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The Critical Point Equation on 3-dimensional α-cosymplectic Manifolds

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Abstract. The object of the present paper is to study the critical point equation (CPE) on 3-dimensional α-cosymplectic manifolds. We prove that if a 3-dimensional connected α-cosymplectic manifold satisfies the Miao-Tam critical point equation, then the manifold is of constant sectional curvature $-\alpha^2$, provided $D\lambda \neq (\xi\lambda)\xi$. We also give several interesting corollaries of the main result.

1. Introduction

In [4], Miao-Tam studied the volume functional on the space of constant scalar curvature metrics with a given boundary metric. They derived a necessary and sufficient condition for a metric to be a critical point as follows:

On a compact Riemannian manifold (M^n, g), $n \geq 3$, with smooth boundary, if there exists a non-zero smooth function $\lambda : M^n \to \mathbb{R}$ (called potential function) such that

$$Hess\lambda - (\Delta\lambda)g - \lambda S = g \quad on \quad M^n \tag{1.1}$$

and $\lambda = 0$ on ∂M^n, where Δ is the Laplacian operator, Hess is the Hessian operator and S is the Ricci tensor with respect to the metric g, then g is said to satisfy the Miao-Tam critical condition.

In particular, if the potential function λ is a non-zero constant, then (1.1) is just

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an Einstein metric. Recently, Hwang [2] proved that the CPE conjecture is also true under certain condition on the bounds of the potential function λ. In 2017, Wang [8] proved that if the metric of a 3-dimensional $(k,\mu)'$-almost Kenmotsu manifold satisfies the Miao-Tam critical condition, then the manifold is locally isometric either to the hyperbolic space $\mathbb{H}^3(-1)$ or to the Riemannian product $\mathbb{H}^2(-4)\times\mathbb{R}$. In [7], Ghosh and Patra considered the CPE in the framework of K-contact manifolds and (k,μ)-contact manifolds.

Motivated by the above studies, in the present paper we study 3-dimensional α-cosymplectic manifolds admitting CPE, i.e. satisfying the relation (1.1). The paper is organized as follows. In section 2, we recall the definition of α-cosymplectic manifolds and some basic formulas and section 3 is devoted to prove our main result, precisely:

Theorem 1.1. *If a 3-dimensional connected α-cosymplectic manifold $(M,\phi,\xi,\eta,g,\alpha)$ satisfies CPE, then the manifold is of constant sectional curvature $-\alpha^2$, provided $D\lambda \neq (\xi\lambda)\xi$, where D denotes the gradient operator with respect to g.*

2. Preliminaries

An almost contact metric structure on a $(2n+1)$-dimensional smooth manifold M consists of a 1-form η, a vector field ξ (called the Reeb field), a $(1,1)$-tensor field ϕ and a Riemannian metric g satisfying the following conditions:

$$\eta(\xi)=1, \qquad \phi^2=-I+\eta\otimes\xi \tag{2.1}$$

and

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{2.2}$$

for any X, $Y\in\chi(M)$. The above relations imply

$$\eta\circ\phi=0, \qquad \phi\xi=0, \tag{2.3}$$

$$g(X,\phi Y)=-g(\phi X,Y), \qquad g(X,\xi)=\eta(X), \tag{2.4}$$

for any X, $Y\in\chi(M)$.

An almost contact metric structure is said to be *normal* if the induced almost complex structure J on the product manifold $M\times\mathbb{R}$ defined by

$$J(X,f\frac{d}{dt}):=(\phi X-f\xi,\eta(X)\frac{d}{dt})$$

is integrable, where $X\in\chi(M)$, t is the coordinate on $\mathbb{R}$ and f is a smooth function on $M\times\mathbb{R}$.

An almost contact metric structure is said to be *a contact metric structure* if

$$g(X,\phi Y)=d\eta(X,Y), \tag{2.5}$$

for any X, $Y \in \chi(M)$. In this case, the 1-form η is called the *contact metric form*. We define a $(1,1)$-tensor field h by $h := \frac{1}{2}\pounds_{\xi}\phi$, where $\pounds$ denotes the Lie derivative in the direction of the vector field ξ. It is symmetric and satisfies $h\phi = -\phi h$. Also, we have $Tr.h = Tr.\phi h = 0$, $h\xi = 0$ and

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2.6}$$

for any $X \in \chi(M)$, where ∇ is the Levi-Civita connection of g.

An almost contact metric manifold is called *Kenmotsu* if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.7}$$

for any X, $Y \in \chi(M)$.

