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Fractional-Order Derivatives and Integrals: Introductory Overview and Recent Developments

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Abstract. The subject of fractional calculus (that is, the calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past over four decades, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of mathematical, physical, engineering and statistical sciences. Various operators of fractional-order derivatives as well as fractional-order integrals do indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables. The main object of this survey-cum-expository article is to present a brief elementary and introductory overview of the theory of the integral and derivative operators of fractional calculus and their applications especially in developing solutions of certain interesting families of ordinary and partial fractional "differintegral" equations. This general talk will be presented as simply as possible keeping the likelihood of *non-specialist* audience in mind.

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1. Introduction, Notations and Preliminaries

Throughout this presentation, we denote by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}_0^-$, and $\mathbb{N}$ the sets of complex numbers, real numbers, real and positive numbers, non-positive integers, and positive integers, respectively.

Fractional calculus, the differentiation and integration of arbitrary (real or complex) order, arises naturally in various areas of science and engineering. For example, very recently, Wang and Zhang [104] investigated a class of nonlinear fractional-order differential impulsive systems with the Hadamard derivative (see also [103, 105, 112]).

The concept of *fractional calculus* (that is, calculus of integrals and derivatives of any arbitrary real or complex order) seems to have stemmed from a question raised in the year 1695 by Marquis de l'Hôpital (1661–1704) to Gottfried Wilhelm Leibniz (1646–1716), which sought the meaning of Leibniz's (currently popular) notation

$$\frac{d^n y}{dx^n}$$

for the derivative of order $n \in \mathbb{N}_0 := \{0, 1, 2, \ ...\}$ when $n = \frac{1}{2}$ (What if $n = \frac{1}{2}$?). In his reply, dated 30 September 1695, Leibniz wrote to l'Hôpital as follows:

> "... *This is an apparent paradox from which, one day, useful consequences will be drawn.* ..."

Subsequent mention of fractional derivatives was made, in some context or the other, by (for example) Euler in 1730, Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Liouville in 1832, Riemann in 1847, Greer in 1859, Holmgren in 1865, Grünwald in 1867, Letnikov in 1868, Laurent in 1884, Nekrassov in 1888, Krug in 1890, and Weyl in 1917. In fact, in his **700-page** textbook, entitled "*Traité du Calcul Différentiel et du Calcul Intégral*" (Second edition; Courcier, Paris, 1819), S. F. Lacroix devoted two pages (pp. 409-410) to fractional calculus, showing eventually that

$$\frac{d^{\frac{1}{2}}}{dv^{\frac{1}{2}}}\, v = \frac{2\sqrt{v}}{\sqrt{\pi}}.$$

In addition, of course, to the theories of differential, integral, and integro-differential equations, and special functions of mathematical physics as well as their extensions and generalizations in one and more variables, some of the areas of present-day applications of fractional calculus include

1. Fluid Flow
2. Rheology
3. Dynamical Processes in Self-Similar and Porous Structures
4. Diffusive Transport Akin to Diffusion
5. Electrical Networks
6. Probability and Statistics
7. Control Theory
8. Viscoelasticity
9. Electrochemistry of Corrosion
10. Chemical Physics
11. Dynamical Systems
12. Mathematical Bio-Sciences

and so on (see, for details, [64, 27, 31]).

The very first work, devoted exclusively to the subject of fractional calculus, is the book by Oldham and Spanier [63]; it was published in the year 1974. Ever since then a significantly large number of books and monographs, edited volumes, and conference proceedings have appeared and continue to appear rather frequently. And, today, there exist **at least eight** international scientific research journals which are devoted *almost entirely* to the subject of fractional calculus and its widespread applications.

2. The Riemann-Liouville and Weyl Operators of Fractional Calculus

We begin by defining the linear integral operators $\mathcal{I}$ and $\mathcal{K}$ by

$$(\mathcal{I}f)(x) := \int_0^x f(t)\,dt \tag{2.1}$$

and

$$(\mathcal{K}f)(x) := \int_x^\infty f(t)\,dt, \tag{2.2}$$

respectively. Then it is easily seen by iteration (and the principle of mathematical induction) that

$$(\mathcal{I}^n f)(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)\,dt \qquad (n \in \mathbb{N}) \tag{2.3}$$

and

$$(\mathcal{K}^n f)(x) = \frac{1}{(n-1)!}\int_x^\infty (t-x)^{n-1} f(t)\,dt \qquad (n \in \mathbb{N}), \tag{2.4}$$

