

## The Maximal Ideal Space of Extended Differentiable Lipschitz Algebras

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ABSTRACT. In this paper, we first introduce new classes of Lipschitz algebras of infinitely differentiable functions which are extensions of the standard Lipschitz algebras of infinitely differentiable functions. Then we determine the maximal ideal space of these extended algebras. Finally, we show that if  $X$  and  $K$  are uniformly regular subsets in the complex plane, then  $R(X, K)$  is natural.

### 1. Introduction

Let  $X$  be a compact Hausdorff space, and let  $C(X)$  be the Banach algebra of all continuous complex-valued functions on  $X$  under the uniform norm,  $\|f\|_X = \sup_{x \in X} |f(x)|$  for  $f$  in  $C(X)$ . A subalgebra  $A$  of  $C(X)$  which separates the points of  $X$ , contains the constants, and which is a Banach algebra with respect to some norm  $\|\cdot\|$ , is a Banach function algebra on  $X$ . If the norm of a Banach function algebra is equivalent to the uniform norm, then it is a uniform algebra. If  $A$  is a function algebra on  $X$ , then  $\overline{A}$ , the uniform closure of  $A$ , is a uniform function algebra on  $X$ . Let  $A$  be a Banach function algebra on  $X$ . Then for every  $x \in X$  the map  $e_x : f \rightarrow f(x), A \rightarrow \mathbb{C}$ , is a complex homomorphism on  $A$  and it is called the point evaluation homomorphism at  $x$ . A Banach function algebra  $A$  on  $X$  is natural if  $M_A$ , the maximal ideal space of  $A$ , is  $X$ ; equivalently, if every homomorphism on  $A$  is given by evaluation at a point of  $X$ . Define the map  $J : X \rightarrow M_A$  by  $J(x) = e_x$ .

Let  $(X, d)$  be a compact metric space,  $K$  be a compact subset of  $X$  and

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$0 < \alpha \leq 1$ . We define

$$p_{\alpha,K}(f) = \sup\left\{\frac{|f(x) - f(y)|}{d^\alpha(x,y)} : x, y \in K, x \neq y\right\},$$

$$Lip(X, K, \alpha) := \{f \in C(X) : p_{\alpha,K}(f) < \infty\}.$$

The subalgebra of those functions  $f$  in  $Lip(X, K, \alpha)$  such that

$$\lim_{d(x,y) \rightarrow 0} \frac{|f(x) - f(y)|}{d^\alpha(x,y)} = 0, \quad (x, y \in K, x \neq y),$$

is denoted by  $lip(X, K, \alpha)$ . It is easy to see that these extended Lipschitz algebras are both Banach algebras under the norm  $\|f\|_{\alpha,K} = \|f\|_X + p_{\alpha,K}(f)$ . In fact,  $Lip(X, K, \alpha)$  is a natural Banach function algebra on  $X$  for  $\alpha \leq 1$  and  $lip(X, K, \alpha)$  is a natural Banach function algebra on  $X$  whenever  $\alpha < 1$ , [7].

It is interesting to note that  $Lip(X, K, \beta)$  is dense in  $lip(X, K, \alpha)$  for each  $\beta$  where  $\alpha < \beta \leq 1$ . This result is similar to the density of  $Lip(X, \beta)$  in  $lip(X, \alpha)$  [2], using the measure theory and duality. For further general facts about Lipschitz algebras the reader is referred to [1, 9, 10].

Let  $X$  be a perfect, compact plane set. A complex-valued function  $f : X \rightarrow \mathbb{C}$  is complex-differentiable at  $a \in X$  if

$$f'(a) = \lim\left\{\frac{f(z) - f(a)}{z - a} : z \in X, z \rightarrow a\right\}$$

exists. We call  $f'(a)$  the complex derivative of  $f$  at  $a$ . Also, we denote the  $n^{th}$  derivative of  $f$  at  $a \in X$  by  $f^{(n)}(a)$ . Now we introduce the type of compact sets which we shall consider next, [3].

**Definition 1.1.** Let  $K$  be a compact plane set which is connected by rectifiable arcs, and suppose  $\delta(z, w)$  is the geodesic metric on  $K$ , the infimum of the lengths of the arcs joining  $z$  and  $w$ .