In 2005, Kim and Pak [3] introduced the notion of *almost α-cosymplectic manifold*, which is an almost contact metric manifold that satisfies $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, for α a real number. Recently, Erken [1] and Öztürk et. al [5, 6] obtained some fundamental properties of almost α-cosymplectic manifolds. *An α-cosymplectic manifold* is a normal almost α-cosymplectic manifold. An α-cosymplectic manifold with $\alpha = 0$ is a cosymplectic manifold and with $\alpha = 1$, it is a Kenmotsu manifold.

On a $(2n+1)$-dimensional α-cosymplectic manifold M, for any X, $Y \in \chi(M)$, the following relations hold:

$$\nabla_X \xi = \alpha[X - \eta(X)\xi], \tag{2.8}$$

$$R(X,Y)\xi = \alpha^2[\eta(X)Y - \eta(Y)X], \tag{2.9}$$

$$S(X,\xi) = -2n\alpha^2\eta(X), \tag{2.10}$$

$$Q\xi = -2n\alpha^2\xi. \tag{2.11}$$

3. Proof of the Main Theorem

Before proving our main result, we recall the following lemma, given by Miao-Tam.

Lemma 3.1.([4, Theorem 7]) *If the metric of a connected Riemannian manifold satisfies the Miao-Tam critical condition, then the scalar curvature is constant.*

It is known that the Riemannian curvature tensor of a 3-dimensional Riemannian manifold (M,g) is given by:

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &\quad -\frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{3.1}$$

for any X, Y, $Z \in \chi(M)$, where S is the Ricci tensor, Q is the Ricci operator and r is the scalar curvature.

Assume that $(M, \phi, \xi, \eta, g, \alpha)$ is a 3-dimensional connected α-cosymplectic manifold which satisfies the Miao-Tam critical condition, i.e. which satisfies (1.1). Putting $Y = Z = \xi$ in (3.1) and using (2.9), (2.10) and (2.11), the Ricci operator can be written as

$$QX = (\alpha^2 + \frac{r}{2})X - (3\alpha^2 + \frac{r}{2})\eta(X)\xi, \tag{3.2}$$

for any $X \in \chi(M)$.

Taking covariant derivative of the above equation with respect to Y and using (2.7) and Lemma 3.1, we obtain

$$(\nabla_Y Q)X = -\alpha(3\alpha^2 + \frac{r}{2})\big[g(X,Y)\xi + \eta(X)Y - 2\eta(X)\eta(Y)\xi\big], \tag{3.3}$$

for any X, $Y \in \chi(M)$.

Taking trace of the equation (1.1), we have

$$\Delta\lambda = -\frac{1}{2}(r\lambda + 3). \tag{3.4}$$

Using (3.4) in (1.1), we obtain

$$\nabla_X D\lambda = \lambda QX + fX, \quad \textit{where } f = -\frac{1}{2}(r\lambda + 1), \tag{3.5}$$

for any $X \in \chi(M)$, where D denotes the gradient operator with respect to g.

Taking the covariant derivative of (3.5) with respect to Y, we get

$$\nabla_Y \nabla_X D\lambda = (Y\lambda)QX + \lambda\nabla_Y QX + (Yf)X + f\nabla_Y X, \tag{3.6}$$

for any X, $Y \in \chi(M)$.

Similarly, we get

$$\nabla_X \nabla_Y D\lambda = (X\lambda)QY + \lambda\nabla_X QY + (Xf)Y + f\nabla_X Y. \tag{3.7}$$

Also

$$\nabla_{[X,Y]} D\lambda = \lambda Q[X,Y] + f[X,Y], \tag{3.8}$$

and using (3.6), (3.7) and (3.8) we have

$$\begin{aligned} R(X,Y)D\lambda &= \nabla_X \nabla_Y D\lambda - \nabla_Y \nabla_X D\lambda - \nabla_{[X,Y]} D\lambda \\ &= (X\lambda)QY - (Y\lambda)QX + \lambda\big[(\nabla_X Q)Y - (\nabla_Y Q)X\big] \\ &\quad +(Xf)Y - (Yf)X. \end{aligned} \tag{3.9}$$

In view of (3.3) and (3.9) yields

$$R(X,Y)D\lambda = (X\lambda)QY - (Y\lambda)QX + \lambda\alpha(3\alpha^2 + \frac{r}{2})\big[\eta(X)Y - \eta(Y)X\big] + (Xf)Y - (Yf)X, \tag{3.10}$$

for any X, $Y \in \chi(M)$.

