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Generalized Hyers-Ulam Stability of Some Cubic-quadratic-additive Type Functional Equations

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Abstract. We will prove the generalized Hyers-Ulam stability of cubic-quadratic-additive type functional equations and general cubic functional equations whose solutions are cubic-quadratic-additive mappings and general cubic mappings, respectively.

1. Introduction

Throughout this paper, we assume that V and W are real vector spaces and k is a real number satisfying $k \not\in \{-1, 0, 1\}$ unless there are specifications for them. For any given mapping $f : V \to W$, we will set

$$\begin{aligned} D_1 f(x,y) &= f(x+ky) - \frac{k^2+k}{2} f(x+y) + (k^2-1) f(x) - \frac{k^2-k}{2} f(x-y) \\ &\quad - f(ky) + \frac{k^2+k}{2} f(y) + \frac{k^2-k}{2} f(-y), \\ D_2 f(x,y) &= f(x+2y) - f(x-2y) - 2f(x+y) + 2f(x-y) + 6f(y) + 2f(-y) \\ &\quad - 2f(2y), \\ D_3 f(x,y) &= f(x+ky) + f(x-ky) - k^2 f(x+y) - k^2 f(x-y) + 2(k^2-1) f(x), \\ D_4 f(x,y) &= f(x+3y) - 5f(x+2y) + 10f(x+y) - 10f(x) + 5f(x-y) \\ &\quad - f(x-2y) + 4f(y) + 4f(-y) - f(2y) - f(-2y), \end{aligned}$$

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$$
\begin{aligned}
D_5 f(x,y) &= f(x+2y) - f(x-2y) - 2f(x+y) + 2f(x-y) - 2f(3y) + 6f(2y) \\
&\quad - 6f(y), \\
D_6 f(x,y,z) &= f(x+y+z) + f(x-y-z) + f(x-y+z) + f(x+y-z) \\
&\quad - 2f(x+y) - 2f(x-y) - 2f(x+z) - 2f(x-z) + 4f(x), \\
D_7 f(x,y,z) &= f(x+y+z) - f(x-y-z) - f(x+y) + f(x-y) - f(x+z) \\
&\quad + f(x-z) - f(y+z) + f(-y-z) + f(y) - f(-y) + f(z) - f(-z)
\end{aligned}
$$

for all $x, y, z \in V$.

Every solution of functional equation

$$\sum_{i=0}^{1} (-1)^{1-i} {}_1C_i f(x+iy) - f(y) = 0$$

is called an *additive mapping*, and each solution of functional equation

$$\sum_{i=0}^{2} (-1)^{2-i} {}_2C_i f(x+iy) - 2! f(y) = 0$$

is called a *quadratic mapping*, while every solution of functional equation

$$\sum_{i=0}^{3} (-1)^{3-i} {}_3C_i f(x+iy) - 3! f(y) = 0$$

is called a *cubic mapping*.

If a mapping can be expressed by the sum of an additive mapping, a quadratic mapping, and a cubic mapping, then we call the mapping a cubic-quadratic-additive mapping. If a mapping can be expressed by the sum of a constant, an additive mapping, a quadratic mapping, and a cubic mapping, then we call the mapping a general cubic mapping. A functional equation is called a cubic-quadratic-additive type functional equation if each solution of that equation is a cubic-quadratic-additive mapping. A functional equation is called a general cubic type functional equation when each of its solutions is a general cubic mapping.

In 1940, Ulam [13] raised an important problem concerning the stability of group homomorphisms: *Under what conditions is the approximate solution of an equation necessarily close to the exact solution of the equation?* Just the following year, Hyers [8] solved the problem of Ulam only in the case of the Cauchy additive functional equation $Af(x,y) = f(x+y) - f(x) - f(y) = 0$. Indeed, Hyers proved the following statement for any previously given constant $\varepsilon > 0$: every solution of inequality $\|Af(x,y)\| \le \varepsilon$ (for all x and y) can be approximated by an exact solution (an additive function). In this case, the Cauchy additive functional equation is said to satisfy the Hyers-Ulam stability.