- (i)  $K$  is called *regular* if for each  $z_0 \in K$  there exists a constant  $C$  such that for all  $z \in K$ ,  $\delta(z, z_0) \leq C|z - z_0|$ .
- (ii)  $K$  is called *uniformly regular* if there exists a constant  $C$  such that for all  $z, w \in K$ ,  $\delta(z, w) \leq C|z - w|$ .

**Definition 1.2.** Let  $K$  and  $X$  be compact plane sets such that  $K \subseteq X$ . Let  $D^n(X, K)$  be the algebra of continuous functions on  $X$  with continuous  $n^{th}$  derivatives on  $K$ . For  $f \in D^n(X, K)$ , we define the norm by

$$\|f\|_n = \|f\|_X + \sum_{j=1}^n \frac{\|f^{(j)}\|_K}{j!}.$$

Let  $K$  and  $X$  be compact plane sets such that  $K \subseteq X$ . If  $K$  is a finite union of regular sets then the following property is holds:

For each  $z_0 \in K$  there exists a constant  $C$  such that for every  $z \in K$  and  $f \in D^1(X, K)$ ,

$$|f(z) - f(z_0)| \leq C|z - z_0|(\|f\|_X + \|(f|_K)'\|_K).$$

This inequality implies that  $D^1(X, K)$  is complete [3]. Throughout this paper we always assume that  $X$  and  $K$  are nonempty compact plane sets,  $K \subseteq X$  and  $0 < \alpha \leq 1$  for Lip and  $0 < \alpha < 1$  for lip.

**Definition 1.3.** The algebra of complex-valued functions  $f$  on  $X$  whose derivatives up to order  $n$  exists on  $K$  and for each  $j(1 \leq j \leq n)$ ,  $f^{(j)} \in Lip(K, \alpha)$  is denoted by  $Lip^n(X, K, \alpha)$ . The algebra  $lip^n(X, K, \alpha)$  is defined in a similar way. For  $f$  in  $Lip^n(X, K, \alpha)$  or in  $lip^n(X, K, \alpha)$ , let

$$\begin{aligned} \|f\|_{\alpha, K, n} &= \|f\|_X + p_{\alpha, K}(f) + \sum_{j=1}^n \frac{\|f^{(j)}\|_{\alpha, K}}{j!} \\ &= \|f\|_X + p_{\alpha, K}(f) + \sum_{j=1}^n \frac{\|f^{(j)}\|_K + p_{\alpha, K}(f^{(j)})}{j!}. \end{aligned}$$

The algebra of functions  $f$  in  $Lip(X, K, \alpha)(lip(X, K, \alpha))$  with derivatives of all orders for which  $f^{(j)} \in Lip(K, \alpha)(f^{(j)} \in lip(K, \alpha))$  for all  $j(j = 1, 2, 3, \dots)$  is denoted by  $Lip^\infty(X, K, \alpha)(lip^\infty(X, K, \alpha))$ . Now, we introduce certain subalgebra of  $Lip^\infty(X, K, \alpha)$  and  $lip^\infty(X, K, \alpha)$  as follows:

Let  $M = \{M_j\}_{j=0}^\infty$  be a sequence of positive numbers such that

$$M_0 = 1 \quad \text{and} \quad \frac{M_j}{M_r \cdot M_{j-r}} \geq \binom{j}{r} \quad (r = 0, 1, \dots, j).$$

Whenever we refer to  $M = \{M_j\}_{j=0}^\infty$  we mean this sequence satisfies the above conditions.