where, just as elsewhere in this presentation,

$$\mathbb{N} := \{1, 2, 3, \cdots\} = \mathbb{N}_0 \setminus \{0\} \qquad (\mathbb{N}_0 := \{0, 1, 2, \cdots\})$$

and

$$\mathbb{Z} := \{0, \pm 1, \pm 2, \cdots\} = \mathbb{N} \cup \mathbb{Z}_0^- \quad \text{and} \quad \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\} \qquad \bigl(\mathbb{Z}^- := \{-1, -2, -3, \cdots\}\bigr).$$

The familiar (Euler's) Gamma function $\Gamma(z)$ which is defined, for $z \in \mathbb{C} \setminus \mathbb{Z}_0^-$, by

$$\Gamma(z) = \begin{cases} \displaystyle\int_0^\infty e^{-t}\, t^{z-1}\, dt & \bigl(\Re(z) > 0\bigr) \\ \dfrac{\Gamma(z+n)}{\prod\limits_{j=0}^{n-1} (z+j)} & \bigl(z \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ n \in \mathbb{N}\bigr) \end{cases}$$

happens to be one of the most fundamental and the most useful special functions of mathematical analysis. It emerged essentially from an attempt by Euler to give a meaning to $x!$ when x is any positive real number, who in 1729 undertook the problem of interpolating $n!$ between the positive integer values of n.

Historically, the origin of the above-defined Gamma function $\Gamma(z)$ can be traced back to two letters from Leonhard Euler (1707–1783) to Christian Goldbach (1690–1764), elaborating upon a simple desire to extend the factorials to values between the integers. The first letter (dated October 13, 1729) dealt with the interpolation problem, while the second letter (dated January 8, 1730) dealt with integration and tied the contents of the two letters together.

Thus, since

$$\Gamma(1) = 1 \qquad \text{and} \qquad \Gamma(z+1) = z\Gamma(z) \qquad (z \in \mathbb{C} \setminus \mathbb{Z}_0^-)$$

so that, obviously,

$$\Gamma(n) = (n-1)(n-2)(n-3) \cdots 3 \cdot 2 \cdot 1 \cdot \Gamma(1) =: (n-1)! \qquad (n \in \mathbb{N}),$$

with a view to interpolating $(n-1)!$ between the *positive* integer values of n, one can set

$$(n-1)! = \Gamma(n) \tag{2.5}$$

in terms of the Gamma function. Thus, in general, Equations (2.3) and (2.4) would lead us *eventually* to the familiar Riemann-Liouville operator $\mathcal{R}^\mu$ and the Weyl operator $\mathcal{W}^\mu$ of fractional integral of order μ $(\mu \in \mathbb{C})$, defined by (*cf.*, *e.g.*, Erdélyi *et al.* [19, Chapter 13])

$$(\mathcal{R}^\mu f)(x) := \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t)\, dt \qquad \bigl(\Re(\mu) > 0\bigr) \tag{2.6}$$

and

$$(\mathcal{W}^\mu f)(x) := \frac{1}{\Gamma(\mu)} \int_x^\infty (t-x)^{\mu-1} f(t)\, dt \qquad \bigl(\Re(\mu) > 0\bigr), \tag{2.7}$$

respectively, it being *tacitly* assumed that the function $f(t)$ is so constrained that the integrals in (2.6) and (2.7) exist.

In the remarkably vast literature on fractional calculus and its fairly widespread applications, there are potentially useful operators of fractional derivatives $\mathcal{D}^{\mu}_{x;0}$ and $\mathcal{D}^{\mu}_{x;\infty}$ of order μ $(\mu \in \mathbb{C})$, which correspond to the above-defined fractional integral operators $\mathcal{R}^{\mu}$ and $\mathcal{W}^{\mu}$, respectively, and we have

$$\left(\mathcal{D}^{\mu}_{x;0} f\right)(x) := \frac{d^m}{dx^m}\left(\mathcal{R}^{m-\mu} f\right)(x) \tag{2.8}$$

$$\big(m-1 \leqq \Re(\mu) < m; \quad m \in \mathbb{N}\big)$$

and

$$\left(\mathcal{D}^{\mu}_{x;\infty} f\right)(x) := \frac{d^m}{dx^m}\left(\mathcal{W}^{m-\mu} f\right)(x) \tag{2.9}$$

$$\big(m-1 \leqq \Re(\mu) < m; \quad m \in \mathbb{N}\big).$$

There also exist, in the considerably extensive literature on the theory and applications of fractional calculus, numerous *further* extensions and generalizations of the operators $\mathcal{R}^{\mu}$, $\mathcal{W}^{\mu}$, $\mathcal{D}^{\mu}_{x;0}$, and $\mathcal{D}^{\mu}_{x;\infty}$, each of which we have chosen to introduce here for the sake of the *non-specialists* in this subject.

