

Hardy Spaces of Certain Convolution Operator

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ABSTRACT. In this article, we determine sufficient conditions on the parameters of a generalized convolution operator to ensure that it belongs to the Hardy space and to the space of bounded analytic functions. We exhibit the utility of these results by deducing several interesting examples.

1. Introduction

Let \mathcal{H} denote the class of analytic functions in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$, and \mathcal{A} be the subclass of \mathcal{H} consisting of normalized analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{D}).$$

For the analytic functions f and g , we say that f is subordinate to g and write $f \prec g$, if there exists an analytic function w in \mathbb{D} such that $w(0) = 0$ and $f(z) = g(w(z))$. In particular, if g is univalent in \mathbb{D} , then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{D}) \iff [f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D})].$$

Further, for functions $f_j \in \mathcal{A}$, given by $f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n$ ($j = 1, 2$), the Hadamard product (or Convolution) of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) := z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n \quad (z \in \mathbb{D}).$$

Let $\mathcal{R}(\beta)$ denote the class of functions $f \in \mathcal{A}$ such that $\Re(f'(z)) > \beta$ ($z \in \mathbb{D}$, $\beta < 1$), and let $\mathcal{P}(\beta)$ denote the class of functions $f \in \mathcal{H}$ such that $f(0) = 1$ and

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$\Re(f(z)) > \beta$ ($z \in \mathbb{D}$, $\beta < 1$). For $\beta = 0$, we denote $\mathcal{R}(\beta)$ and $\mathcal{P}(\beta)$ simply by \mathcal{R} and \mathcal{P} , respectively.

Let \mathcal{H}^∞ denote the space of all bounded analytic functions in \mathbb{D} . This is Banach algebra with respect to the norm $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$. For $f \in \mathcal{H}$, we define

$$(1.1) \quad M_p(r, f) = \begin{cases} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} & (0 < p < \infty), \\ \max_{|z| \leq r} |f(z)| & (p = \infty). \end{cases}$$

The function $f \in \mathcal{H}$ belongs to the Hardy space \mathcal{H}^p ($0 < p \leq \infty$), if $M_p(r, f)$ is bounded for all $r \in [0, 1)$. Clearly, we have

$$\mathcal{H}^\infty \subset \mathcal{H}^q \subset \mathcal{H}^p \quad \text{for } 0 < p < q < \infty$$

(see [3, p. 2]). For $1 \leq p \leq \infty$, \mathcal{H}^p is a Banach space with the norm defined by

$$(1.2) \quad \|f\|_p = \lim_{r \rightarrow 1^-} M_p(r, f) \quad (1 \leq p \leq \infty)$$

(see [3, p. 23]). Following are two widely known results (see [8]) for the Hardy space \mathcal{H}^p :

$$(1.3) \quad \begin{aligned} \Re(f'(z)) > 0 &\implies f' \in \mathcal{H}^p \text{ for all } p < 1 \\ &\implies f \in \mathcal{H}^{q/(1-q)} \text{ for all } 0 < q < 1. \end{aligned}$$

In [10], Ponnusamy studied the Hardy space of hypergeometric functions. Further, Baricz [1] obtained the conditions for the generalized Bessel functions such that it belongs to Hardy space and Yagmur and Orhan [16] studied the same problem for the generalized Struve functions.

In [6] (see also [7]), Ibrahim studied the following generalized fractional integral operator in the complex plane \mathbb{C} :

$$(1.4) \quad I_z^{\alpha, \mu} f(z) = \frac{(\mu + 1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^z (z^{\mu+1} - \xi^{\mu+1})^{\alpha-1} \xi^\mu f(\xi) d\xi \quad (\alpha, \mu \in \mathbb{R}, \alpha > 0, \mu \neq -1),$$

where the function $f(z)$ is analytic in a simply connected region of \mathbb{C} containing the origin, and the multiplicity of $(z^{\mu+1} - \xi^{\mu+1})^{\alpha-1}$ is removed by requiring $\log(z^{\mu+1} - \xi^{\mu+1})$ to be real when $(z^{\mu+1} - \xi^{\mu+1}) > 0$. We observe that, if we take $\mu = 0$ in (1.4), we arrive at the *Srivastva-Owa* fractional integral operator [13] (see also [11]) and when $\mu \rightarrow -1^+$ in (1.4), we obtain the *Hadamard* fractional integral operator [5].