**Definition 1.4.** Let

$$Lip(X, K, M, \alpha) = \{f \in Lip^\infty(X, K, \alpha) : (\|f\|_X + p_{\alpha, K}(f) + \sum_{j=1}^\infty \frac{\|f^{(j)}\|_{\alpha, K}}{M_j}) < \infty\},$$

$$lip(X, K, M, \alpha) = \{f \in lip^\infty(X, K, \alpha) : (\|f\|_X + p_{\alpha, K}(f) + \sum_{j=1}^\infty \frac{\|f^{(j)}\|_{\alpha, K}}{M_j}) < \infty\}.$$

For  $f$  in  $Lip(X, K, M, \alpha)$  or in  $lip(X, K, M, \alpha)$ , set

$$\|f\| = \|f\|_X + p_{\alpha, K}(f) + \sum_{j=1}^\infty \frac{\|f^{(j)}\|_{\alpha, K}}{M_j}.$$

For convenience, we regard  $Lip^n(X, K, \alpha)$  and  $lip^n(X, K, \alpha)$  as being algebras of the type  $Lip(X, , K, M, \alpha)$  and  $lip(X, K, M, \alpha)$ , respectively, by setting  $M_j = j!$  ( $j = 0, 1, 2, \dots, n$ ) and  $\frac{1}{M_j} = 0$  ( $j = n + 1, \dots$ ). In this paper, we determine the maximal ideal of  $Lip(X, K, M, \alpha)$  and  $lip(X, K, M, \alpha)$ . Finally by using a interesting method we obtain the maximal ideal of new classes of uniform algebras generated by rational functions, which defined by T. G. Honary and S. Moradi in [8].

## 2. The Maximal Ideal Space of Extended Differentiable Lipschitz Algebras

We show that, the completeness of  $D^1(X, K)$  implies that  $Lip^n(X, K, \alpha)$  and  $lip^n(X, K, \alpha)$  are Banach function algebras.

**Theorem 2.1.** *Let  $X$  be a compact plane set which is connected by rectifiable arcs and sequence  $M = \{M_k\}_{k=0}^{\infty}$  satisfies the condition above and  $D^1(X, K)$  complete under norm  $\|f\|_1 = \|f\|_X + \|(f|_K)'\|_K$ . Then  $Lip(X, K, M, \alpha)$  or  $lip(X, K, M, \alpha)$  is Banach functions algebra under norm  $\|f\| = \|f\|_X + p_{\alpha, K}(f) + \sum_{j=1}^{\infty} \frac{\|f^{(j)}\|_{\alpha, K}}{M_j}$  on  $X$ .*

*Proof.* Obviously  $Lip(X, K, M, \alpha)$ , is normed algebra and contains the constant functions and separates the points of  $X$ . Now, we prove that the algebra  $Lip(X, K, M, \alpha)$  is complete. Let  $\{f_n\}_{n=1}^{\infty}$  be a Cauchy sequence in  $Lip(X, K, M, \alpha)$ , so  $\{f_n\}_{n=1}^{\infty}$  is a Cauchy sequence in  $Lip(X, K, \alpha)$ . By completeness of  $Lip(X, K, \alpha)$ , there exists  $f \in Lip(X, K, \alpha)$  such that  $\|f_n - f\|_X + p_{\alpha, K}(f_n - f) \rightarrow 0$  as  $n \rightarrow \infty$ . On the other hand  $\{f_n^{(j)}\}_{n=1}^{\infty}$  is a Cauchy sequence in  $Lip(K, \alpha)$  for all  $j(j = 1, 2, 3, \dots)$ . Since  $Lip^n(K, \alpha)$  is complete [5], there exists  $g_j \in Lip(K, \alpha)$  such that  $\|f_n^{(j)} - g_j\|_{\alpha, K} \rightarrow 0$  as  $n \rightarrow \infty$  for all  $j(j = 1, 2, 3, \dots)$ . Consequently by the completeness of  $D^1(X, K)$   $f^{(j)} = g_j$  and  $f^{(j)} \in Lip(K, \alpha)$  for all  $j$ . We show that  $f \in Lip(X, K, M, \alpha)$  and  $\|f_n - f\|_{Lip(X, K, M, \alpha)} \rightarrow 0$  as  $n \rightarrow \infty$ . For  $\varepsilon = 1$  there exists  $N_1 \in \mathbb{N}$  such that for every  $m, n \geq N_1$  and  $i \in \mathbb{N}$ ,

$$\begin{aligned} & (\|f_n - f_m\|_X + p_{\alpha, K}(f_n - f_m) + \sum_{j=1}^i \frac{\|f_n^{(j)} - f_m^{(j)}\|_{\alpha, K}}{M_j}) \\ & \leq (\|f_n - f_m\|_X + p_{\alpha, K}(f_n - f_m) + \sum_{j=1}^{\infty} \frac{\|f_n^{(j)} - f_m^{(j)}\|_{\alpha, K}}{M_j}) < 1. \end{aligned}$$