By setting $X = \xi$ in the above equation and using (2.10) and (3.2) we get

$$R(\xi,Y)D\lambda = (\xi\lambda)\big[(\alpha^2 + \frac{r}{2})Y - (3\alpha^2 + \frac{r}{2})\eta(Y)\xi\big] + 2\alpha^2(Y\lambda)\xi + \lambda\alpha(3\alpha^2 + \frac{r}{2})[Y - \eta(Y)\xi] + (\xi f)Y - (Yf)\xi. \tag{3.11}$$

Taking inner product with ξ in the above equation, we easily compute

$$g(R(\xi,Y)\xi, D\lambda) = 2\alpha^2[(\xi\lambda)\eta(Y) - (Y\lambda)] - (\xi f)\eta(Y) + (Yf). \tag{3.12}$$

On the other hand, from (2.9) we have

$$g(R(\xi,Y)\xi, D\lambda) = \alpha^2[g(Y, D\lambda) - \eta(Y)\eta(D\lambda)]. \tag{3.13}$$

Making use of (3.12) and (3.13) we get

$$\alpha^2[g(Y,D\lambda) - \eta(Y)\eta(D\lambda)] = 2\alpha^2[(\xi\lambda)\eta(Y) - (Y\lambda)] - (\xi f)\eta(Y) + (Yf). \tag{3.14}$$

Removing Y from both sides in the above equation, we obtain

$$2\alpha^2(\xi\lambda)\xi - (\xi f)\xi + Df = \alpha^2[3D\lambda - \eta(D\lambda)\xi]. \tag{3.15}$$

From $f = -\frac{1}{2}(r\lambda + 1)$, we get

$$Df = -\frac{r}{2}(D\lambda) \ \ and \ \ \xi f = -\frac{r}{2}(\xi\lambda). \tag{3.16}$$

Using the above relations in (3.15) we easily have

$$(3\alpha^2 + \frac{r}{2})[(\xi\lambda)\xi - D\lambda] = 0. \tag{3.17}$$

If $D\lambda = (\xi\lambda)\xi$, then taking the covariant derivative with respect to X and using (3.5) we obtain

$$\lambda QX - \frac{1}{2}(r\lambda + 1)X = X(\xi\lambda)\xi + \alpha(\xi\lambda)[X - \eta(X)\xi].$$

Then taking trace we get

$$\xi(\xi\lambda) = -2\alpha(\xi\lambda) - \frac{1}{2}(r\lambda + 3). \tag{3.18}$$

From (1.1) we get

$$\nabla_X D\lambda = (\Delta\lambda)X + \lambda QX + X,$$

hence, for $X = \xi$, together with (3.4) and (2.11) imply

$$\nabla_\xi D\lambda = -\frac{1}{2}(4\alpha^2\lambda + r\lambda + 1)\xi.$$

Since $\nabla_\xi D\lambda = \xi(\xi\lambda)\xi$, we deduce that

$$\xi(\xi\lambda) = -\frac{1}{2}(4\alpha^2\lambda + r\lambda + 1)$$

which together with (3.18) imply

$$\alpha^2\lambda - \alpha(\xi\lambda) - \frac{1}{2} = 0.$$

Then we can state:

Proposition 3.1. *If a 3-dimensional connected α-cosymplectic manifold $(M, \phi, \xi, \eta, g, \alpha)$ satisfies CPE and $r \neq -6\alpha^2$, then the gradient of λ is collinear with ξ. Moreover*

$$(\xi\lambda) = \alpha\lambda - \frac{1}{2\alpha}.$$

Corollary 3.1. *If a 3-dimensional connected α-cosymplectic manifold $(M, \phi, \xi, \eta, g, \alpha)$ satisfies CPE and $r \neq -6\alpha^2$, then it can not be a cosymplectic manifold.*

If $3\alpha^2 + \frac{r}{2} = 0$, then $r = -6\alpha^2$. Putting the value of $r = -6\alpha^2$ in (3.1) and in view of (3.2), we find that manifold is of constant sectional curvature $-\alpha^2$.

Hence we can state the following:

Theorem 3.1. *If a 3-dimensional connected α-cosymplectic manifold $(M, \phi, \xi, \eta, g, \alpha)$ satisfies CPE, then the manifold is of constant sectional curvature $-\alpha^2$, provided $D\lambda \neq (\xi\lambda)\xi$, where D denotes the gradient operator with respect to g.*

If $\alpha = 0$, then the manifold is a cosymplectic manifold and we have the following:

Corollary 3.2. *If a 3-dimensional connected cosymplectic manifold (M, ϕ, ξ, η, g) satisfies CPE, then the manifold is flat, provided $D\lambda \neq (\xi\lambda)\xi$.*

If $\alpha = 1$, then the manifold is a Kenmotsu manifold and we have the following:

Corollary 3.3. *If a 3-dimensional connected Kenmotsu manifold (M, ϕ, ξ, η, g) satisfies CPE, then the manifold is locally isometric to the hyperbolic space $H^3(-1)$, provided $D\lambda \neq (\xi\lambda)\xi$.*

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