About three decades later, Rassias [12] generalized Hyers' result and then Găvruta [7] extended Rassias' result by allowing unbounded control functions. The concept of stability introduced by Rassias and Găvruta is known today as the *generalized Hyers-Ulam stability* of functional equations.

Jun et al. [9] and Lee [10] investigated the stability of the general cubic functional equation $D_3 f(x, y) = 0$ for $k = 2$. Independently from them, Gordji [3] investigated the stability of the cubic-quadratic-additive functional equation $D_5 f(x, y) = 0$. Moreover, Gordji et al. [2, 3, 4, 5, 6] investigated the stability of the general cubic functional equation $D_3 f(x, y) = 0$ for any integer $k \notin \{-1, 0, 1\}$. However, they could not prove the uniqueness of the exact solution because they divided the related function into even and odd parts and proved their stability separately, while we prove in this paper the stability in an integrated way. At the same time, we prove the uniqueness of the exact solution. This is an advantage of this paper in comparison with the papers [2, 3, 4, 5, 6] of other mathematicians.

In this paper, we will prove the generalized Hyers-Ulam stability of the functional equations $D_m f(x, y) = 0$ for $m \in \{1, 2, 3, 4, 5\}$ and $D_m f(x, y, z) = 0$ for $m \in \{6, 7\}$.

2. Main Results

The following theorem is a special version of Baker's theorem when $\delta = 0$ (refer to [1]).

Theorem 2.1.([1, Theorem 1]) *Given an $m \in \mathbb{N}$, assume that V and W are vector spaces over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ and that $\alpha_0, \beta_0, \ldots, \alpha_m, \beta_m$ are scalars satisfying $\alpha_j \beta_\ell - \alpha_\ell \beta_j \neq 0$ whenever $0 \leq j < \ell \leq m$. If the functions $f_\ell : V \to W$, $\ell \in \{0, 1, \ldots, m\}$, satisfy the equation*

$$\sum_{\ell=0}^{m} f_\ell(\alpha_\ell x + \beta_\ell y) = 0$$

for all $x, y \in V$, then each f_ℓ is a generalized polynomial mapping of degree at most $m - 1$.

Baker [1] also states that if $f : V \to W$ is a generalized polynomial mapping of degree at most $m - 1$, then f can be expressed as $f(x) = x_0 + \sum\limits_{\ell=1}^{m-1} a_\ell^*(x)$ for $x \in V$, where a_ℓ^* is a monomial mapping of degree ℓ and f has a property $f(rx) = x_0 + \sum\limits_{\ell=1}^{m-1} r^\ell a_\ell^*(x)$ for $x \in V$ and $r \in \mathbb{Q}$. The monomial mapping of degree 1, 2 and 3 are called an additive mapping, a quadric mapping, and a cubic mapping, respectively. The generalized polynomial mapping of degree 1, 2 and 3, on the other hand, are called a Jensen mapping, a general quadric mapping, and a general cubic mapping, respectively.

In summary, the following corollary can be obtained from Baker's theorem.

Corollary 2.2. *Let V and W be vector spaces over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, and let r be a rational number satisfying $r \notin \{-1, 0, 1\}$. Given an $m \in \mathbb{N}$, assume that $n_1, \ldots, n_m$ are positive integers and that $c_{\ell,i}, d_{\ell,i}, \alpha_0, \beta_0, \ldots, \alpha_m, \beta_m$ ($\ell \in \{1, \ldots, m\}$ and $i \in$*

$\{1,\ldots,n_\ell\}$) are scalars satisfying $\alpha_j\beta_\ell-\alpha_\ell\beta_j\neq 0$ whenever $0\le j<\ell\le m$. If a mapping $f:V\to W$ satisfies the equation $f(rx)=r^kf(x)$ for all $x\in V$ and the equation

$$f(\alpha_0x+\beta_0y)+\sum_{\ell=1}^{m}\sum_{i=1}^{n_\ell}c_{\ell,i}f(d_{\ell,i}(\alpha_\ell x+\beta_\ell y))=0$$

for all $x,y\in V$, then f is a monomial mapping of degree k.