By letting  $m \rightarrow \infty$  for every  $n \geq N_1$  and  $i \in \mathbb{N}$  we have

$$(\|f_n - f\|_X + p_{\alpha, K}(f_n - f) + \sum_{j=1}^i \frac{\|f_n^{(j)} - f^{(j)}\|_{\alpha, K}}{M_j}) \leq 1.$$

Therefore for every  $n \geq N_1$ ,

$$(\|f_n - f\|_X + p_{\alpha,K}(f_n - f) + \sum_{j=1}^{\infty} \frac{\|f_n^{(j)} - f^{(j)}\|_{\alpha,K}}{M_j}) \leq 1,$$

which implies  $(f_n - f) \in Lip(X, K, M, \alpha)$  and consequently  $f = ((f - f_{N_1}) + f_{N_1}) \in Lip(X, K, M, \alpha)$ . It is clearly  $\|f_n - f\|_{Lip(X, K, M, \alpha)} \rightarrow 0$  as  $n \rightarrow \infty$ . By similarly way the algebra  $lip(X, K, M, \alpha)$  is Banach function algebra.  $\square$

In the following let  $J : X \rightarrow M_{Lip(X, K, M, \alpha)}$  be the evaluation map, defined by  $J(z) = e_z$ , where  $e_z$  is the evaluation homomorphism on  $Lip(X, K, M, \alpha)$ . Let also

$$\widetilde{M}_{Lip(K, M, \alpha)} = \{\tilde{\phi} : \tilde{\phi}(f) = \phi(f|_K), \phi \in M_{Lip(K, M, \alpha)}, f \in Lip(X, K, M, \alpha)\}.$$

It is easy to see that  $\widetilde{M}_{Lip(K, M, \alpha)} \subseteq M_{Lip(X, K, M, \alpha)}$ .

**Theorem 2.2.** *Let  $K$  and  $X$  be perfect compact subsets of the complex plane such that  $K \subseteq X$  and  $Lip(K, M, \alpha)$  is complete. Then  $M_{Lip(X, K, M, \alpha)} = \widetilde{M}_{Lip(K, M, \alpha)} \cup J(X \setminus K)$ .*

*Proof.* By the above, the inclusion  $\widetilde{M}_{Lip(K, M, \alpha)} \cup J(X \setminus K) \subseteq M_{Lip(X, K, M, \alpha)}$  is clear. Conversely, let  $\psi \in M_{Lip(X, K, M, \alpha)}$ . If  $A = \{f \in C(X) : f|_K = 0\}$  then  $C_0(X \setminus K) = A|_{X \setminus K}$  [6] and  $M(A) \cong X \setminus K$ . Moreover,  $A \subseteq Lip(X, K, M, \alpha)$ . There are two cases for  $\psi$  :

**Case 1:**  $\psi|_A = 0$ . In this case we prove that if  $\phi(f|_K) = \psi(f)$  for  $f \in Lip(X, K, M, \alpha)$  then  $\phi \in M_{Lip(K, M, \alpha)}$ . We first show that  $\phi$  is well-defined. Let  $f_1|_K = f_2|_K$ , where  $f_1, f_2 \in Lip(X, K, M, \alpha)$ . Obviously  $f_1 - f_2 \in A$  so  $\psi(f_1 - f_2) = 0$  hence  $\psi(f_1) = \psi(f_2)$ . Therefore,  $\phi$  is well-defined. Clearly,  $\phi$  is a homomorphism and hence  $\phi \in M_{Lip(K, M, \alpha)}$ . It follows that  $\psi = \tilde{\phi} \in \widetilde{M}_{Lip(K, M, \alpha)}$ .