For $\alpha > 0$ and $\mu > -1$, we define a fractional integral operator $\Omega_z^{\alpha, \mu} : \mathcal{A} \rightarrow \mathcal{A}$

by

$$\begin{aligned}
 \Omega_z^{\alpha,\mu} f(z) &= \frac{(\mu + 1)^\alpha \Gamma\left(\frac{1}{\mu+1} + \alpha + 1\right)}{\Gamma\left(\frac{1}{\mu+1} + 1\right)} z^{-\alpha(\mu+1)} I_z^{\alpha,\mu} f(z) \\
 (1.5) \quad &= z + \sum_{n=2}^{\infty} \frac{\Gamma\left(\frac{1}{\mu+1} + \alpha + 1\right) \Gamma\left(\frac{n}{\mu+1} + 1\right)}{\Gamma\left(\frac{n}{\mu+1} + \alpha + 1\right) \Gamma\left(\frac{1}{\mu+1} + 1\right)} a_n z^n.
 \end{aligned}$$

Note that $\Omega_z^{0,0} f(z) = f(z)$, $\Omega_z^{1,0} f(z) = \mathcal{L}f(z)$ and $\Omega_z^{\alpha,0} f(z) = L(2, \alpha + 2)f(z)$, where \mathcal{L} and $L(a, c)$ denotes the Libera integral operator [9] and Carlson-Shaffer operator [2], respectively. Also, we observe that the operator $\Omega_z^{\alpha,\mu}$ satisfies the following recurrence relation

$$(1.6) \quad z(\Omega_z^{\alpha,\mu} f(z))' = (1 + \alpha(\mu + 1))\Omega_z^{\alpha-1,\mu} f(z) - \alpha(\mu + 1)\Omega_z^{\alpha,\mu} f(z) \quad (\alpha > 1, \mu > -1).$$

Corresponding to fractional integral operator $\Omega_z^{\alpha,\mu} f(z)$, a fractional differential operator $\Phi^{\alpha,\mu} f(z)$ was studied in [6, 7] to obtain its boundedness in Bergman space and certain geometric properties were also discussed.

2. Main Results

In order to derive our main results, we recall here the following lemmas:

Lemma 2.1.([15]) *Let F and G be analytic functions in \mathbb{D} and $F(0) = G(0)$. If $H(z) = zG'(z)$ is starlike function in \mathbb{D} and $zF'(z) \prec zG'(z)$, then*

$$F(z) \prec G(z) = G(0) + \int_0^z \frac{H(t)}{t} dt.$$

Lemma 2.2.([14]) *For $\alpha < 1$, $\beta < 1$, we have $\mathcal{P}(\alpha) * \mathcal{P}(\beta) \subset \mathcal{P}(\delta)$, where $\delta = 1 - 2(1 - \alpha)(1 - \beta)$. The value of δ is best possible.*

Our first main result is given by Theorem 2.3 below.