Now, for the Riemann-Liouville fractional derivative operator $\mathcal{D}^{\mu}_{x;0}$ defined by (2.8), it is easily seen from (2.6) that

$$\mathcal{D}^{\mu}_{x;0}\left\{x^{\lambda}\right\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}\, x^{\lambda-\mu} \qquad \big(\Re(\lambda) > -1\big). \tag{2.10}$$

Thus, upon setting $\lambda = 1$ and $\mu = \frac{1}{2}$, this last formula (2.10) readily yields

$$\mathcal{D}^{\frac{1}{2}}_{x;0}\{x\} = \frac{\Gamma(2)}{\Gamma\left(\frac{3}{2}\right)}\, x^{\frac{1}{2}}. \tag{2.11}$$

Observing that

$$\Gamma(2) = 1 \cdot \Gamma(1) = 1 \qquad \text{and} \qquad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2},$$

since

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

the fractional derivative formula (2.10) assumes the following simple form:

$$\mathcal{D}^{\frac{1}{2}}_{x;0}\{x\} = \frac{2\sqrt{x}}{\sqrt{\pi}}. \tag{2.12}$$

In fact, it is the fractional derivative formula (2.12) in its equivalent form:

$$\frac{d^{\frac{1}{2}}}{dv^{\frac{1}{2}}}\, v = \frac{2\sqrt{v}}{\sqrt{\pi}},$$

which was derived in two pages (pp. 409–410) by S. F. Lacroix in his **700-page** textbook, entitled "*Traité du Calcul Différentiel et du Calcul Intégral*" (Second edition; Courcier, Paris, 1819).

3. Initial-Value Problems Based Upon Fractional Calculus

If we define, as usual, the Laplace transform operator $\mathcal{L}$ by

$$\mathcal{L}\{f(t):s\} := \int_0^\infty e^{-st} f(t)\, dt =: F(s), \tag{3.1}$$

provided that the integral exists, for the Riemann-Liouville fractional derivative operator $\mathcal{D}_{t;0}^{\mu}$ of order μ, we have

$$\mathcal{L}\left\{\left(\mathcal{D}_{t;0}^{\mu}\, f\right)(t):s\right\} = s^{\mu} F(s) - \sum_{k=0}^{n-1} s^k \left.\left(\mathcal{D}_{t;0}^{\mu-k-1}\right) f(t)\right|_{t=0} \tag{3.2a}$$

$$\left(n-1 \leqq \Re(\mu) < n; \quad n \in \mathbb{N}\right).$$

On the other hand, for the nth ordinary derivative $f^{(n)}(t)$ $(n \in \mathbb{N}_0)$, it is well known that

$$\mathcal{L}\left\{f^{(n)}(t):s\right\} = s^n F(s) - \sum_{k=0}^{n-1} s^k \left. f^{(n-k-1)}(t)\right|_{t=0} \qquad (n \in \mathbb{N}_0) \tag{3.2b}$$

or, equivalently,

$$\mathcal{L}\left\{f^{(n)}(t):s\right\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1}\, f^{(k)}(0+) \qquad (n \in \mathbb{N}_0), \tag{3.2c}$$

where, *and in what follows*, an *empty* sum is to be interpreted as nil.

Upon comparing the Laplace transform formulas (3.2a) and (3.2b), we see that such initial values as those occurring in (3.2a) are usually *not* interpretable physically in a given initial-value problem. This situation is overcome at least partially by making use of the so-called *Liouville-Caputo fractional derivative* which was introduced in the earlier work published in **1832** by Joseph Liouville (1809-1882) [49, p. 10] *and* which arose recently in several important works, dated 1969 onwards, by Michele Caputo (see, for details, [64, p. 78 *et seq.*]; see also [31, p. 90 *et seq.*]).