**Case 2:** Let  $\psi|_A \neq 0$ . So  $\psi|_A$  is a non-zero homomorphism on  $A$ . By  $C_0(X \setminus K) = A|_{X \setminus K}$ , there exists an  $z_0 \in X \setminus K$  such that for every  $f \in A$ ,  $\psi(f) = f(z_0)$ . We take an open set  $U$  in  $X$  such that  $K \subset U$  and  $z_0 \in X \setminus U$ . By Urysohn's Lemma there exists a continuous function  $h$  on  $X$  such that  $0 \leq h \leq 1$  and

$$h(z) = \begin{cases} 1 & z \in K \\ 0 & z \in X \setminus U. \end{cases}$$

Obviously  $h \in Lip(X, K, M, \alpha)$  and moreover,

$$\psi(h) = \psi(1 - (1 - h)) = \psi(1) - \psi(1 - h) = 1 - (1 - h)(z_0) = 1 - 1 = 0.$$

So for every  $f \in Lip(X, K, M, \alpha)$ ,

$$\begin{aligned} \psi(f) &= \psi(fh - f(1 - h)) = \psi(fh) + \psi(f(1 - h)) = \psi(f)\psi(h) + f(1 - h)(z_0) \\ &= \psi(f) \cdot 0 + f(z_0) = f(z_0). \end{aligned}$$

Hence  $\psi = e_{z_0}$ , for  $z_0 \in X \setminus K$ . Therefore,  $M_{Lip(X, K, M, \alpha)} = \widetilde{M}_{Lip(K, M, \alpha)} \cup J(X \setminus K)$ .  $\square$

**Corollary 2.3.** *With the same assumptions in the theorem, if moreover,  $Lip(K, M, \alpha)$  is natural then  $Lip(X, K, M, \alpha)$  is also natural.*

*Proof.* By the naturality of  $Lip(K, M, \alpha)$

$$\widetilde{M}_{Lip(K, M, \alpha)} = \{\widetilde{\phi} : \widetilde{\phi}(f) = e_z(f|_K) = f(z), f \in Lip(X, K, M, \alpha)\} = J(K).$$

Hence

$$M_{Lip(X, K, M)} = J(K) \cup J(X \setminus K) = J(X) \cong X.$$

By similarly way the algebra  $lip(X, K, M, \alpha)$  is natural.  $\square$

**Corollary 2.4.** *The algebras  $Lip^n(X, K, \alpha)$  and  $lip^n(X, K, \alpha)$  both are natural.*

**Lemma 2.5.** *Let  $X$  and  $K$  be uniformly regular subsets of the complex plane such that  $K \subseteq X$ . For  $n \geq 0$ ,*

- (i)  $D^{n+1}(X, K) \subseteq Lip^n(X, K, 1) \subseteq lip^n(X, K, \alpha)$ ,
- (ii) *The standard norms of  $D^{n+1}(X, K)$  and  $Lip^n(X, K, 1)$  are equivalent on  $D^{n+1}(X, K)$ ,*
- (iii)  $D^{n+1}(X, K)$  *is close subalgebra of  $Lip^n(X, K, 1)$ .*

*Proof.* (i) Obviously  $Lip^n(X, K, 1) \subseteq lip^n(X, K, \alpha)$ . If  $f \in D^{n+1}(X, K)$  then  $f \in C(X)$  and  $f^{(j)} \in D^1(K)$  for all  $j = 0, 1, 2, \dots, n$ . Since  $K$  is uniformly regular, there exists  $C > 0$  such that for all  $j = 0, 1, 2, \dots, n$  and  $z, w$  in  $K$  we have

$$|f^{(j)}(z) - f^{(j)}(w)| \leq C \|f^{(j+1)}\|_K |z - w|$$

consequently,

$$p_{1,K}(f^{(j)}) = \sup\left\{\frac{|f^{(j)}(z) - f^{(j)}(w)|}{|z - w|} : z, w \in K, z \neq w\right\} \leq C \|f^{(j+1)}\|_K.$$

Hence  $f \in Lip(X, K, 1)$  and  $f^{(j)} \in Lip(K, 1)$  for all  $j = 1, 2, \dots, n$ . It follows  $f \in Lip^n(X, K, 1)$  therefore  $D^{n+1}(X, K) \subseteq Lip^n(X, K, 1)$ .