Proof. Put

$$f_0(\alpha_0x+\beta_0y)=f(\alpha_0x+\beta_0y)$$

and

$$f_\ell(\alpha_\ell x+\beta_\ell y)=\sum_{i=1}^{n_\ell}c_{\ell,i}f(d_{\ell,i}(\alpha_\ell x+\beta_\ell y))$$

for all $\ell\in\{1,\ldots,m\}$. Then $f_0,\ldots,f_m$ satisfy the conditions of Theorem 2.1. So f is a generalized polynomial mapping of degree at most $m-1$. In addition, since f satisfies the equation $f(rx)=r^kf(x)$ for all $x\in V$, f is a monomial mapping of degree k. □

According to Theorem 2.1, the functional equations $D_1f(x,y)=0$, $D_2f(x,y)=0$, $D_4f(x,y)=0$, $D_5f(x,y)=0$, $D_6f(x,y,y)=0$, and $D_7f(x,y,y)=0$ are cubic-quadratic-additive type functional equations, and the functional equation $D_3f(x,y)=0$ is a general cubic type functional equation. Since $D_6f(x,y,y)=0$ and $D_7f(x,y,y)=0$ are cubic-quadratic-additive type functional equations, $D_6f(x,y,z)=0$ and $D_7f(x,y,z)=0$ are cubic-quadratic-additive type functional equations.

Hereafter, let Y be a real Banach space. For any mapping $f:V\to Y$ and any function $\varphi:V\times V\to[0,\infty)$, we use the following notations:

$$\begin{aligned}f_e(x)&=\frac{f(x)+f(-x)}{2},\\ f_o(x)&=\frac{f(x)-f(-x)}{2},\\ \varphi_e(x,y)&=\frac{\varphi(x,y)+\varphi(-x,-y)}{2}.\end{aligned}$$

Lemma 2.3. *Let V be a real vector space and let $(Y,\|\cdot\|)$ be a real Banach space. Given an $m\in\{1,2,3,4,5\}$, assume that a mapping $f:V\to Y$ satisfies $f(0)=0$ and inequality*

$$\|D_mf(x,y)\|\le\varphi(x,y) \tag{2.1}$$

for all $x,y\in V$. Then the following inequalities

$$\begin{aligned}&\|f_e(2x)-4f_e(x)\|\le\mu_m(x),\\ &\|f_o(4x)-10f_o(2x)+16f_o(x)\|\le\nu_m(x)\end{aligned} \tag{2.2}$$