In many recent works, especially in the theory of viscoelasticity and in hereditary solid mechanics, the following definition of Liouville (1832) and Caputo (1969) is adopted for the fractional derivative of order $\alpha > 0$ of a *causal* function $f(t)$ (*i.e.*, $f(t) = 0$ for $t < 0$):

$$\text{(3.3)}\quad \frac{d^\alpha}{dt^\alpha}\, f(t) := \begin{cases} f^{(n)}(t) & (\alpha = n \in \mathbb{N}_0) \\[2ex] \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau & (n-1 < \alpha < n;\ n \in \mathbb{N}), \end{cases}$$

where $f^{(n)}(t)$ denotes the usual (ordinary) derivative of order n and Γ is the Gamma function occurring already in (2.6) and (2.7). One can apply the above notion in order to generalize some basic topics of classical mathematical physics, which are treated by simple, linear, ordinary or partial, differential equations, since [*cf.* Equation (3.2a) *and* Definition (3.3)]

(3.4)

$$\mathcal{L}\left\{ \frac{d^\alpha}{dt^\alpha} f(t) : s \right\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0+) \quad (n-1 < \alpha \leqq n;\ n \in \mathbb{N}_0),$$

which, just as the Laplace transform formulas (3.2b) or (3.2c), is obviously *more* suited for initial-value problems than the Laplace transform formula (3.2a). See, for details, Gorenflo *et al.* [24], Podlubny [64] and Kilbas *et al.* [31].

In the theory of ordinary differential equations, the following first- and second-order differential equations:

$$\frac{dy}{dt} + cy = 0 \quad (c > 0) \qquad \text{and} \qquad \frac{d^2y}{dt^2} + cy = 0 \quad (c > 0)$$

are usually referred to as the *relaxation equation* and the *oscillation equation*, respectively. On the other hand, in the theory of partial differential equations, the following partial differential equations:

$$\frac{\partial^2 u}{\partial x^2} = k\frac{\partial u}{\partial t} \quad (k > 0) \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} = k\frac{\partial^2 u}{\partial t^2} \quad (k > 0)$$

are known as the *diffusion* (*or heat*) *equation* and the *wave equation*, respectively.

The basic processes of relaxation, diffusion, oscillations and wave propagation have been generalized by several authors by introducing fractional derivatives in the governing (ordinary or partial) differential equations. This leads to *superslow* or *intermediate processes* that, in mathematical physics, we may refer to as *fractional phenomena*. Our analysis of each of these phenomena, carried out by means of fractional calculus and Laplace transforms, leads to certain special functions in one variable of Mittag-Leffler and Fox-Wright types. These useful special functions are investigated systematically as relevant cases of the general class of functions which are popularly known as Fox's H-function after Charles Fox (1897-1977) who initiated a detailed study of these functions as symmetrical Fourier kernels (see, for details, Srivastava *et al.* [87, 88]).

We choose to summarize below some recent investigations by Gorenflo *et al.* [24] who did indeed make references to numerous earlier *closely-related* works on this subject.

I. The Fractional (Relaxation-Oscillation) Ordinary Differential Equation

$$\frac{d^\alpha u}{dt^\alpha} + c^\alpha \; u\,(t;\alpha) = 0 \qquad (c > 0;\; 0 < \alpha \leqq 2) \tag{3.5}$$

Case I.1: Fractional Relaxation $(0 < \alpha \leqq 1)$

Initial Condition: $u\,(0+;\alpha) = u_0$

Case I.2: Fractional Oscillation $(1 < \alpha \leqq 2)$

Initial Conditions: $u\,(0+;\alpha) = u_0$

$\dot{u}\,(0+;\alpha) = v_0$

with $v_0 \equiv 0$ for continuous dependence of the solution on the parameter α also in the transition from $\alpha = 1-$ to $\alpha = 1+$.

