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Existence and Behavior Results for a Nonlocal Nonlinear Parabolic Equation with Variable Exponent

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Abstract. In this article, we study the solvability of the Cauchy-Dirichlet problem for a class of nonlinear parabolic equations with nonstandard growth and nonlocal terms. We prove the existence of weak solutions of the considered problem under more general conditions. In addition, we investigate the behavior of the solution when the problem is homogeneous.

1. Introduction

This paper deals with the existence and behavior of the solution of a nonlinear parabolic Dirichlet-type boundary value problem whose model example is the following:

$$(1.1)\quad \begin{cases} \frac{\partial u}{\partial t} - \sum\limits_{i=1}^{n} D_i \left(|u|^{p_0-2} D_i u \right) + a\left(x,t,u\right) + g\left(x,t\right) \|u\|^s_{L^p(\Omega)}\left(t\right) = h\left(x,t\right), \\ u\left(x,0\right) = 0 = u_0\left(x\right), \quad u\mid_{\Gamma_T} = 0 \end{cases}$$

where $(x,t) \in Q_T := \Omega \times (0,T)$, $T > 0$, $\Gamma_T := \partial\Omega \times [0,T]$, $\Omega \subset \mathbb{R}^n \ (n \geq 3)$ is a bounded domain with sufficiently smooth boundary (at least Lipschitz), $D_i \equiv \partial/\partial x_i$ and $p_0 \geq 2, p, s \geq 1$ and $a : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$, $a\left(x,t,\tau\right)$ is a function with variable nonlinearity in τ, (for example, $a\left(x,t,\tau\right) = a_0\left(x,t\right)|\tau|^{\alpha(x,t)-1} + a_1\left(x,t\right)$) and g is a real valued measurable function which is not zero on the cylinder Q_T.

Recently, nonlinear parabolic equations with nonlocal terms have been well

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studied ([2, 4, 8, 9, 10, 11, 12, 13, 16]). Here, "nonlocal term" denotes a functional depending on the unknown function. There are numerous nonlocal mathematical models studied by many authors to express processes in physics and engineering. For example, Galaktionov and Levine [18] presented a general approach to critical Fujita exponents for nonlinear parabolic problems with nonlocal nonlinearities. Pao [24] considered a nonlocal model obtained from combustion theory. The degenerate parabolic equations with a nonlocal term which appear in a population dynamics model that communicates through chemical means, were studied in [4, 12, 17].

The equation in (1.1) is nonlinear with respect to the solution, and for the case $p_0 = 2$, this equation is a nonlocal reaction-diffusion equation which describes an ignition model for a compressible reactive gas (see, [4, 7]). In this case the existence, uniqueness and blow-up of nonnegative solutions to problem (1.1) have been studied in [23, 27, 34, 35]. Models similar to (1.1) also arise in biology to describe the density of some biological species. In such models the nonlocal term and the absorption term cooperate and communicate during the diffusion.

Boundary-value problems of type (1.1) are a case of the Newtonian filtration equation which can be given in the following general form:

$$u_t = \Delta\varphi(u) + f.$$

Equation (1.1) is a parabolic equation with implicit degeneracy which is similar to the equation of Newtonian polytropic filtration [15, 19, 21, 22] i.e.

$$u_t = \Delta\left(|u|^{m-1} u\right) + f,$$

where $m > 1$. This equation is parabolic for u different from 0 and degenerates when $u = 0$. Under the condition $m > 1$, the above equation describes the non-stationary flow of a compressible Newtonian fluid in a porous medium under polytropic conditions.

Over the past decade, there has been an increasing interest in the study of degenerate parabolic equations that involve variable exponents [3, 5, 6]. In this paper, we investigate the parabolic equation with such an additional term f, together with variable nonlinearity and nonlocal terms. If we rearrange the main part of the equation, we arrive at

$$u_t = \Delta\left(|u|^{p_0-2} u\right) + F\left(x, t, u, \|u\|_{L^p(\Omega)}, h\right).$$

To the best of our knowledge, there have not been any studies on the existence of solutions for the parabolic equations of type (1.1) providing a function F whose argument depends on nonlinear nonlocal term $\|u\|^s_{L^p(\Omega)}(t)$ and a separate $|u|$ with variable nonlinearity. We stress that the nonlinearity of nonlocal term $g(x,t)\|u\|^s_{L^p(\Omega)}(t)$ is independent from the local nonlinearity. This causes some difficulties in studying the uniqueness and behavior of the solution of problem (1.1).

We apply the general solvability theorem of [31], i.e. Theorem 2.6, to prove the existence of weak solution of (1.1). We study problem (1.1) on the domain of the operator generated by the addressed problem and verify the existence of a sufficiently smooth solution of the problem under more general (weak) conditions. Investigating a boundary-value problem on its own space yields better results. Therefore in this work, we analyse the considered problem on its own space. Apart from linear boundary value problems, the sets generated by nonlinear problems are subsets of linear spaces which do not have linear structure (see [28, 29, 30, 31, 32, 33] and references therein).

This paper is organized as follows. In the next section, we recall some useful results on the generalized Orlicz-Lebesgue spaces and results on nonlinear spaces (pn-spaces). In Section 3, we present the assumptions, define the weak solution, and then prove the existence of weak solution to problem (1.1). In Section 4, we examine the behavior of the solution of (1.1) when the problem is homogeneous.

2. Preliminaries

In this section, we begin with some available facts from the theory of the generalized Lebesgue spaces which are also called Orlicz-Lebesgue spaces. We present these facts without proof; proofs can be found in [1, 14, 20, 25].

Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$ such that $|\Omega| > 0$. (Throughout the paper, we denote by $|\Omega|$ the Lebesgue measure of Ω). Let $p(x,t) \geq 1$ be a measurable bounded function defined on the cylinder $Q_T = \Omega \times (0,T)$ i.e.

$$1 \leq p^- \equiv \underset{Q_T}{ess} \inf |p(x,t)| \leq \underset{Q_T}{ess} \sup |p(x,t)| \equiv p^+ < \infty. \tag{2.1}$$

On the set of all functions on Q_T define the functional σ_p and $\|.\|_p$ by

$$\sigma_p(u) \equiv \int\limits_{Q_T} |u|^{p(x,t)}\, dxdt$$

and

$$\|u\|_{L^{p(x,t)}(Q_T)} \equiv \inf\left\{\lambda > 0 |\ \sigma_p\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

The Generalized Lebesgue space is defined as follows:

$$L^{p(x,t)}(Q_T) := \{u : u \text{ is a measurable real-valued function in } Q_T,\ \sigma_p(u) < \infty\}.$$

The space $L^{p(x,t)}(Q_T)$ becomes a Banach space under the norm $\|.\|_{L^{p(x,t)}(Q_T)}$ which is so-called Luxemburg norm.

We present the following results for these spaces (see [20, 25, 26]):

Lemma 2.1. *If $0 < |\Omega| < \infty$, and p_1 and p_2 fulfill (2.1), then*

$$L^{p_1(x,t)}(Q_T) \subset L^{p_2(x,t)}(Q_T) \iff p_2(x,t) \leq p_1(x,t) \text{ for a.e } (x,t) \in Q_T.$$

Lemma 2.2. *The dual space of* $L^{p(x,t)}(Q_T)$ *is* $L^{p^*(x,t)}(Q_T)$ *if and only if* $p \in L^{\infty}(Q_T)$. *The space* $L^{p(x,t)}(Q_T)$ *is reflexive if and only if*

$$1 < p^- \leq p^+ < \infty$$

here $p^*(x,t) \equiv \frac{p(x,t)}{p(x,t)-1}$.

For $u \in L^{p(x,t)}(Q_T)$ and $v \in L^{q(x,t)}(Q_T)$ where p, q satisfy (2.1) and $\frac{1}{p(x,t)} + \frac{1}{q(x,t)} = 1$, the following inequalities hold:

$$\int\limits_{Q_T} |uv|\, dxdt \leq 2\,\|u\|_{L^{p(x,t)}(Q_T)}\,\|v\|_{L^{q(x,t)}(Q_T)}$$

and for all $u \in L^{p(x,t)}(\Omega)$, we have

$$\min\{\|u\|^{p^-}_{L^{p(x,t)}(Q_T)}, \|u\|^{p^+}_{L^{p(x,t)}(Q_T)}\} \leq \sigma_p(u) \leq \max\{\|u\|^{p^-}_{L^{p(x,t)}(Q_T)}, \|u\|^{p^+}_{L^{p(x,t)}(Q_T)}\}.$$

We introduce certain nonlinear function spaces (pn-spaces) which are complete metric spaces and directly connected to the problem under consideration. We also give some embedding results for these spaces [33, 32, 30, 31] (see also references cited therein).

Definition 2.3. Let $\alpha \geq 0$, $\beta \geq 1$, $\varrho = (\varrho_{1,\ldots,}\varrho_n)$ be multi-index, $m \in \mathbb{Z}^+$ and $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be bounded domain with sufficiently smooth boundary.

$$S_{m,\alpha,\beta}(\Omega) \equiv \left\{ u \in L^1(\Omega) \mid [u]^{\alpha+\beta}_{S_{m,\alpha,\beta}(\Omega)} \equiv \sum_{0\leq|\varrho|\leq m} \left(\int\limits_{\Omega} |u|^{\alpha}\, |D^{\varrho}u|^{\beta}\, dx \right) < \infty \right\}$$

in particular,

$$\mathring{S}_{1,\alpha,\beta}(\Omega) \equiv \left\{ u \in L^1(\Omega) \mid [u]^{\alpha+\beta}_{\mathring{S}_{1,\alpha,\beta}(\Omega)} \equiv \sum_{i=1}^{n} \left(\int\limits_{\Omega} |u|^{\alpha}\, |D_i u|^{\beta}\, dx \right) < \infty \right\} \cap \{u\mid_{\partial\Omega} \equiv 0\}$$

and for $p \geq 1$,

$$L^p\left(0,T; \mathring{S}_{1,\alpha,\beta}(\Omega)\right) \equiv \left\{ u \in L^1(Q_T) \mid [u]^p_{L^p\left(0,T;\mathring{S}_{1,\alpha,\beta}(\Omega)\right)} \equiv \int\limits_0^T [u]^p_{\mathring{S}_{1,\alpha,\beta}(\Omega)}\, dt < \infty \right\}.$$

These spaces are called *pn-spaces*.*

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*$S_{1,\alpha,\beta}(\Omega)$ is a complete metric space with the following metric: $\forall u, v \in S_{1,\alpha,\beta}(\Omega)$

$$d_{S_{1,\alpha,\beta}}(u,v) = \left\| |u|^{\frac{\alpha}{\beta}}\, u - |v|^{\frac{\alpha}{\beta}}\, v \right\|_{W^{1,\beta}(\Omega)}$$

Theorem 2.4. *Let $\alpha \geq 0$, $\beta \geq 1$ then $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, $\varphi(t) \equiv |t|^{\frac{\alpha}{\beta}} t$ is a homeomorphism between $S_{1,\alpha,\beta}(\Omega)$ and $W^{1,\beta}(\Omega)$.*

Theorem 2.5. *The following embeddings hold:*

(i) *Let α, $\alpha_1 \geq 0$ and $\beta_1 \geq 1$, $\beta \geq \beta_1$, $\frac{\alpha_1}{\beta_1} \geq \frac{\alpha}{\beta}$, $\alpha_1 + \beta_1 \leq \alpha + \beta$ then we have*

$$\mathring{S}_{1,\alpha,\beta}(\Omega) \subseteq \mathring{S}_{1,\alpha_1,\beta_1}(\Omega).$$

(ii) *Let $\alpha \geq 0$, $\beta \geq 1$, $n > \beta$ and $\frac{n(\alpha+\beta)}{n-\beta} \geq r$ then there is a continuous embedding*

$$\mathring{S}_{1,\alpha,\beta}(\Omega) \subset L^r(\Omega).$$

Furthermore for $\frac{n(\alpha+\beta)}{n-\beta} > r$ the embedding is compact.