hold for all $x \in V$, where $\mu_m, \nu_m : V \to \mathbb{R}$ are defined by

$$
\begin{aligned}
(2.3)\quad \mu_1(x) &:= \frac{1}{|4k^3-4k|}\times \\
&\times \Bigg(|k-2|\Big(\varphi_e(x,2x)+\varphi_e((k+1)x,x)+\frac{|k^2+k|}{2\varphi_e(2x,x)}\Big) \\
&\qquad + |k+2|\Big(\varphi_e(x,-2x)+\varphi_e((k-1)x,x)+\frac{|k^2-k|}{2\varphi_e(-2x,x)}\Big) \\
&\qquad + 4\varphi_e(kx,x)+|2k^3+k^2-k-2|\varphi_e(-x,x) \\
&\qquad + |k^3-4k^2-3k+2|\varphi_e(x,x)\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\mu_2(x) &:= \frac{1}{2}\varphi_e(0,x), \\
\mu_3(x) &:= \frac{1}{k^4-k^2}\times \\
&\times \Big(\varphi_e(kx,x)+k^2\varphi_e(x,x)+\frac{1}{2}\varphi_e(0,2x)+|k^2-1|\varphi_e(0,x)\Big), \\
\mu_4(x) &:= \frac{1}{4}\big(\varphi_e(0,x)+\varphi_e(-x,x)\big), \quad \mu_5(x) := \frac{1}{10}\big(\varphi_e(-x,x)+3\varphi_e(0,x)\big), \\
\nu_1(x) &:= \frac{1}{|k^4-k^2||k+2|}\times \\
&\times \Bigg((k+2)\Big(2\varphi_e((k-2)x,x)+2\varphi_e((k+2)x,x) \\
&\qquad\qquad + 2\varphi_e(2x,2x)-2\varphi_e(-2x,2x) \\
&\qquad\qquad - (k^2+k)\varphi_e(3x,x)+|k^2-k|\varphi_e(-3x,x) \\
&\qquad\qquad + 2|k^2-1|\varphi_e(-2x,x)\Big) \\
&\qquad + 16\varphi_e(x,2x)+16\varphi_e(kx,x)+16\varphi_e((k+1)x,x) \\
&\qquad + 2|k^3-2k^2-k-6|\varphi_e(2x,x) \\
&\qquad + |k^3+11k^2-6k|\varphi_e(-x,x) \\
&\qquad + |k^3-23k^2-10k-16|\varphi_e(x,x)\Bigg), \\
\nu_2(x) &:= \varphi_e(2x,x)+2\varphi_e(x,x), \\
\nu_3(x) &:= \frac{1}{|k^4-k^2|}\times \\
&\times \Big(|4k^2-3|\varphi_e(x,x)+2k^2\varphi_e(2x,x)+2k^2\varphi_e(x,2x) \\
&\qquad + 2\varphi_e((k+1)x,x)+2\varphi_e((k-1)x,x)+k^2\varphi_e(2x,2x) \\
&\qquad + \varphi_e(x,3x)+\varphi_e((2k+1)x,x)+\varphi_e((2k-1)x,x)\Big), \\
\nu_4(x) &:= \varphi_e(x,x)+5\varphi_e(0,x), \quad \nu_5(x) := \varphi_e(2x,x)+2\varphi_e(0,x).
\end{aligned}
$$

Proof. Some somewhat long and tedious calculations yield the following equalities:

$$
\begin{aligned}
&f_e(2x) - 2^2 f_e(x)\\
&\quad = \frac{1}{4k^3-4k}\Bigg((k-2)\bigg(D_1f_e(x,2x) - D_1f_e((k+1)x,x) - \frac{k^2+k}{2D_1f_e(2x,x)}\bigg)\\
&\qquad\qquad - (k+2)\bigg(D_1f_e(x,-2x) - D_1f_e((k-1)x,x) - \frac{k^2-k}{2D_1f_e(-2x,x)}\bigg)\\
&\qquad\qquad - 4D_1f_e(kx,x) + (2k^3+k^2-k-2)D_1f_e(-x,x)\\
&\qquad\qquad + (k^3-4k^2-3k+2)D_1f_e(x,x)\Bigg)\\
&\quad = -\frac{D_2f_e(0,x)}{2}\\
&\quad = \frac{D_3f_e(kx,x) + k^2D_3f_e(x,x) - \frac{1}{2}D_3f_e(0,2x) - (k^2-1)D_3f_e(0,x)}{k^4-k^2}\\
&\quad = \frac{D_4f_e(0,x) + D_4f_e(-x,x)}{4}\\
&\quad = \frac{3D_5f_e(0,x) - 2D_5f_e(-x,x)}{10}\\
&f_o(4x) - 10f_o(2x) + 16f_o(x)\\
&\quad = \frac{1}{(k^4-k^2)(k+2)}\Bigg((k+2)\Big(2D_1f_o((k-2)x,x) - 2D_1f_o((k+2)x,x)\\
&\qquad\qquad + 2D_1f_o(2x,2x) - 2D_1f_o(-2x,2x)\\
&\qquad\qquad - (k^2+k)D_1f_o(3x,x) + (k^2-k)D_1f_o(-3x,x)\\
&\qquad\qquad - 2(k^2-1)D_1f_o(-2x,x)\Big)\\
&\qquad\qquad + 16D_1f_o(x,2x) + 16D_1f_o(kx,x) - 16D_1f_o((k+1)x,x)\\
&\qquad\qquad + 2(k^3-2k^2-k-6)D_1f_o(2x,x)\\
&\qquad\qquad + (k^3+11k^2-6k)D_1f_o(-x,x)\\
&\qquad\qquad - (k^3-23k^2-10k-16)D_1f_o(x,x)\Bigg)\\
&\quad = D_2f_o(2x,x) + 2D_2f_o(x,x)\\
&\quad = \frac{1}{k^4-k^2}\Big((4k^2-3)D_3f_o(x,x) - 2k^2D_3f_o(2x,x) + 2k^2D_3f_o(x,2x)\\
&\qquad\qquad - 2D_3f_o((k+1)x,x) + 2D_3f_o((k-1)x,x) - k^2D_3f_o(2x,2x)\\
&\qquad\qquad + D_3f_o(x,3x) - D_3f_o((2k+1)x,x) + D_3f_o((2k-1)x,x)\Big)\\
&\quad = D_4f_o(x,x) + 5D_4f_o(0,x)
\end{aligned}
$$