Explicit Solution (in *both* cases):

$$\begin{aligned} u\,(t;\alpha) &= u_0 \; E_\alpha\big(-(ct)^\alpha\big) \\ &= u_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\alpha n + 1)}\,(ct)^{\alpha n} \\ &= \begin{cases} u_0\left(1 - \dfrac{(ct)^\alpha}{\Gamma(1+\alpha)}\right) \;\approx\; u_0 \exp\left(-\dfrac{(ct)^\alpha}{\Gamma(1+\alpha)}\right) & (t \to 0+) \\[2ex] \dfrac{u_0}{(ct)^\alpha \Gamma(1-\alpha)} & (t \to \infty), \end{cases} \end{aligned}$$

where $E_\alpha\,(z)$ denotes the familiar Mittag-Leffler function defined by (*cf.*, *e.g.*, Srivastava and Kashyap [88, p. 42, Equation II.5 (23)])

$$E_\alpha\,(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\,(\alpha n + 1)} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} \frac{\zeta^{\alpha-1} e^{\zeta}}{\zeta^\alpha - z}\,d\zeta \qquad (\alpha > 0;\; z \in \mathbb{C})\,.$$

II. The Fractional (Diffusion-Wave) Partial Differential Equation

(3.6)
$$\frac{\partial^{2\beta} u}{\partial t^{2\beta}} = k\,\frac{\partial^2 u}{\partial x^2}$$
$$(k > 0;\ \ -\infty < x < \infty;\ 0 < \beta \leqq 1),$$

where $u = u(x, t; \beta)$ is assumed to be a *causal* function of time $(t > 0)$ *with*

$$u(\mp\infty, t; \beta) = 0.$$

Case II.1: Fractional Diffusion $\left(0 < \beta \leqq \frac{1}{2}\right)$

Initial Condition: $u(x, 0+; \beta) = f(x)$

Case II.2: Fractional Wave $\left(\frac{1}{2} < \beta \leqq 1\right)$

Initial Conditions: $u(x, 0+; \beta) = f(x)$

$\dot{u}(x, 0+; \beta) = g(x)$

with $g(x) \equiv 0$ for continuous dependence of the solution on the parameter β also in the transition from $\beta = \frac{1}{2}-$ to $\beta = \frac{1}{2}+$.

Explicit Solution (in *both* cases):

(3.7)
$$u(x, t; \beta) = \int_{-\infty}^{\infty} \mathcal{G}_c(\xi, t; \beta)\ f(x - \xi)\, d\xi,$$

where the Green function $\mathcal{G}_c(x, t; \beta)$ is given by

(3.8)
$$|x|\, \mathcal{G}_c(x, t; \beta) = \frac{z}{2} \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(1 - \beta - \beta n)} \qquad \left(z = \frac{|x|}{\sqrt{k}t^{\beta}};\ 0 < \beta < 1\right),$$

which can readily be expressed in terms of Wright's (generalized Bessel) function $J_\nu^\mu(z)$ defined by (*cf.*, *e.g.*, Srivastava and Kashyap [89, p. 42, Equation II.5(22)])

(3.9)
$$J_\nu^\mu(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(1 + \nu + \mu n)}.$$

4. Fractional Kinetic Equations

During the past several years, fractional kinetic equations of different forms have been widely used in describing and solving several important problems of physics and astrophysics. Saxena *et al.* [75] introduced the solution of the generalized fractional kinetic equation associated with the generalized Mittag-Leffler function. Subsequently, Saxena *et al.* [78] developed an alternative derivation of the generalized fractional kinetic equations in terms of special functions with the Sumudu transform. More recently, Kumar *et al.* [37] gave the solution of a generalized fractional kinetic equation involving the Bessel function of the first kind; Choi and Kumar [17] (see also [35]) presented the solution of the generalized fractional kinetic equations involving the Aleph function. In fact, as observed recently by V. P. Saxena [71], the so-called Aleph function (which was claimed to be a generalization of the familiar I-function) is a redundant variant of the I-function itself. The I-function does indeed provide a generalization of Fox's H-function (see, for details, [20]; see also [11] for the closely-related $\overline{H}$-function). For other results involving various classes of fractional kinetic equations and their solutions, one may refer to such works as (for example) [15, 26, 34, 68, 73, 74, 75, 76, 78, 97]. In particular, Tomovski *et al.* [97] presented the *corrected* version of an obviously erroneous solution of a certain fractional kinetic equation which was given by Saxena and Kalla [73, p. 508, Eq. (3.2)] and also derived the solution of a much more general family of fractional kinetic equations (see, for details, [97, p. 813, Remark 3 *and* Theorem 10]).

Here, in this presentation, we propose to investigate solution of a certain generalized fractional kinetic equation associated with the generalized Mittag-Leffler function (see [72]). It is also pointed out that the result presented here is general enough to be specialized to include many known solutions for fractional kinetic equations.