(iii) *If $\alpha \geq 0$, $\beta \geq 1$ and $p \geq \alpha + \beta$ then*

$$W_0^{1,p}(\Omega) \subset \mathring{S}_{1,\alpha,\beta}(\Omega)$$

holds.

In the following, we present the general solvability theorem of [31], whose proof relies on Galerkin approximation (see also for similar theorems [33, 30]). We will employ this theorem to demonstrate the existence of a weak solution of problem (1.1).

Theorem 2.6. *Let X and Y be Banach spaces with dual spaces X^* and Y^* respectively, Y be a reflexive Banach space, $M_0 \subseteq X$ be a weakly complete "reflexive" pn-space, $X_0 \subseteq M_0 \cap Y$ be a separable vector topological space. Let the following conditions be fulfilled:*

(i) *$f : S_0 \longrightarrow L^q(0,T;Y)$ is a weakly compact (weakly continuous) mapping, where*

$$S_0 := L^p(0,T;M_0) \cap W^{1,q}(0,T;Y) \cap \{x(t) : x(0) = 0\}$$

$1 < \max\{q, q'\} \leq p < \infty$, $q' = \frac{q}{q-1}$;

(ii) *there is a linear continuous operator $A : W^{s,m}(0,T;X_0) \longrightarrow W^{s,m}(0,T;Y^*)$, $s \geq 0$, $m \geq 1$ such that A commutes with $\frac{\partial}{\partial t}$ and the conjugate operator A^* has $ker(A^*) = 0$;*

(iii) *operators f and A generate, in generalized sense, a coercive pair on space $L^p(0,T;X_0)$, i.e. there exist a number $r > 0$ and a function $\Psi : \mathbb{R}^1_+ \longrightarrow \mathbb{R}^1_+$ such that $\Psi(\tau)/\tau \nearrow \infty$ as $\tau \nearrow \infty$ and for any $x \in L^p(0,T;X_0)$ such that $[x]_{L^p(M_0)} \geq r$ following inequality holds:*

$$\int_0^T \langle f(t, x(t)), Ax(t)\rangle \, dt \geq \Psi\left([x]_{L^p(M_0)}\right);$$

(iv) *there exists some constants* $C_0 > 0$, $C_1, C_2 \geq 0$ *and* $\nu > 1$ *such that the inequalities*

$$\int_0^T \langle \xi(t), A\xi(t) \rangle \, dt \geq C_0 \|\xi\|^{\nu}_{L^q(0,T;Y)} - C_2,$$

$$\int_0^t \left\langle \frac{\partial x}{\partial \tau}, Ax(\tau) \right\rangle d\tau \geq C_1 \|x\|^{\nu}_{Y}(t) - C_2, \quad \text{a.e. } t \in [0,T]$$

hold for any $x \in W^{1,p}(0,T;X_0)$ *and* $\xi \in L^p(0,T;X_0)$.

Assume that that conditions (i)-(iv) are fulfilled. Then the Cauchy problem

$$\frac{dx}{d\tau} + f(t, x(t)) = y(t), \quad y \in L^q(0,T;Y); \quad x(0) = 0$$

is solvable in S_0 *in the following sense*

$$\int_0^T \left\langle \frac{dx}{d\tau} + f(t, x(t)), y^*(t) \right\rangle dt = \int_0^T \langle y(t), y^*(t) \rangle, \quad \forall y^* \in L^{q'}(0,T;Y^*),$$

for any $y \in L^q(0,T;Y)$ *satisfying the inequality*

$$\sup \left\{ \frac{1}{[x]_{L^p(0,T;M_0)}} \int_0^T \langle y(t), Ax(t) \rangle \, dt : x \in L^p(0,T;X_0) \right\} < \infty.$$

3. Statement of the Problem and the Main Result

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. We study the problem

$$\begin{cases} \frac{\partial u}{\partial t} - \sum_{i=1}^{n} D_i \left(|u|^{p_0 - 2} D_i u \right) + a(x,t,u) + g(x,t) \|u\|^s_{L^p(\Omega)}(t) = h(x,t), & (x,t) \in Q_T \\ u(x,0) = 0 = u_0(x), \quad u|_{\Gamma_T} = 0 \end{cases}$$

under the following conditions:

$p_0 \geq 2$, $p, s \geq 1$, $g : Q_T \to \mathbb{R}$ is a measurable function satisfying $g(x,t) \neq 0$ for a.e. $(x,t) \in Q_T$ and $a : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$, $a(x,t,\tau)$ is a Carathédory function with variable nonlinearity in τ (see inequality (3.1)).

Let the function $a(x,t,\tau)$ in problem (1.1) fulfill the following conditions:

(U1) *There exists a measurable function* $\alpha : \Omega \times (0,T) \longrightarrow \mathbb{R}$, $1 < \alpha^- \leq \alpha(x,t) \leq \alpha^+ < \infty$ *such that* $a(x,t,\tau)$ *satisfies the inequalities*

$$|a(x,t,\tau)| \leq a_0(x,t) |\tau|^{\alpha(x,t)-1} + a_1(x,t) \tag{3.1}$$

and

$$a(x,t,\tau)\,\tau \geq a_2(x,t)\,|\tau|^{\alpha(x,t)} - a_3(x,t), \tag{3.2}$$

a.e. $(x,t,\tau) \in Q_T \times \mathbb{R}$.

Here a_i, $i = 0,1,2,3$ are nonnegative, measurable functions defined on Q_T and $a_2(x,t) \geq A_0 > 0$ *a.e.* $(x,t) \in Q_T$.

