiplying both sides of equation (1.1) by g^{il} yields

$$\begin{aligned}\mathcal{W}^\star_{jk} &= g^{il}\mathcal{W}^\star_{ijkl} \\ &= \frac{4}{3}\left(R_{jk} - \frac{R}{4}g_{jk}\right).\end{aligned}$$

Thus, a $\mathcal{W}^\star_{jk}-$curvature flat space-time is Einstein, i.e.,

$$R_{jk} = \frac{R}{4}g_{jk}. \tag{5.1}$$

Now, equation (1.1) becomes

$$\mathcal{W}^\star_{ijkl} = R_{ijkl} - \frac{R}{12}[g_{ik}g_{jl} - g_{jl}g_{jk}].$$

Theorem 5.1. *A space-time manifold M is Einstein if and only if $\mathcal{W}^\star_{jk} = 0$. Moreover, a $\mathcal{W}^*-$flat space-time has a constant curvature.*

A vector field ξ is said to be a *conformal vector field* if

$$\mathcal{L}_\xi g = 2\phi g,$$

where $\mathcal{L}_\xi$ denotes the Lie derivative along the flow lines of ξ and ϕ is a scalar. ξ is called *Killing* if $\phi = 0$. Let T_{ij} be the energy-momentum tensor defined on M. ξ is said to be a *matter inheritance collineation* if

$$\mathcal{L}_\xi T = 2\phi T.$$

The tensor T_{ij} is said to have a symmetry inheritance property along the flow lines of ξ. ξ is called a *matter collineation* if $\phi = 0$. A Killing vector field ξ is a matter collineation. However, a matter collineation is not generally Killing.

Theorem 5.2. *Assume that M is a $\mathcal{W}^{\star}-$flat space-time. Then, ξ is conformal if and only if $\mathcal{L}_{\xi}T = 2\phi T$.*

Proof. Using equations (5.1) and (2.3), we have

$$\left(\Lambda - \frac{R}{4}\right) g_{ij} = kT_{ij}. \tag{5.2}$$

Then

$$\left(\Lambda - \frac{R}{4}\right) \mathcal{L}_{\xi} g = k\mathcal{L}_{\xi}T. \tag{5.3}$$

Assume that ξ is conformal. The above two equations lead to

$$2\phi\left(\Lambda - \frac{R}{4}\right) g = k\mathcal{L}_{\xi}T$$
$$2\phi T = \mathcal{L}_{\xi}T.$$

Conversely, suppose that the energy-momentum tensor has a symmetry inheritance property along ξ. It is easy to show that ξ is a conformal vector field. □

Corollary 5.3. *Assume that M is a $\mathcal{W}^{\star}-$flat space-time. Then, M admits a matter collineation ξ if and only if ξ is Killing.*

Equations (5.1) and (2.3) imply

$$\left(\Lambda - \frac{R}{4}\right) g_{ij} = kT_{ij}. \tag{5.4}$$

Taking the covariant derivative of 5.4 we get

$$\nabla_l T_{ij} = \frac{1}{k}\nabla_l \left(\Lambda - \frac{R}{4}\right) g_{ij}. \tag{5.5}$$

Since a $\mathcal{W}^{\star}-$curvature flat space-time has $\nabla_l R = 0$, $\nabla_l T_{ij} = 0$.

Theorem 5.4. *The energy-momentum tensor of a $\mathcal{W}^{\star}-$flat space-time is covariantly constant.*

Let M be a space-time and $\mathcal{W}^{*i}_{klm} = g^{ij}\mathcal{W}^{*}_{jklm}$ be a $(1,3)$ curvature tensor. According to [3], there exists a unique traceless tensor $\mathcal{B}^{i}_{klm}$ and three unique $(0,2)$ tensors $\mathcal{C}_{kl}$, $\mathcal{D}_{kl}$, $\mathcal{E}_{kl}$ such that

$$\mathcal{W}^{*i}_{klm} = \mathcal{B}^{i}_{klm} + \delta^{i}_{k}\mathcal{C}_{lm} + \delta^{i}_{l}\mathcal{D}_{km} + \delta^{i}_{m}\mathcal{E}_{kl}.$$