Fractional kinetic equations have gained popularity during the past decade or so due mainly to the discovery of their relation with the theory of CTRW (Continuous Time Random Walks) in [29]. These equations are investigated in order to determine and interpret certain physical phenomena which govern such processes as diffusion in porous media, reaction and relaxation in complex systems, anomalous diffusion, and so on (see also [28, 36]).

Consider an arbitrary reaction characterized by a time-dependent quantity $N = N(t)$. It is possible to calculate the rate of change $\frac{dN}{dt}$ to be a balance between the destruction rate $\mathfrak{d}$ and the production rate $\mathfrak{p}$ of N, that is,

$$\frac{dN}{dt} = -\mathfrak{d} + \mathfrak{p}.$$

In general, through feedback or other interaction mechanism, destruction and production depend on the quantity N itself, that is,

$$\mathfrak{d} = \mathfrak{d}(N) \qquad \text{and} \qquad \mathfrak{p} = \mathfrak{p}(N).$$

This dependence is complicated, since the destruction or the production at a time t depends not only on $N(t)$, but also on the past history $N(\eta)$ $(\eta < t)$ of the variable N. This may be formally represented by the following equation (see [26]):

$$\frac{dN}{dt} = -\mathfrak{d}\left(N_t\right) + \mathfrak{p}\left(N_t\right), \tag{4.1}$$

where N_t denotes the function defined by

$$N_t\left(t^*\right) = N\left(t - t^*\right) \qquad (t^* > 0).$$

Haubold and Mathai [26] studied a special case of the equation (4.1) in the following form:

$$\frac{dN_i}{dt} = -c_i\, N_i\left(t\right) \tag{4.2}$$

with the initial condition that $N_i\left(t = 0\right) = N_0$ is the number density of species i at time $t = 0$ and the constant $c_i > 0$. This is known as a standard kinetic equation. The solution of the equation (4.2) is easily seen to be given by

$$N_i\left(t\right) = N_0\, e^{-c_i t}. \tag{4.3}$$

Integration gives an alternative form of the equation (4.2) as follows:

$$N\left(t\right) - N_0 = c \cdot\ {}_0D_t^{-1} N\left(t\right), \tag{4.4}$$

where ${}_0D_t^{-1}$ is the standard integral operator and c is a constant.

The fractional-calculus generalization of the equation (4.4) is given as in the following form (see [26]):

$$N\left(t\right) - N_0 = c^{\nu}\, {}_0D_t^{-\nu} N\left(t\right), \tag{4.5}$$

where ${}_0D_t^{-\nu}$ is the familiar Riemann-Liouville fractional integral operator (see, *e.g.*, [31, 52]; see also [14]) defined by

$${}_0D_t^{-\nu}\, f(t) = \frac{1}{\Gamma\left(\nu\right)} \int_0^t \left(t - u\right)^{\nu - 1} f\left(u\right) du \qquad \big(\Re\left(\nu\right) > 0\big). \tag{4.6}$$

In terms of the generalized Bessel function $\omega_{l,b,c}\left(t\right)$ of the first kind, Kumar *et al.* [37] studied the following equation:

$$N\left(t\right) - N_0\ \omega_{l,b,c}\left(t\right) = -\mathfrak{d}^{\nu}\, {}_0D_t^{-\nu}\, N\left(t\right), \tag{4.7}$$

whose solution is given by

$$N\left(t\right) = N_0 \sum_{k=0}^{\infty} \frac{\left(-c\right)^k \Gamma\left(2k + l + 1\right)}{k!\, \Gamma\left(l + k + \frac{b+1}{2}\right)} \left(\frac{t}{2}\right)^{2k+l} \quad E_{\nu, 2k+l+1}\left(-\mathfrak{d}^{\nu} t^{\nu}\right), \tag{4.8}$$

where $E_{\nu,2k+l+1}(\cdot)$ is the above-mentioned generalized Mittag-Leffler function (see [53, 111]; see also [82]).