We investigate problem (1.1) for the functions $h \in L^{q_0}\left(0,T;W^{-1,q_0}(\Omega)\right) + L^{\alpha^*(x,t)}(Q_T)$ where α^* is conjugate of α i.e. $\alpha^*(x,t) := \frac{\alpha(x,t)}{\alpha(x,t)-1}$ and the dual space $W^{-1,q_0}(\Omega) := \left(W_0^{1,p_0}(\Omega)\right)^*$, $q_0 := \frac{p_0}{p_0-1}$.

Let us denote S_0 by

$$\begin{aligned} S_0 := L^{p_0}\left(0,T;\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\right) &\cap L^{\alpha(x,t)}(Q_T) \\ &\cap W^{1,q_0}\left(0,T;W^{-1,q_0}(\Omega)\right) \cap \{u : u(x,0) = 0\}. \end{aligned}$$

We understand the solution of the problem under consideration in the following sense:

Definition 3.1. A function $u \in S_0$, is called the *generalized solution* (*weak solution*) of problem (1.1) if it satisfies the equality

$$\begin{aligned} &\int_0^T\int_\Omega \frac{\partial u}{\partial t} w\,dx\,dt + \sum_{i=1}^n \int_0^T\int_\Omega \left(|u|^{p_0-2} D_i u\right) D_i w\,dx\,dt \\ &+ \int_0^T\int_\Omega a(x,t,u)\,w\,dx\,dt + \int_0^T\int_\Omega g(x,t)\,\|u\|_{L^p(\Omega)}^s\,w\,dx\,dt = \int_0^T\int_\Omega h w\,dx\,dt \end{aligned}$$

for all $w \in L^{p_0}\left(0,T;W_0^{1,p_0}(\Omega)\right) \cap L^{\alpha(x,t)}(Q_T) \cap W^{1,q_0}\left(0,T;W^{-1,q_0}(\Omega)\right)$.

We are ready to proceed to the main theorem of this section but first define the followings. For sufficiently small $\eta \in (0,1)$

$$Q_{1,T} := \{(x,t) \in Q_T |\ \alpha(x,t) \in [1, p_0 - \eta)\},$$

$$Q_{2,T} := \left\{(x,t) \in Q_T |\ \alpha(x,t) \in [p_0 - \eta, \alpha^+]\right\}$$

and

$$\beta(x,t) := \begin{cases} \frac{p_0\alpha^*(x,t)}{p_0-\alpha(x,t)} & \text{if } (x,t) \in Q_{1,T}, \\ \infty & \text{if } (x,t) \in Q_{2,T}. \end{cases}$$

Also, $\tilde{p_0} := \frac{np_0}{n-q_0}$ which is critical exponent in Theorem 2.5 and its conjugate is $\tilde{p_0}^* = \frac{\tilde{p_0}}{\tilde{p_0}-1}$.

Theorem 3.2.(Existence Theorem) *Let **(U1)** be satisfied;* $1 \leq s < p_0 - 1$ *and* $p \leq p_0$. *If* $a_0 \in L^{\beta(x,t)}(Q_T)$, $a_1 \in L^{\alpha^*(x,t)}(Q_T)$, $a_2 \in L^{\infty}(Q_T)$, $a_3 \in L^1(Q_T)$ *and* $g \in L^{\frac{p_0}{p_0-(s+1)}}\left(0,T;L^{\tilde{p_0}^*}(\Omega)\right)$ *then for all* $h \in L^{q_0}\left(0,T;W^{-1,q_0}(\Omega)\right) + L^{\alpha^*(x,t)}(Q_T)$ *problem* (1.1) *has a generalized solution in the space* S_0 *and* $\partial u/\partial t$ *belongs to* $L^{q_0}\left(0,T;W^{-1,q_0}(\Omega)\right)$.

The proof of Theorem 3.2 is based on the general existence theorem (Theorem 2.6). We introduce the following spaces and mappings in order to apply Theorem 2.6 to prove Theorem 3.2.

$$S_0 := L^{p_0}\left(0,T;\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\right) \cap L^{\alpha(x,t)}(Q_T)$$
$$\cap W^{1,q_0}\left(0,T;W^{-1,q_0}(\Omega)\right) \cap \{u : u(x,0)=0\},$$
$$f : S_0 \longrightarrow L^{q_0}\left(0,T;W^{-1,q_0}(\Omega)\right) + L^{\alpha^*(x,t)}(Q_T),$$
$$f(u) := -\sum_{i=1}^{n} D_i\left(|u|^{p_0-2} D_i u\right) + a(x,t,u) + g(x,t)\|u\|^s_{L^p(\Omega)}(t),$$
$$A : L^{p_0}\left(0,T;W_0^{1,p_0}(\Omega)\right) \cap L^{\alpha(x,t)}(Q_T) \subset S_0 \longrightarrow L^{p_0}\left(0,T;W_0^{1,p_0}(\Omega)\right) \cap L^{\alpha(x,t)}(Q_T),$$
$$A := Id.$$

We prove some lemmas to show that all conditions of Theorem 2.6 are fulfilled under the conditions of Theorem 3.2.

Lemma 3.3. *Under the conditions of Theorem 3.2, f and A generate a "coercive pair" on* $L^{p_0}\left(0,T;W_0^{1,p_0}(\Omega)\right) \cap L^{\alpha(x,t)}(Q_T)$.

Proof. Since $A \equiv Id$, being "coercive pair" equals to order coercivity of f on the space $L^{p_0}\left(0,T;W_0^{1,p_0}(\Omega)\right) \cap L^{\alpha(x,t)}(Q_T)$.