All of these tensors are given by

$$\mathcal{C}_{ml} = \frac{1}{33}\left[10\mathcal{W}^{*t}_{tml} - 2\left(\mathcal{W}^{*t}_{mtl} + \mathcal{W}^{*t}_{lmt}\right)\right] = 0,$$

$$\begin{aligned}\mathcal{D}_{km} &= \frac{1}{33}\left[-2\left(\mathcal{W}^{*t}_{tkm}+\mathcal{W}^{*t}_{mkt}\right)+10\mathcal{W}^{*t}_{ktm}\right]\\ &= \frac{1}{9}[R_{km}-\frac{g_{km}}{4}R],\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_{kl} &= \frac{1}{33}\left[10\mathcal{W}^{*t}_{klt}-2\left(\mathcal{W}^{*t}_{tlk}+\mathcal{W}^{*t}_{ltk}\right)\right]\\ &= \frac{-1}{9}\left[R_{kl}-\frac{g_{kl}}{4}R\right].\end{aligned}$$

Assume that the $\mathcal{W}^*-$curvature tensor is traceless. Then

$$\mathcal{C}_{kl}=\mathcal{D}_{kl}=\mathcal{E}_{kl}=0,$$

and consequently

$$R_{ml}=\frac{g_{ml}}{4}R.$$

Theorem 5.5. *Assume that M is a space-time admitting a traceless $\mathcal{W}^*-$curvature tensor. Then, M is an Einstein space-time.*

For a perfect fluid space-time with the energy density μ and isotropic pressure p, we have

$$T_{ij}=(\mu+p)\,u_iu_j+pg_{ij}, \tag{5.6}$$

where u_i is the velocity of the fluid flow with $g_{ij}u^j=u_i$ and $u_iu^i=-1$ [10, 6, 7]. In [2, Theorem 2.2], a characterization of such space-times is given. This result leads us to the following.

Theorem 5.6. *Assume that the perfect fluid space-time M is $\mathcal{W}^\star-$semi-symmetric. Then, M is regarded as inflation and this fluid acts as a cosmological constant. Moreover, the perfect fluid represents the quintessence barrier.*

Using Equations (5.2), we have

$$\left(\Lambda-kp-\frac{R}{4}\right)g_{ij}=k\,(\mu+p)\,u_iu_j. \tag{5.7}$$

Multiplying the both sides by g^{ij} we get

$$R=4\Lambda+k\,(\mu-3p)\,. \tag{5.8}$$

For $\mathcal{W}^\star-$curvature flat space-times, the scalar curvature is constant and consequently

$$\mu-3p=\text{constant}. \tag{5.9}$$

Again, a contraction of equation (5.7) with u^i leads to

$$R = 4\left(k\mu + \Lambda\right). \tag{5.10}$$

The comparison between (5.8) and (5.10) gives

$$\mu + p = 0, \tag{5.11}$$

i.e., the perfect fluid performs as a cosmological constant. Then equation (5.6) implies

$$T_{ij} = pg_{ij}. \tag{5.12}$$

For a $\mathcal{W}^{\star}-$flat space-time, the scalar curvature is constant. Thus $\mu =$ constant and consequently $p =$ constant. Therefore, the covariant derivative of equation (5.12) implies $\nabla_l T_{ij} = 0$.

Theorem 5.7. *Let M be a perfect fluid $\mathcal{W}^{\star}-$flat space-time obeying equation (2.3), then the μ and p are constants and $\mu + p = 0$ i.e. the perfect fluid performs as a cosmological constant. Moreover, $\nabla_l T_{ij} = 0$.*

The following results are two direct consequences of being $\mathcal{W}^{\star}-$curvature flat.

Corollary 5.8. *A $\mathcal{W}^{\star}-$flat space-time M obeying equation (4.7) is a Euclidean space.*

Corollary 5.9. *Let M be a dust fluid $\mathcal{W}^{\star}-$flat space-time satisfying equation (2.3) (i.e. $T_{ij} = \mu u_i u_j$). Then M is a vacuum space-time(i.e. $T_{ij} = 0$).*

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