Srivastava and Tomovski [96] introduced the following generalization of the Mittag-Leffler function:

$$E_{\alpha,\beta}^{\gamma,\kappa}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} \tag{4.9}$$

$$\big(\alpha, \beta, \gamma \in \mathbb{C};\ \Re(\alpha) > \max\{0, \Re(\kappa) - 1\};\ \min\{\Re(\beta), \Re(\kappa)\} > 0\big),$$

where, in terms of the Gamma function $\Gamma(z)$, the widely-used Pochhammer symbol $(\lambda)_\nu$ $(\lambda,\ \nu \in \mathbb{C})$ is defined, in general, by (see, for details, [88, 91]; see also [85])

$$\begin{aligned}(\lambda)_\nu &:= \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} \qquad \big(\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-\big) \\ &= \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases}\end{aligned} \tag{4.10}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient in (4.10) exists. A special case of the generalized Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,\kappa}(z)$ when $\kappa = \mathfrak{q} \in (0,1) \cup \mathbb{N}$ was studied earlier by Shukla and Prajapati (see [81]).

Saxena and Nishimoto [77] studied a further generalization of the generalized Mittag-Leffler function (4.9) in the following form:

$$E_{\gamma,\kappa}\left[(\alpha_1,\beta_1),\cdots,(\alpha_m,\beta_m);z\right] = \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}}{\prod_{j=1}^{m}\Gamma(n\alpha_j+\beta_j)} \frac{z^n}{n!} \tag{4.11}$$

$$\left(\alpha_j, \beta_j, \gamma, \kappa \in \mathbb{C};\ \min\{\Re(\kappa), \Re(\alpha_j), \Re(\beta_j)\} > 0 \quad (j \in \{1, \cdots, m\});\right.$$
$$\left.\Re\left(\sum_{j=1}^{m} \alpha_j\right) > \max\{0, \Re(\kappa) - 1\}\right).$$

The special case of (4.11) when $\gamma = \kappa = 1$ reduces to the following multi-index Mittag-Leffler function (see [33]; see also [16]):

$$E_{1,1}\left[(\alpha_1,\beta_1),\cdots,(\alpha_m,\beta_m);z\right] = \sum_{n=0}^{\infty} \frac{z^n}{\prod_{j=1}^{m}\Gamma(n\alpha_j+\beta_j)} \tag{4.12}$$

$$\big(\alpha_j, \beta_j \in \mathbb{C};\ \min\{\Re(\alpha_j), \Re(\beta_j)\} > 0 \quad (j \in \{1, \cdots, m\})\big).$$

The Mittag-Leffler function $E_\alpha(z)$, the generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$, and *all* of their aforementioned extensions and generalizations are obviously contained as special cases in the well-known Fox-Wright function ${}_p\Psi_q$ defined by (see, for details, [88, p. 21]; see also [31, p. 56])

$$\begin{aligned} {}_p\Psi_q\,[z] = {}_p\Psi_q \left[\begin{array}{c} (a_1,\alpha_1)\,,\cdots,(a_p,\alpha_p)\,; \\ \\ (b_1,\beta_1)\,,\cdots,(b_q,\beta_q)\,; \end{array} z \right] &= {}_p\Psi_q \left[\begin{array}{c} (a_i,\alpha_i)_{1,p}\,; \\ \\ (b_j,\beta_j)_{1,q}\,; \end{array} z \right] \\ &= \sum_{n=0}^{\infty} \frac{\prod\limits_{i=1}^{p} \Gamma\left(a_i+\alpha_i n\right)}{\prod\limits_{j=1}^{q} \Gamma\left(b_j+\beta_j n\right)} \frac{z^n}{n!}. \end{aligned} \tag{4.13}$$

Suppose that $f(t)$ is a real- (or complex-) valued function of the (time) variable $t > 0$ and s is a real or complex parameter. The Laplace transform of the function $f(t)$ is defined by

$$\begin{aligned} F\left(s\right) = \mathfrak{L}\left\{f\left(t\right):s\right\} &= \int_0^\infty e^{-st}\, f\left(t\right) dt \\ &= \lim_{\tau\to\infty} \int_0^\tau e^{-st}\, f\left(t\right) dt, \end{aligned} \tag{4.14}$$

whenever the limit exits (as a finite number). The convolution of two functions $f(t)$ and $g(t)$, which are defined for $t > 0$, plays an important rôle in a number of different physical applications. The Laplace convolution of the functions $f(t)$ and $g(t)$ is given by the following integral:

$$(f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau = (g*f)(t), \tag{4.15}$$

which exists if the functions f and g are at least piecewise continuous. One of the very significant properties possessed by the convolution in connection with the Laplace transform is that the Laplace transform of the convolution of two functions is the product of their transforms (see, *e.g.*, [80]).