For $u \in L^{p_0}\left(0,T;W_0^{1,p_0}(\Omega)\right) \cap L^{\alpha(x,t)}(Q_T)$, we have the following equation:

$$\langle f(u), u\rangle_{Q_T} = \sum_{i=1}^{n}\left(\int_0^T\int_\Omega |u|^{p_0-2}|D_i u|^2\,dxdt\right)$$
$$+\int_{Q_T} a(x,t,u)\,udxdt + \int_0^T\int_\Omega g(x,t)\|u\|^s_{L^p(\Omega)}\,udxdt.$$

By using (3.2), we obtain

$$\langle f(u), u\rangle_{Q_T} \geq \sum_{i=1}^{n}\left(\int_0^T\int_\Omega |u|^{p_0-2}|D_i u|^2\,dxdt\right) + \int_{Q_T}|a_2(x,t)|\,|u|^{\alpha(x,t)}\,dxdt$$
$$-\int_{Q_T}|a_3(x,t)|\,dxdt - \int_0^T\int_\Omega |g(x,t)|\,\|u\|^s_{L^p(\Omega)}\,|u|\,dxdt. \tag{3.3}$$

If we employ **(U1)** to estimate the second integral in (3.3) and by applying Hölder inequality together with the embedding $\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega) \subset L^p(\Omega)$ (see Theorem 2.5) to estimate the fourth integral then we get,

$$\begin{aligned} \langle f(u), u\rangle_{Q_T} \geq{} & [u]^{p_0}_{L^{p_0}\left(0,T;\mathring{S}_{1,(p_0-2),2}(\Omega)\right)} + A_0 \int\limits_{Q_T} |u|^{\alpha(x,t)}\, dxdt \\ & - C \int\limits_0^T [u]^{s}_{\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)} \|u\|_{L^{\tilde{p_0}}(\Omega)} \|g\|_{L^{\tilde{p_0}^*}(\Omega)}\, dt \\ & - \|a_3\|_{L^1(Q_T)}. \end{aligned} \tag{3.4}$$

By taking account the embeddings (see Theorem 2.5)

$$\mathring{S}_{1,(p_0-2),2}(\Omega) \subset \mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)$$

and

$$\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega) \subset L^{\tilde{p_0}}(\Omega)$$

into (3.4) to estimate the pseudo-norm and third integral respectively, we attain

$$\begin{aligned} \langle f(u), u\rangle_{Q_T} \geq{} & C_0\, [u]^{p_0}_{L^{p_0}\left(0,T;\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\right)} + A_0 \int\limits_{Q_T} |u|^{\alpha(x,t)}\, dxdt \\ & - C_1 \int\limits_0^T [u]^{s+1}_{\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)} \|g\|_{L^{\tilde{p_0}^*}(\Omega)}\, dt - \|a_3\|_{L^1(Q_T)}. \end{aligned} \tag{3.5}$$

By utilizing Young's inequality to the third integral in (3.5), we have

$$\langle f(u), u\rangle_{Q_T} \geq C_2 \left([u]^{p_0}_{L^{p_0}\left(0,T;\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\right)} + \|u\|^{\alpha^-}_{L^{\alpha(x,t)}(Q_T)} \right) - K.$$

Here, $K = K\left(\|a_3\|_{L^1(Q_T)}, \|g\|_{L^{\frac{p_0}{p_0-(s+1)}}\left(0,T;L^{\tilde{p_0}^*}(\Omega)\right)} \right)$, $C_2 = C_2(p_0, s, A_0, |\Omega|)$ are positive constants. So the proof is completed. □

Lemma 3.4. *Under the conditions of Theorem 3.2, f is bounded from S_0 into $L^{q_0}\left(0,T;W^{-1,q_0}(\Omega)\right) + L^{\alpha^*(x,t)}(Q_T)$.*

Proof. First, we define the mappings

$$\begin{aligned} f_1(u) &:= \sum_{i=1}^n -D_i\left(|u|^{p_0-2} D_i u \right) + g(x,t)\, \|u\|^s_{L^p(\Omega)}(t), \\ f_2(u) &:= a(x,t,u). \end{aligned}$$

We need to show that these mappings are both bounded from $L^{p_0}\left(0,T;\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\right)\cap L^{\alpha(x,t)}(Q_T)$ into $L^{q_0}\left(0,T;W^{-1,q_0}(\Omega)\right)+L^{\alpha^*(x,t)}(Q_T)$.

Let us show that f_1 is bounded: For $u\in L^{p_0}\left(0,T;\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\right)$ and $v\in L^{p_0}\left(0,T;W_0^{1,p_0}(\Omega)\right)$,

$$\left|\left\langle f_1(u),v\right\rangle_{Q_T}\right|\le\sum_{i=1}^{n}\left(\int_0^T\int_\Omega |u|^{p_0-2}|D_iu|\,|D_iv|\,dxdt\right)+\int_0^T\int_\Omega |g(x,t)|\,\|u\|^s_{L^p(\Omega)}|v|\,dxdt.$$

Using the embedding $\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\subset L^p(\Omega)$ and Hölder's inequality above we find,

$$\le\left[\sum_{i=1}^{n}\left(\int_0^T\int_\Omega |u|^{(p_0-2)q_0}|D_iu|^{q_0}\,dxdt\right)\right]^{\frac{1}{q_0}}\left[\sum_{i=1}^{n}\left(\int_0^T\int_\Omega |D_iv|^{p_0}\,dxdt\right)\right]^{\frac{1}{p_0}}$$
$$+\tilde{C}\int_0^T[u]^s_{\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)}\|g\|_{L^{\frac{np_0}{n(p_0-1)+p_0}}(\Omega)}\|v\|_{W_0^{1,p_0}(\Omega)}\,dt.$$

Estimating the second integral above by Hölder's inequality ($\frac{p_0}{s}>1$), we obtain

$$\left|\left\langle f_1(u),v\right\rangle_{Q_T}\right|\le\Psi([u]_{L^{p_0}\left(0,T;\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\right)})\,\|v\|_{L^{p_0}\left(0,T;W_0^{1,p_0}(\Omega)\right)}$$

where

$$\Psi([u]_{L^{p_0}\left(0,T;\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\right)})=[u]^{p_0-1}_{L^{p_0}\left(0,T;\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\right)}$$
$$+\tilde{C}_1\,[u]^s_{L^{p_0}\left(0,T;\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\right)}\|g\|_{L^{\frac{p_0}{p_0-(s+1)}}\left(0,T;L^{\tilde{p_0}^*}(\Omega)\right)}.$$

By the last inequality, boundedness of f_1 is achieved.