(ii) Let  $f \in Lip^n(X, K, 1)$ . Then we have

$$\begin{aligned} \|f\|_{Lip^n(X, K, 1)} &= \|f\|_X + p_{1,K}(f) + \sum_{j=1}^n \frac{\|f^{(j)}\|_K + p_{1,K}(f^{(j)})}{j!} \\ &= \|f\|_X + p_{1,K}(f) + \sum_{j=1}^n \frac{\|f^{(j)}\|_K}{j!} + \sum_{j=1}^n \frac{p_{1,K}(f^{(j)})}{j!} \\ &\leq \|f\|_X + \sum_{j=1}^n \frac{\|f^{(j)}\|_K}{j!} + \sum_{j=0}^{n-1} \frac{C \|f^{(j+1)}\|_K}{j!} + \frac{C \|f^{(n+1)}\|_K}{n!} \end{aligned}$$

$$\begin{aligned}
&= \|f\|_X + \sum_{j=1}^n \frac{\|f^{(j)}\|_K}{j!} + \sum_{j=1}^n \frac{C\|f^{(j)}\|_K}{(j-1)!} + \frac{C\|f^{(n+1)}\|_K}{n!} \\
&= \|f\|_X + \sum_{j=1}^n \frac{(1+Cj)\|f^{(j)}\|_K}{j!} + \frac{C\|f^{(n+1)}\|_K}{n!} \\
&\leq (1+(n+1)C)(\|f\|_X + \sum_{j=1}^{n+1} \frac{\|f^{(j)}\|_K}{j!}) = (1+(n+1)C)\|f\|_{D^{n+1}}.
\end{aligned}$$

Now for  $z_0 \in K$  and for all  $z \in K$  which ( $z \neq z_0$ ) and  $f \in D^{n+1}(X, K)$  we have,

$$\frac{|f^{(n)}(z) - f^{(n)}(z_0)|}{|z - z_0|} \leq p(f^{(n)}),$$

and hence,  $|f^{(n+1)}(z_0)| \leq p_{1,K}(f^{(n)})$  and because  $z_0$  is ordinary on  $K$ , thus

$$\|f^{(n+1)}\|_K \leq p_{1,K}(f^{(n)}) \quad (f \in D^{n+1}(X, K)).$$

Hence for all  $f$  in  $D^{n+1}(X, K)$ ,

$$\begin{aligned}
\|f\|_{n+1} &= \|f\|_X + \sum_{j=1}^{n+1} \frac{\|f^{(j)}\|_K}{j!} \\
&= \|f\|_X + \sum_{j=1}^n \frac{\|f^{(j)}\|_K}{j!} + \frac{\|f^{(n+1)}\|_K}{(n+1)!} \\
&\leq \|f\|_X + \sum_{j=1}^{n-1} \frac{\|f^{(j)}\|_K}{j!} + \frac{\|f^{(n)}\|_K}{n!} + \frac{p_{1,K}(f)}{(n+1)!} \\
&= \|f\|_X + \sum_{j=1}^{n-1} \frac{\|f^{(j)}\|_K}{j!} + \frac{(n+1)\|f^{(n)}\|_K + p_{1,K}(f)}{(n+1)!} \\
&\leq \|f\|_X + p_{1,K}(f) + \sum_{j=1}^{n-1} \frac{\|f^{(j)}\|_K}{j!} + (n+1) \frac{\|f^{(n)}\|_K + p_{1,K}(f)}{n!} \\
&\leq (n+1)(\|f\|_X + p_{1,K}(f) + \sum_{j=1}^n \frac{\|f^{(j)}\|_{\alpha,K}}{j!}) \\
&= (n+1)\|f\|_{Lip^n(X, K, 1)}.
\end{aligned}$$

Therefore standard norms of  $D^{n+1}(X, K)$  and  $Lip^n(X, K, 1)$  are equivalent on  $D^{n+1}(X, K)$ .

(iii) It follows immediate from (ii).  $\square$

**Definition 2.6.** Let  $K$  and  $X$  be compact subset of  $\mathbb{C}$  such that  $K \subseteq X$ . We define

$$R(X, K) = \{f \in C(X) : f|_K \in R(K)\}$$

where  $R(K)$  is the uniform closure of  $R_0(K)$ , the algebra of all rational functions with poles off  $K$ , [4, 8].