and

$$f_o(4x) - 10f_o(2x) + 16f_o(x) = D_5f_o(2x,x) - 2D_5f_o(0,x)$$

for all $x \in V$. Thus, we can easily obtain the inequalities in (2.2). □

Recall that Y is a real Banach space.

Lemma 2.4. *Given $m \in \{1,2,3,4,5\}$, assume that a mapping $f : V \to Y$ satisfies $f(0) = 0$ and $D_mf(x,y) = 0$ for all $x,y \in V$. Then the equalities*

$$f_1(2x) = 2f_1(x), \quad f_e(2x) = 4f_e(x), \quad f_3(2x) = 8f_3(x) \tag{2.4}$$

are true for all $x \in V$, where

$$f_o(x) := \frac{f(x) - f(-x)}{2}, \qquad f_e(x) := \frac{f(x) + f(-x)}{2},$$
$$f_1(x) := \frac{8f_o(x) - f_o(2x)}{6}, \quad f_3(x) := -\frac{2f_o(x) - f_o(2x)}{6}.$$

Proof. According to inequalities of (2.2), we obtain the equalities

$$f_e(2x) - 4f_e(x) = 0 \quad \text{and} \quad f_o(4x) - 10f_o(2x) + 16f_o(x) = 0$$

for all $x \in V$. We can easily derive equalities of (2.4) from the last equalities. □

Using [11, Theorems 4.1–4.4] for the case of $a = 2$ and $n = 2$, we can prove the following theorems.

Theorem 2.5. *Let $m \in \{1,2,3,4,5\}$ be fixed and let $\varphi : V \times V \to [0,\infty)$ be a function satisfying the condition*

$$\sum_{i=0}^{\infty} \frac{\varphi(2^ix, 2^iy)}{2^i} < \infty \tag{2.5}$$

for all $x,y \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and inequality (2.1) for all $x,y \in V$, then there exists a unique mapping $F : V \to Y$ satisfying

$$D_mF(x,y) = 0 \tag{2.6}$$

for all $x,y \in V$ and

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \left(\frac{\mu_m(2^ix)}{4^{i+1}} + \frac{(4^{i+1}-1)\nu_m(2^ix)}{6 \cdot 8^{i+1}} \right) \tag{2.7}$$

for all $x \in V$.

Proof. On account of Lemma 2.3, the following inequalities

$$\|f_e(2x) - 4f_e(x)\| \leq \mu_m(x) \quad \text{and} \quad \|f_o(4x) - 10f_o(2x) + 16f_o(x)\| \leq \nu_m(x)$$

hold for all $x \in V$. Due to [11, Theorems 3.1], there exists a unique mapping $F : V \to Y$ satisfying equality (2.6) for all $x, y \in V$, inequality (2.7) for all $x \in V$, and equalities of (2.4) for all $x \in V$. Since equalities of (2.4) can be derived from (2.6), we conclude that there exists a unique mapping $F : V \to Y$ satisfying equality (2.6) for all $x, y \in V$ and inequality (2.7) for all $x \in V$. □

Theorem 2.6. *Let $m \in \{1, 2, 3, 4, 5\}$ be fixed and let $\varphi : V \times V \to [0, \infty)$ be a function satisfying the condition*

$$\sum_{i=0}^{\infty} 8^i \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty \tag{2.8}$$

for all $x, y \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and inequality (2.1) *for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ satisfying equality* (2.6) *for all $x, y \in V$ and*

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \left(4^i \mu_m\left(\frac{x}{2^{i+1}}\right) + \frac{8^{i+1} - 2^{i+1}}{6} \nu_m\left(\frac{x}{2^{i+2}}\right)\right) \tag{2.9}$$

for all $x \in V$.