Similarly, from (3.1) and Theorem 2.5, for all $u\in S_0$, we have the following estimate

$$\begin{aligned}\sigma_{\alpha^*}(f_2(u))&=\sigma_{\alpha^*}(a(x,t,u))\\&=\int_0^T\int_\Omega |a(x,t,u)|^{\alpha^*(x,t)}\,dxdt\\&\le C_3\left(\sigma_\alpha(u)+[u]^{p_0}_{L^{p_0}\left(0,T;\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\right)}\right)+C_4,\end{aligned}$$

here $C_3=C_3\left(\alpha^+,\alpha^-,\|a_0\|_{L^{\beta(x,t)}(Q_T)}\right)$, $C_4=C_4\left(\sigma_\beta(a_0),\sigma_{\alpha^*}(a_1),|\Omega|\right)>0$ are constants. That yields $f_2:L^{p_0}\left(0,T;\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\right)\cap L^{\alpha(x,t)}(Q_T)\to L^{\alpha^*(x,t)}(Q_T)$ is bounded. $\square$

Lemma 3.5. *Under the conditions of Theorem 3.2, f is weakly compact from S_0 into $L^{q_0}\left(0,T;W^{-1,q_0}(\Omega)\right)+L^{\alpha^*(x,t)}(Q_T)$.*

Proof. First we verify the weak compactness of f_0, where $f_0(u) := -\sum_{i=1}^{n} D_i\left(|u|^{p_0-2}D_iu\right)$. Let $\{u_m(x,t)\}_{m=1}^{\infty}\subset S_0$ be bounded and $u_m \overset{S_0}{\rightharpoonup} u_0$. It is sufficient to show a subsequence of $\{u_{m_j}\}_{m=1}^{\infty}\subset\{u_m\}_{m=1}^{\infty}$ which satisfies $f_0\left(u_{m_j}\right) \overset{L^{q_0}\left(0,T;W^{-1,q_0}(\Omega)\right)}{\rightharpoonup} f_0(u_0)$.

Since for a.e. $t\in(0,T)$, $u_m(\cdot,t)\in \mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)$, and by existence of an one-to-one correspondence between the classes (Theorem 2.4)

$$\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega) \underset{\varphi^{-1}}{\overset{\varphi}{\longleftrightarrow}} W_0^{1,q_0}(\Omega)$$

with the homeomorphism

$$\varphi(\tau)\equiv|\tau|^{p_0-2}\tau,\ \varphi^{-1}(\tau)\equiv|\tau|^{-\frac{p_0-2}{p_0-1}}\tau,$$

for all $m\geq 1$ we have

$$|u_m|^{p_0-2}u_m\in L^{q_0}\left(0,T;W_0^{1,q_0}(\Omega)\right)$$

is bounded. Due to the fact $L^{q_0}\left(0,T;W_0^{1,q_0}(\Omega)\right)$ is a reflexive space, there exists a subsequence $\{u_{m_j}\}_{m=1}^{\infty}\subset\{u_m\}_{m=1}^{\infty}$ such that

$$\left|u_{m_j}\right|^{p_0-2}u_{m_j} \overset{L^{q_0}\left(0,T;W_0^{1,q_0}(\Omega)\right)}{\rightharpoonup} \xi.$$

Now, we show that $\xi=|u_0|^{p_0-2}u_0$. According to compact embedding [33],

$$L^{p_0}\left(0,T;\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\right)\cap W^{1,q_0}\left(0,T;W^{-1,q_0}(\Omega)\right)\hookrightarrow L^{p_0}(Q_T) \tag{3.6}$$

$$\exists\left\{u_{m_{j_k}}\right\}_{m=1}^{\infty}\subset\{u_{m_j}\}_{m=1}^{\infty},\ u_{m_{j_k}} \overset{L^{p_0}(Q_T)}{\to} u_0$$

which implies

$$u_{m_{j_k}} \underset{a.e}{\overset{Q_T}{\to}} u_0$$

by the continuity of $\varphi(\tau)$, we get

$$\left|u_{m_{j_k}}\right|^{p_0-2}u_{m_{j_k}} \underset{a.e}{\overset{Q_T}{\to}} |u_0|^{p_0-2}u_0$$

that yields $\xi=|u_0|^{p_0-2}u_0$.

From this, we deduce that for each $v \in L^{p_0}\left(0, T; W_0^{1,p_0}(\Omega)\right)$

$$\langle f_0\left(u_{m_{j_k}}\right), v\rangle_{Q_T} = \sum_{i=1}^{n} \langle -D_i\left(\left|u_{m_{j_k}}\right|^{p_0-2} D_i u_{m_{j_k}}\right), v\rangle_{Q_T}$$
$$\underset{m_j \nearrow \infty}{\longrightarrow} \sum_{i=1}^{n} \langle -D_i\left(|u_0|^{p_0-2} D_i u_0\right), v\rangle_{Q_T} = \langle f_0(u_0), v\rangle_{Q_T}$$

whence, the result is obtained.