Theorem 2.3 *Let $\alpha > 1$ and $\mu > -1$. If $f(z) \in \mathcal{R}$, then $\Omega_z^{\alpha,\mu} f(z) \in \mathcal{R} \cap \mathcal{H}^\infty$.*

Proof. From the definition of operator $\Omega_z^{\alpha,\mu} f(z)$, we have

$$(2.1) \quad \Re\{(\Omega_z^{\alpha,\mu} f(z))'\} = \frac{(\mu + 1)^\alpha \Gamma\left(\frac{1}{\mu+1} + \alpha + 1\right)}{\Gamma(\alpha)\Gamma\left(\frac{1}{\mu+1} + 1\right)} \int_0^1 u^\mu (1 - u^{\mu+1})^{\alpha-1} \Re\{f'(zu)\} du.$$

By hypothesis $f(z) \in \mathcal{R}$, hence it follows that $\Omega_z^{\alpha,\mu} f(z) \in \mathcal{R}$. By first implication of (1.3), we have $(\Omega_z^{\alpha,\mu} f(z))' \in \mathcal{H}^q$ for all $q < 1$. Further, by second implication of (1.3), we have $\Omega_z^{\alpha,\mu} f \in \mathcal{H}^{q/(1-q)}$ for all $0 < q < 1$, or equivalently, $\Omega_z^{\alpha,\mu} f \in \mathcal{H}^p$ for all $0 < p < \infty$.

Since $f \in \mathcal{R}$, then using the well known bound for *Caratheodory functions* in \mathbb{D} , we have $|a_n| \leq 2/n$, $n \geq 2$. Hence

$$(2.2) \quad |\Omega_z^{\alpha, \mu} f(z)| \leq |z| + \frac{\Gamma(\frac{1}{\mu+1} + \alpha + 1)}{\Gamma(\frac{1}{\mu+1} + 1)} \sum_{n=2}^{\infty} \frac{\Gamma(\frac{n}{\mu+1} + 1)}{\Gamma(\frac{n}{\mu+1} + \alpha + 1)} |a_n| |z|^n \\ < 1 + 2 \sum_{n=2}^{\infty} \theta(n),$$

where $\theta(n)$ is given by

$$(2.3) \quad \theta(n) = \frac{\Gamma(\frac{1}{\mu+1} + \alpha + 1) \Gamma(\frac{n}{\mu+1} + 1)}{n \Gamma(\frac{n}{\mu+1} + \alpha + 1) \Gamma(\frac{1}{\mu+1} + 1)}.$$

Evidently, $\theta(1) = 1$ and $\theta(n) > 0$ for all $n \in \mathbb{N}$. It is well known that the digamma function $\Psi(x) = \Gamma'(x)/\Gamma(x)$ is increasing for all $x > 0$ (see [12, II, sec. 3/ eqn. 4 on p. 723]). Therefore $\Psi(x + \varepsilon) > \Psi(x)$ for $\varepsilon > 0$. The auxiliary function $\tilde{\Gamma}(x) := \Gamma(x + \varepsilon)/\Gamma(x)$ has a positive derivative as

$$(2.4) \quad \tilde{\Gamma}'(x) = \frac{\Gamma'(x + \varepsilon)\Gamma(x) - \Gamma(x + \varepsilon)\Gamma'(x)}{[\Gamma(x)]^2} \\ = \frac{\Gamma(x + \varepsilon)}{\Gamma(x)} \left(\frac{\Gamma'(x + \varepsilon)}{\Gamma(x + \varepsilon)} - \frac{\Gamma'(x)}{\Gamma(x)} \right) > 0 \quad (x > 0, \varepsilon > 0).$$

Thus $\tilde{\Gamma}(x)$ is an increasing function, so that

$$(2.5) \quad \frac{\Gamma(x + \varepsilon)}{\Gamma(x)} \geq \frac{\Gamma(y + \varepsilon)}{\Gamma(y)} \quad \text{whenever } x \geq y > 0.$$

Hence

$$\frac{\theta(n)}{\theta(n+1)} = \frac{(n+1) \Gamma(1 + \frac{n}{\mu+1}) \Gamma(1 + \alpha + \frac{n+1}{\mu+1})}{n \Gamma(1 + \frac{n+1}{\mu+1}) \Gamma(1 + \alpha + \frac{n}{\mu+1})} \geq 1,$$

which shows that $0 < \theta(n+1) < \theta(n) \leq \theta(2)$ for each $n \geq 2$. Hence $\theta(n)$ is a non-increasing function of $n \geq 2$.