Proof. By Lemma 2.3, we obtain

$$\|f_e(2x) - 4f_e(x)\| \leq \mu_m(x) \quad \text{and} \quad \|f_o(4x) - 10f_o(2x) + 16f_o(x)\| \leq \nu_m(x)$$

for all $x \in V$. By [11, Theorems 3.2], there exists a unique mapping $F : V \to Y$ satisfying equality (2.6) for all $x, y \in V$ and inequality (2.9) for all $x \in V$. □

Theorem 2.7. *Let $m \in \{1, 2, 3, 4, 5\}$ be fixed and let $\varphi : V \times V \to [0, \infty)$ be a function satisfying the conditions*

$$\sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y)}{4^i} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} 2^i \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty \tag{2.10}$$

for all $x, y \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and inequality (2.1) *for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ satisfying equality* (2.6) *for all $x, y \in V$ and*

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \frac{\mu_m(2^i x)}{4^{i+1}} + \frac{1}{6} \sum_{i=0}^{\infty} \left(\frac{\nu_m(2^i x)}{8^{i+1}} + 2^i \nu_m\left(\frac{x}{2^{i+1}}\right)\right) \tag{2.11}$$

for all $x \in V$.

Proof. Using Lemma 2.3, we have

$$\|f_e(2x) - 4f_e(x)\| \le \mu_m(x) \quad \text{and} \quad \|f_o(4x) - 10f_o(2x) + 16f_o(x)\| \le \nu_m(x)$$

for all $x \in V$. On account of [11, Theorems 3.3], there exists a unique mapping $F : V \to Y$ satisfying equality (2.6) for all $x, y \in V$ and inequality (2.11) for all $x \in V$. □

Recall that Y is a real Banach space.

Theorem 2.8. *Let $m \in \{1, 2, 3, 4, 5\}$ be fixed and let $\varphi : V \times V \to [0, \infty)$ be a function satisfying the conditions*

$$\sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y)}{8^i} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} 4^i \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty \tag{2.12}$$

for all $x, y \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and inequality (2.1) *for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ satisfying equality* (2.6) *for all $x, y \in V$ as well as inequality*

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \left(4^i \mu_m\left(\frac{x}{2^{i+1}}\right) + \frac{1}{6}\left(\frac{\nu_m(2^i x)}{8^{i+1}} + 2^i \nu_m\left(\frac{x}{2^{i+1}}\right)\right)\right) \tag{2.13}$$

for all $x \in V$.

Proof. In view of Lemma 2.3, the following inequalities

$$\|f_e(2x) - 4f_e(x)\| \le \mu_m(x) \quad \text{and} \quad \|f_o(4x) - 10f_o(2x) + 16f_o(x)\| \le \nu_m(x)$$

hold for all $x \in V$. Due to [11, Theorems 3.4], there exists a unique mapping $F : V \to Y$ satisfying equality (2.6) for any $x, y \in V$ and inequality (2.11) for each $x \in V$. □

Lemma 2.9. *Given an $m \in \{6, 7\}$ and a function $\varphi : V^3 \to [0, \infty)$, assume that a mapping $f : V \to Y$ satisfies $f(0) = 0$ and*

$$\|D_m f(x, y, z)\| \le \varphi(x, y, z) \tag{2.14}$$

for all $x, y, z \in V$. Then inequalities of (2.2) *are true for all $x \in V$, where $\mu_m, \nu_m : V \to \mathbb{R}$ are defined by*

$$\begin{aligned}
&\mu_6(x) := \frac{2\varphi_e(0, x, x)}{2}, \\
&\nu_6(x) := \varphi_e(2x, x, x) + 4\varphi_e(x, x, x), \\
&\mu_7(x) := \varphi_e\left(x, \frac{x}{2}, \frac{x}{2}\right) + 2\varphi_e\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right), \\
&\nu_7(x) := \varphi_e(2x, x, x) + 2\varphi_e(x, x, x)
\end{aligned} \tag{2.15}$$

for all $x \in V$.