We shall show the weak compactness of f_2. Since

$$a : L^{p_0}\left(0, T; \mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\right) \cap L^{\alpha(x,t)}(Q_T) \to L^{\alpha^*(x,t)}(Q_T)$$

is bounded by Lemma 3.4, then for $m \geq 1$, $f_2(u_m) = \{a(x,t,u_m)\}_{m=1}^{\infty} \subset L^{\alpha^*(x,t)}(Q_T)$. Also $L^{\alpha^*(x,t)}(Q_T)$ $(1 < (\alpha^*)^- < \infty)$ is a reflexive space thus $\{u_m\}_{m=1}^{\infty}$ has a subsequence $\{u_{m_j}\}_{m=1}^{\infty}$ such that

$$a\left(x, t, u_{m_j}\right) \overset{L^{\alpha^*(x,t)}(Q_T)}{\rightharpoonup} \psi.$$

We deduce from the compact embedding (3.6) that

$$\exists \left\{u_{m_{j_k}}\right\}_{m=1}^{\infty} \subset \left\{u_{m_j}\right\}_{m=1}^{\infty}, \; u_{m_{j_k}} \overset{L^{p_0}(Q_T)}{\rightarrow} u_0$$

thus

$$u_{m_{j_k}} \underset{a.e}{\overset{Q_T}{\rightarrow}} u_0.$$

Accordingly, the continuity of $a(x,t,.)$ for almost $(x,t) \in Q_T$ implies that

$$a(x,t,u_{m_{j_k}}) \underset{a.e}{\overset{Q_T}{\rightarrow}} a(x,t,u_0),$$

so, we arrive at $\psi = a(x,t,u_0)$ i.e. $f_2(u_{m_{j_k}}) \overset{L^{q_0}\left(0,T;W^{-1,q_0}(\Omega)\right)+L^{\alpha^*(x,t)}(Q_T)}{\rightharpoonup} f_2(u_0)$.

Let $a_1(x,t,u) := g(x,t)\|u\|_{L^p(\Omega)}(t)$. Using the compact imbedding (3.6) and $p \leq p_0$, we attain

$$g(x,t)\left\|u_{m_j}\right\|_{L^p(\Omega)}^s(t) \overset{L^{q_0}\left(0,T;W^{-1,q_0}(\Omega)\right)}{\longrightarrow} g(x,t)\|u_0\|_{L^p(\Omega)}^s(t).$$

Therefore, a_1 is weakly compact from S_0 into $L^{q_0}\left(0,T;W^{-1,q_0}(\Omega)\right) + L^{\alpha^*(x,t)}(Q_T)$. As a conclusion, f is weakly compact from S_0 into $L^{q_0}\left(0,T;W^{-1,q_0}(\Omega)\right) + L^{\alpha^*(x,t)}(Q_T)$. $\square$

Now, we give the proof of main theorem of this section.

Proof of Theorem 3.2. Since $A = Id$, obviously it is a linear bounded map and satisfies the conditions (ii) of Theorem 2.6. Furthermore for any $u \in W_0^{1,p_0}(Q_T)$ the following inequalities are valid:

$$\int_0^T \langle u, u\rangle_\Omega \, dt = \int_0^T \|u\|^2_{L^2(\Omega)} \, dt \geq M \|u\|^2_{L^{q_0}(0,T;W^{-1,q_0}(\Omega))}$$

and

$$\int_0^t \left\langle \frac{\partial u}{\partial \tau}, u \right\rangle_\Omega d\tau = \frac{1}{2} \|u\|^2_{L^2(\Omega)}(t) \geq M \frac{1}{2} \|u\|^2_{W^{-1,q_0}(\Omega)}(t),$$

a.e. $t \in [0,T]$ (constant $M > 0$ comes from embedding inequality). Thus condition (iv) of Theorem 2.6 is satisfied as well. Consequently from Lemma 3.3-Lemma 3.5, it follows that the mappings f and A fulfill all the conditions of Theorem 2.6. Employing this theorem to problem (1.1), we find that (1.1) is solvable in S_0 for any $h \in L^{q_0}\left(0,T;W^{-1,q_0}(\Omega)\right) + L^{\alpha^*(x,t)}(Q_T)$ satisfying the following inequality

$$\sup\left\{ \frac{1}{[u]_{L^{p_0}\left(0,T;\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\right)} + \|u\|_{L^{\alpha(x,t)}(Q_T)}} \int_0^T \langle h, u\rangle_\Omega \, dt : u \in Q_0 \right\} < \infty$$

where $Q_0 := L^{p_0}\left(0,T;W_0^{1,p_0}(\Omega)\right) \cap L^{\alpha(x,t)}(Q_T)$. Considering the norm definition of h in $L^{q_0}\left(0,T;W^{-1,q_0}(\Omega)\right) + L^{\alpha^*(x,t)}(Q_T)$, we conclude that (1.1) is solvable in S_0 for any $h \in L^{q_0}\left(0,T;W^{-1,q_0}(\Omega)\right) + L^{\alpha^*(x,t)}(Q_T)$. In order to complete the proof, it remains to remark that (1.1) can be written in the form

$$\frac{\partial u}{\partial t} = h(x,t) - F(x,t,u,D_i u),$$

and under the conditions of Theorem 3.2, right hand belongs to $L^{q_0}\left(0,T;W^{-1,q_0}(\Omega)\right)$ which implies $\partial u/\partial t \in L^{q_0}\left(0,T;W^{-1,q_0}(\Omega)\right)$. $\square$

Remark 3.6. We note that if the function $\alpha(x,t)$ in (3.1) satisfies the inequality $\alpha^+ < p_0$ then the existence of a solution of the problem (1.1) can be shown under more general (weaker) conditions. This is verified in the following theorem.