Now, we shall show that $\lim_{n \rightarrow \infty} |\theta(n)|^{1/n} = 1$. Using the asymptotic formula for the Gamma function [4, sec. 1.18, (4)], we have

$$\lim_{n \rightarrow \infty} |\theta(n)|^{1/n} = \lim_{n \rightarrow \infty} \left[\frac{\Gamma(1 + \alpha + \frac{1}{\mu+1})}{\Gamma(1 + \frac{1}{\mu+1})} \right]^{1/n} \cdot \lim_{n \rightarrow \infty} \left[\frac{\Gamma(1 + \frac{n}{\mu+1})}{n \Gamma(1 + \alpha + \frac{n}{\mu+1})} \right]^{1/n} \\ \sim \lim_{n \rightarrow \infty} \left[\left(\frac{n}{\mu+1} \right)^{-\alpha} \right]^{1/n} = \lim_{n \rightarrow \infty} (n^{1/n})^{-\alpha} \left[\frac{1}{(\mu+1)^\alpha} \right]^{1/n} = 1.$$

This implies that the series in (2.2) converges for $|z| < 1$. Further, applying Raabe's test for convergence, we deduce that the power series for $\Omega_z^{\alpha,\mu} f$ converges absolutely for $|z| = 1$ for all $\alpha > 0$ and $\mu > -1$.

Also, using the known result [3, Theorem 3.11], $(\Omega_z^{\alpha,\mu} f)' \in \mathcal{H}^q$ implies continuity of $\Omega_z^{\alpha,\mu} f$ on the compact set \mathbb{D} . Since the continuous function $\Omega_z^{\alpha,\mu} f$ on the compact set \mathbb{D} is bounded, thus $\Omega_z^{\alpha,\mu} f$ is a bounded analytic function in \mathbb{D} . Therefore $\Omega_z^{\alpha,\mu} f \in \mathcal{H}^\infty$ and this completes the proof. \square

Remark 2.4. If we consider $\Psi(z) = -z - 2\log(1 - z) = z + 2\sum_{n=2}^\infty \frac{z^n}{n}$, then from the geometrical descriptions of image domains of functions in the class \mathcal{R} , we can easily verify that $\Psi \in \mathcal{R}$ and not belongs to \mathcal{H}^∞ , but as per the Theorem 2.3, function $\Omega_z^{\alpha,\mu}\Psi(z) \in \mathcal{H}^\infty \cap \mathcal{R}$, where

$$\Omega_z^{\alpha,\mu}\Psi(z) = z + 2 \sum_{n=2}^\infty \frac{\Gamma\left(\frac{1}{\mu+1} + \alpha + 1\right) \Gamma\left(\frac{n}{\mu+1} + 1\right)}{n \Gamma\left(\frac{n}{\mu+1} + \alpha + 1\right) \Gamma\left(\frac{1}{\mu+1} + 1\right)} z^n \quad (\alpha > 1, \mu > -1, z \in \mathbb{D}).$$

Theorem 2.5 *Let $f \in \mathcal{A}, \alpha > 1$ and $\mu > -1$. If*

$$(2.6) \quad \left| \frac{\Omega_z^{\alpha-1,\mu} f(z) - \Omega_z^{\alpha,\mu} f(z)}{z} \right| < \frac{1}{1 + \alpha(\mu + 1)} \quad (z \in \mathbb{D}),$$

then $\frac{\Omega_z^{\alpha,\mu} f(z)}{z} \in \mathcal{P}$.