Proof. Since the equalities

$$f_e(2x) - 2^2 f_e(x) = \frac{D_6 f_e(0,x,x)}{2} = D_7 f_e\left(x, \frac{x}{2}, \frac{x}{2}\right) + 2D_7 f_e\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right)$$

and

$$\begin{aligned} f_o(4x) - 10 f_o(2x) + 16 f_o(x) &= D_6 f_o(2x,x,x) + 4D_6 f_o(x,x,x) \\ &= D_7 f_o(2x,x,x) + 2D_7 f_o(x,x,x) \end{aligned}$$

are true for any $x \in V$, we can easily obtain inequalities in (2.2). □

Lemma 2.10. *Given an $m \in \{6,7\}$ and a function $\varphi : V^3 \to [0,\infty)$, assume that a mapping $f : V \to Y$ satisfies $f(0) = 0$ and $D_m f(x,y,z) = 0$ for all $x,y,z \in V$. Then equalities in* (2.4) *are true for all $x \in V$.*

Proof. Due to inequalities of (2.2), we get

$$f_e(2x) - 4f_e(x) = 0 \quad \text{and} \quad f_o(4x) - 10f_o(2x) + 16f_o(x) = 0$$

for any $x \in V$.

Hence, we can derive equalities of (2.4) from the above equalities. □

By applying [11, Theorems 4.1–4.4] for the case of $a = 2$ and $n = 3$, we can prove the following theorems.

Theorem 2.11. *Let $m \in \{6,7\}$ be fixed and let $\varphi : V^3 \to [0,\infty)$ be a function satisfying the condition*

$$\sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y, 2^i z)}{2^i} < \infty$$

for all $x,y,z \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and inequality (2.14) *for all $x,y,z \in V$, then there exists a unique mapping $F : V \to Y$ satisfying*

$$D_m F(x,y,z) = 0 \tag{2.16}$$

for all $x,y,z \in V$ as well as inequality (2.7) *for all $x \in V$.*

Again, Y is a real Banach space.

Theorem 2.12. *Let $m \in \{6,7\}$ be fixed and let $\varphi : V^3 \to [0,\infty)$ be a function satisfying the condition*

$$\sum_{i=0}^{\infty} 8^i \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}, \frac{z}{2^i}\right) < \infty$$

for all $x,y,z \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and inequality (2.14) *for all $x,y,z \in V$, then there exists a unique mapping $F : V \to Y$ satisfying equality* (2.16) *for all $x,y,z \in V$ and inequality* (2.9) *for all $x \in V$.*

Theorem 2.13. *Let $m \in \{6,7\}$ be fixed and let $\varphi : V^3 \to [0,\infty)$ be a function satisfying the conditions*

$$\sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y, 2^i z)}{4^i} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} 2^i \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}, \frac{z}{2^i}\right) < \infty$$

for all $x, y, z \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and inequality (2.14) for all $x, y, z \in V$, then there exists a unique mapping $F : V \to Y$ satisfying equality (2.16) for all $x, y, z \in V$ and inequality (2.11) for all $x \in V$.

Theorem 2.14. *Let $m \in \{6,7\}$ be fixed and let $\varphi : V^3 \to [0,\infty)$ be a function satisfying the conditions*

$$\sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y, 2^i z)}{8^i} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} 4^i \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}, \frac{z}{2^i}\right) < \infty$$

for all $x, y, z \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and inequality (2.14) for all $x, y, z \in V$, then there exists a unique mapping $F : V \to Y$ satisfying equality (2.16) for all $x, y \in V$ and inequality (2.13) for all $x \in V$.