Theorem 3.7. *Assume that (3.1) and inequalities* $1 \leq s < p_0 - 1$, $p \leq p_0$ *are satisfied. If* $1 < \alpha^- \leq \alpha(x,t) \leq \alpha^+ < p_0$, $(x,t) \in Q_T$ *and* $g \in L^{\frac{p_0}{p_0-(s+1)}}\left(0,T;L^{\tilde{p_0}^*}(\Omega)\right)$, $a_0 \in L^{\beta_1(x,t)}(Q_T)$, $a_1 \in L^{\alpha^*(x,t)}(Q_T)$ *where* $\beta_1(x,t) := \frac{p_0\alpha^*(x,t)}{p_0-\alpha(x,t)}$ *then* $\forall h \in L^{q_0}\left(0,T;W^{-1,q_0}(\Omega)\right)$ *problem (1.1) has a generalized solution in the space* $L^{p_0}\left(0,T;\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\right) \cap W^{1,q_0}\left(0,T;W^{-1,q_0}(\Omega)\right)$.

Proof. We deduce from inequality (3.1) that

$$\langle f(u), u\rangle_{Q_T} \geq \sum_{i=1}^{n}\left(\int_0^T\int_\Omega |u|^{p_0-2}|D_i u|^2\,dxdt\right) - \int_{Q_T}|a_0(x,t)|\,|u|^{\alpha(x,t)}\,dxdt$$
$$-\int_{Q_T}|a_1(x,t)|\,dxdt - \int_0^T\int_\Omega |g(x,t)|\,\|u\|^s_{L^p(\Omega)}\,|u|\,dxdt.$$

For arbitrary $\epsilon > 0$ estimating the second integral above by Young's inequality and using $L^{p_0}\left(0,T;\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\right) \subset L^{p_0}(Q_T)$, we attain the following inequality which gives the coercivity of f,

$$\langle f(u), u\rangle_{Q_T} \geq C_5\,[u]^{p_0}_{L^{p_0}\left(0,T;\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\right)} - \tilde{K}.$$

here $C_5 = C_5(p_0, |\Omega|, s)$ and

$$\tilde{K} = \tilde{K}\left(\epsilon, \|a_0\|_{L^{\beta_1(x,t)}(Q_T)}, \|a_1\|_{L^{\alpha^*(x,t)}(Q_T)}, \|g\|_{L^{\frac{p_0}{p_0-(s+1)}}\left(0,T;L^{\tilde{p_0}^*}(\Omega)\right)}\right).$$

By the embedding

$$L^{p_0}\left(0,T;\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\right) \subset L^{p_0}(Q_T) \subset L^{\alpha(x,t)}(Q_T),$$

weak compactness and boundedness of
$f : L^{p_0}\left(0,T;\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)\right) \cap W^{1,q_0}\left(0,T;W^{-1,q_0}(\Omega)\right) \to L^{q_0}\left(0,T;W^{-1,q_0}(\Omega)\right)$ follows from Lemma 3.4 and Lemma 3.5. Thus by the virtue of the proof of Theorem 3.2, we get the desired result.

4. Homogeneous Case

In this section, we analyze problem (1.1) in homogeneous case. We establish sufficient conditions which ensure that problem (1.1) has only trivial solution under these conditions.

Theorem 4.1. *Let conditions of Theorem 3.2 be fulfilled with the following assumptions:*

(i) *Let $h(x,t) = 0$ and $p = 2$, $p_0 > 2$.*

(ii) *Condition (3.2) is satisfied with $a_3(x,t) = 0$.*

(iii) *The functional $\|g\|_{L^2(\Omega)}(t)$ is bounded for almost every $t \in \mathbb{R}^+$,*

then problem (1.1) *has only trivial solution.*

Proof. Conditions of Theorem 4.1 provide that (1.1) has a solution in S_0. It follows from Definition 3.1 that every weak solution satisfies the following relation,

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)}+\sum_{i=1}^{n}\int\limits_{\Omega}\left(|u|^{p_0-2}\left(D_iu\right)^2\right)dx+\int\limits_{\Omega}a\left(x,t,u\right)udx+\int\limits_{\Omega}g\left(x,t\right)\|u\|^s_{L^2(\Omega)}udx=0$$

then we get

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)}+\frac{4}{p_0^2}\sum_{i=1}^{n}\int\limits_{\Omega}(D_i(|u|^{\frac{p_0}{2}}))^2dx+\int\limits_{\Omega}a\left(x,t,u\right)udx+\int\limits_{\Omega}g\left(x,t\right)\|u\|^s_{L^2(\Omega)}udx=0$$

by using imbedding inequality and condition (ii), we deduce that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)}+\frac{4}{p_0^2c}\int\limits_{\Omega}|u|^{p_0}\,dx+\int\limits_{\Omega}g(x,t)\|u\|^s_{L^2(\Omega)}udx\leq 0$$

employing Hölder inequality and condition (iii) to the last term we find that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)}+\frac{4}{p_0^2c}\int\limits_{\Omega}|u|^{p_0}\,dx-K\|u\|^{s+1}_{L^2(\Omega)}\leq 0$$

where $K>0$ is a constant. From embedding inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)}+\frac{4}{p_0^2c\,|\Omega|^{\frac{p_0-2}{2}}}\|u\|^{p_0}_{L^2(\Omega)}-K\|u\|^{s+1}_{L^2(\Omega)}\leq 0,$$

whence denoting by $y=\|u\|^2_{L^2(\Omega)}$ and $\mu=\frac{p_0}{2}$, we have

$$\frac{1}{2}\frac{dy}{dt}+\frac{4}{p_0^2c\,|\Omega|^{\frac{p_0-2}{2}}}y^{\mu}-Ky^{\frac{s+1}{2}}\leq 0$$

by utilizing Young inequality to the last term in the above equation, we attain

$$\frac{1}{2}\frac{dy}{dt}+(\frac{4}{p_0^2c\,|\Omega|^{\frac{p_0-2}{2}}}-K\varepsilon)y^{\mu}-Kc(\varepsilon)y\leq 0$$

where $\varepsilon<\frac{4}{Kp_0^2c|\Omega|^{\frac{p_0-2}{2}}}$ from here, we conclude

$$\frac{1}{2}\frac{dy}{dt}\leq Kc(\varepsilon)y.$$

Integrating the last inequality and considering $y(0)=0$, we arrive at the desired result. □

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