Proof. The inequality (2.6) is equivalent to

$$\frac{\Omega_z^{\alpha-1,\mu} f(z) - \Omega_z^{\alpha,\mu} f(z)}{z} \prec \frac{z}{1 + \alpha(\mu + 1)} \quad (z \in \mathbb{D}),$$

which in view of (1.6), can be written as $z \left(\frac{\Omega_z^{\alpha,\mu} f(z)}{z} \right)' \prec z(1+z)'$. Now applying

Lemma 2.1, for $F(z) = \frac{\Omega_z^{\alpha,\mu} f(z)}{z}$ and $G(z) = 1 + z$, the desired result follows. \square

Theorem 2.6. *Let $f \in \mathcal{A}, \alpha > 1$ and $\mu > -1$. If $g \in \mathcal{R}(1/2)$ and $\Omega_z^{\alpha,\mu} f$ satisfy the inequality (2.6), then $\Omega_z^{\alpha,\mu} f(z) * g(z) \in \mathcal{R}$.*

Proof. Let $u(z) = \Omega_z^{\alpha,\mu} f(z) * g(z)$, then $u'(z) = \frac{\Omega_z^{\alpha,\mu} f(z)}{z} * g'(z)$. In view of the hypotheses and Theorem 2.5, we have $\frac{\Omega_z^{\alpha,\mu} f(z)}{z} \in \mathcal{P}$ and $g'(z) \in \mathcal{P}(1/2)$. Now using Lemma 2.2, we obtain that $u'(z) \in \mathcal{P}$, which is equivalent to $u(z) \in \mathcal{R}$, hence the desired result follows. \square

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References

- [1] Á. Baricz, *Bessel transforms and Hardy space of generalized Bessel functions*, *Mathematica*, **48(71)**(2006), 127–136.
- [2] B. C. Carlson and D. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, *SIAM J. Math. Anal.*, **15**(1984), 737–745.
- [3] P. L. Duren, *Theory of \mathcal{H}^p space*, *Pure and Applied Mathematics* **38**, Academic Press, New York/London, 1970.
- [4] A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions, Vol. I*, McGraw–Hill Book Company, New York, 1953.
- [5] J. Hadamard, *Essai sur l'étude des fonctions données par leur développement de Taylor*, *J. Math. Pures Appl.*, **8**(1892), 101–186.
- [6] R. W. Ibrahim, *On generalized Srivastva-Owa fractional operators in the unit disk*, *Adv. Difference Equ.*, (2011), 2011:55, 10 pp.
- [7] R. W. Ibrahim, *Studies on generalized fractional operators in complex domain*, *Mathematics Without Boundaries*, 273–284, Springer, New York, 2014.
- [8] Y. Komatu, *On a one-parameter additive family of operators defined on analytic functions regular in the unit disk*, *Bull. Fac. Sci. Engrg. Chou Univ.*, **22**(1979), 1–22.
- [9] R. J. Libera, *Some classes of regular univalent functions*, *Proc. Amer. Math. Soc.*, **16**(1965), 755–758.
- [10] S. Ponnusamy, *The Hardy spaces of hypergeometric functions*, *Complex Var. Theory Appl.*, **29(1)**(1996), 83–96.
- [11] J. K. Prajapat, R. K. Raina and J. Sokol, *Dependence conditions for analytic functions under fractional differintegral operators*, *Math. Sci. Res. J.*, **15(12)**(2011), 333–340.
- [12] A. Prudnikov, Yu. Brychkov and O. Marichev, *Integral and series, Vol. 3. More special functions*, Gordon & Breach Science Publishers, New York, 1990.
- [13] H. M. Srivastva and S. Owa, *Univalent functions, fractional calculus and their applications*, John Wiley & Sons, New York, 1989.
- [14] J. Stankiewicz and Z. Stankiewicz, *Some applications of Hadamard convolution in the theory of functions*, *Ann. Univ. Mariae Curie-Skłodowska Sect. A*, **40**(1986), 251–265.
- [15] T. Suffridge, *Some remarks on convex maps of the unit disk*, *Duke Math. J.*, **37(4)**(1970), 775–777.
- [16] N. Yağmur and H. Orhan, *Hardy space of generalized Struve functions*, *Complex Var. Elliptic Equ.*, **59(7)**(2014), 929–936.