3. Discussion

Subjects similar to those covered in this paper have been studied previously (see [3, 5, 6, 4, 2]). However, in these works, the proof of the uniqueness of the exact solution was not possible, because the stability of the related functions was proved separately for the even and odd parts. We feel that the uniqueness of the solution is important, and that the division into even and odd parts is somewhat unnatural.

In this paper, we were able to prove the stability of the related function in an integrated way, without dividing the related function into even and odd parts. This allows us to prove the uniqueness of the exact solution. We see this a significant improvement over the results of [3, 5, 6, 4, 2].

Because of space constraints, let us look at only one example that uses the results of this paper. If we put $m = 7$, $n = 2$, $c_1 = 1$, $c_2 = -1$, $c_3 = -2$, $c_4 = 2$, $c_5 = 6$, $c_6 = 2$, $c_7 = -2$, $a_{11} = 1$, $a_{12} = 2$, $a_{21} = 1$, $a_{22} = -2$, $a_{31} = 1$, $a_{32} = 1$, $a_{41} = 1$, $a_{42} = -1$, $a_{51} = 0$, $a_{52} = 1$, $a_{61} = 0$, $a_{62} = -1$, $a_{71} = 0$, $a_{72} = 2$ in (1.1) in [11], then the expression (1.1) in [11] becomes $D_2 f(x,y)$ in this paper. We have proved the (generalized) Hyers-Ulam stability of the functional equation $D_2 f(x,y) = 0$ in Theorem 2.5 of this paper. If we set $\varphi(x,y) = \varepsilon > 0$, then it follows from Theorem 2.5 that there exists a unique function $F : V \to Y$ satisfying $D_2 f(x,y) = 0$ for all $x, y \in V$ and

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \left(\frac{\mu_2(2^i x)}{4^{i+1}} + \frac{(4^{i+1}-1)\nu_2(2^i x)}{6 \cdot 8^{i+1}} \right) = \frac{25}{42}\varepsilon$$

for all $x \in V$. In the text of Lemma 2.3, we can see the definitions of μ_2 and ν_2.

References

[1] J. Baker, *A general functional equation and its stability*, Proc. Amer. Math. Soc., **133(6)**(2005), 1657–1664.

[2] M. E. Gordji, M. Ghanifard, H. Khodaei and C. Park, *Fixed points and the random stability of a mixed type cubic, quadratic and additive functional equation*, J. Comput. Anal. Appl., **15**(2013), 612–621.

[3] M. E. Gordji, S. K. Gharetapeh, J. M. Rassias and S. Zolfaghari, *Solution and stability of a mixed type additive, quadratic, and cubic functional equation*, Adv. Difference Equ., (2009), Art. ID 826130, 17 pages.

[4] M. E. Gordji, M. Kamyar, H. Khodaei, D. Y. Shin and C. Park, *Fuzzy stability of generalized mixed type cubic, quadratic, and additive functional equation*, J. Inequal. Appl., (2011), 2011:95, 22 pp.

[5] M. E. Gordji and H. Khodaie, *Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces*, Nonlinear Anal., **71**(2009), 5629–5643.

[6] M. E. Gordji, M. B. Savadkouhi and Th. M. Rassias, *Stability of generalized mixed type additive-quadratic-cubic functional equation in non-Archimedean spaces*, (2009), preprint, arXiv:0909.5692.

[7] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184**(1994), 431–436.

[8] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA, **27**(1941), 222–224.

[9] K.-W. Jun and H.-M. Kim, *On the Hyers-Ulam-Rassias stability of a general cubic functional equation*, Math. Inequal. Appl., **6**(2003), 289–302.

[10] Y.-H. Lee, *On the generalized Hyers-Ulam stability of the generalized polynomial function of degree* 3, Tamsui Oxf. J. Math. Sci., **24(4)**(2008), 429–444.

[11] Y.-H. Lee, S.-M. Jung and M. Th. Rassias, *Uniqueness theorems on functional inequalities concerning cubic-quadratic-additive equation*, J. Math. Inequal., **12**(2018), 43–61.

[12] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72**(1978), 297–300.

[13] S. M. Ulam, *Problems in modern mathematics*, Wiley & Sons, New York, 1964.