It is easy to see that  $R(X, K)$  is uniform algebra and hence it is Banach function algebra.

**Theorem 2.7.** Let  $X$  be a compact Hausdorff space, and let  $A$  be a Banach algebra on  $X$ . Then  $M_A$  is homoeomorphic to  $M_{\overline{A}}$  if and only if  $\|\widehat{f}\| = \|f\|_X$  for all  $f \in A$ .

*Proof.* See [5].  $\square$

**Lemma 2.8.** Let  $n \geq 0$  and  $K$  and  $X$  be uniformly regular subsets of the complex plane such that  $K \subseteq X$ . Then  $\overline{D^{n+1}(X, K)} = \overline{\text{lip}^n(X, K, \alpha)} = \overline{\text{Lip}^n(X, K, \alpha)} = R(X, K)$ .

*Proof.* For  $n \geq 0$ , we have

$$D^{n+1}(X, K) \subseteq \text{lip}^n(X, K, \alpha) \subseteq D^1(X, K) \subseteq R(X, K),$$

and it is also known that

$$R_0(X, K) \subseteq D^1(X, K) \subseteq R(X, K),$$

therefore

$$\overline{D^{n+1}(X, K)} = \overline{\text{lip}^n(X, K, \alpha)} = \overline{\text{Lip}^n(X, K, \alpha)} = R(X, K),$$

and the proof is complete.  $\square$

**Corollary 2.9.** Let  $K$  and  $X$  be uniformly regular subsets of the complex plane such that  $K \subseteq X$ . Then  $R(X, K)$  is natural.

*Proof.* Straightforward calculations show that

$$\|(f^m)^{(j)}\|_K \leq 2^{n^2} \delta^n m^n \|f^m\|_K^{m-n}, \quad (0 \leq j \leq n)$$

$$p_{\alpha, K}((f^m)^{(j)}) \leq 2^{n^2} \delta^{n-1} m^{n+1} \lambda \|f^m\|_K^{m-n}, \quad (0 \leq j \leq n)$$

for all  $m > n$ , where  $\delta$  and  $\lambda$  are constants independent of  $m$  and  $j$ . Therefore for  $f \in \text{Lip}^n(X, K, \alpha)$  we have,

$$\begin{aligned} \|f^m\|_{\alpha, K, n} &= \|f^m\|_X + p_{\alpha, K}(f^m) + \sum_{j=1}^n \frac{\|(f^m)^{(j)}\|_{\alpha, K}}{j!}, \\ &= \|f^m\|_X + p_{\alpha, K}(f^m) + \sum_{j=1}^n \frac{\|(f^m)^{(j)}\|_K + p_{\alpha, K}((f^m)^{(j)})}{j!}, \end{aligned}$$



$$\begin{aligned}
&\leq \|f^m\|_X + \sum_{j=1}^n \frac{1}{j!} 2^{n^2} \delta^n m^n \|f^m\|_K^{m-n} + \sum_{j=0}^n \frac{1}{j!} 2^{n^2} \delta^{n-1} m^{n+1} \lambda \|f^m\|_K^{m-n}, \\
&\leq \|f\|_X^m + \|f^m\|_K^{m-n} ((n+2)2^{n^2} \delta^{n-1} m^n) (\delta + \lambda m), \\
&\leq \|f\|_X^m + \|f^m\|_X^{m-n} ((n+2)2^{n^2} \delta^{n-1} m^n) (\delta + \lambda m), \\
&\leq \|f\|_X^m (1 + (n+2)2^{n^2} \delta^{n-1} m^n) (\delta + \lambda m),
\end{aligned}$$

hence

$$\|f^m\|_{\alpha, K, n}^{\frac{1}{m}} \leq (\|f\|_X^m)^{\frac{1}{m}} (1 + (n+2)2^{n^2} \delta^{n-1} m^n)^{\frac{1}{m}} (\delta + \lambda m)^{\frac{1}{m}}.$$

Therefore  $\|\widehat{f}\| = \lim_{m \rightarrow \infty} \|f^m\|_{\alpha, K, n}^{\frac{1}{m}} \leq \|f\|$ . By Theorem 2.7 and Lemma 2.8 completes the proof of corollary.  $\square$

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