The Laplace Convolution Theorem. *If f and g are piecewise continuous on $[0,\infty)$ and of exponential order α when $t\to\infty$, then*

$$\mathfrak{L}\left\{(f*g)(t):s\right\} = \mathfrak{L}\left\{f(t):s\right\}\cdot\mathfrak{L}\left\{g(t):s\right\} \qquad \big(\Re(s)>\alpha\big). \tag{4.16}$$

The so-called Sumudu transform is an integral transform which was defined and studied by Watugala [109] to facilitate the process of solving differential and integral equations in the time domain. The Sumudu transform has been used in various applications of system engineering and applied physics. For some fundamental properties of the Sumudu transform, one may refer to the works including (for example) [2, 9, 10, 86, 109]. It turns out that the Sumudu transform has very special properties which are useful in solving problems involving kinetic equations in science and engineering.

Let $\mathfrak{A}$ be the class of exponentially bounded functions $f : \mathbb{R} \to \mathbb{R}$, that is,

$$|f(t)| < \begin{cases} M \exp\left(-\dfrac{t}{\tau_1}\right) & (t \leqq 0) \\[2ex] M \exp\left(\dfrac{t}{\tau_2}\right) & (t \geqq 0), \end{cases} \tag{4.17}$$

where M, τ_1 and τ_2 are some positive real constants. The Sumudu transform defined on the set $\mathfrak{A}$ is given by the following formula (see [109]; see also [17])

$$G(u) = \mathcal{S}[f(t); u] := \int_0^\infty e^{-t} f(ut)\, dt \qquad (-\tau_1 < u < \tau_2). \tag{4.18}$$

The Sumudu transform given in (4.18) can also be derived directly from the Fourier integral. Moreover, it can be easily verified that the Sumudu transform is a linear operator and the function $G(u)$ in (4.18) keeps the same units as $f(t)$; that is, for any real or complex number λ, we have

$$\mathcal{S}[f(\lambda t); u] = G(\lambda u).$$

The Sumudu transform $G(u)$ and the Laplace transform $F(s)$ exhibit a duality relation that may be expressed as follows:

$$G\left(\frac{1}{s}\right) = s\, F(s) \qquad \text{or} \qquad G(u) = \frac{1}{u}\, F\left(\frac{1}{u}\right). \tag{4.19}$$

The Sumudu transform has been shown to be the theoretical dual of the Laplace transform. It is also connected to the s-multiplied Laplace transform (see [51]). The use of the convolution theorem for the Sumudu transform in (4.6) gives us the following identity:

$$\mathcal{S}\left[{}_0D_t^{-\nu} f(t); u\right] = \mathcal{S}\left[\frac{t^{\nu-1}}{\Gamma(\nu)}; u\right] \cdot \mathcal{S}[f(t); u] = u^\nu\, G(u). \tag{4.20}$$

In connection with the definition (4.18), in case the parameter u takes on negative or complex values, the dualities such as those described in (4.19) do not hold true, in general, because (after the change of variables) the contour of integration in the Laplace integral changes accordingly.

In our present investigation, we have chosen to make use of the Sumudu transform instead of the classical Laplace transform. In fact, for the various problems considered here, the Sumudu transform has not only been found to be more convenient to use, but the closed-form results derived here also appear to be remarkably simpler (see also [86]).

Throughout this presentation, it is *tacitly* assumed the all involved complex powers of (for example) complex numbers take on their *principal* values.

5. Solution of Generalized Fractional Kinetic Equations by Using the Laplace Transform

We first find the solution of the generalized fractional kinetic equation involving the generalized Mittag-Leffler function (4.11) by applying the Laplace transform technique. We begin by stating and proving the following lemma.

Lemma 5.1. *Let* $\min\{\Re(\lambda), \Re(\rho), \Re(s)\} > 0$. *Then the following Laplace transform of* $E_{\gamma,\kappa}[(\alpha_1,\beta_1), \cdots, (\alpha_m,\beta_m); z]$ *holds true*:

$$\begin{aligned}\mathfrak{L}\left\{t^{\lambda-1}\, E_{\gamma,\kappa}\left[(\alpha_1,\beta_1), \cdots, (\alpha_m,\beta_m); t^{\rho}\right] : s\right\} \\ = \frac{s^{-\lambda}}{\Gamma(\gamma)}\, {}_2\Psi_m\left[\begin{array}{c} (\gamma,\kappa), (\lambda,\rho); \\ \\ (\beta_j,\alpha_j)_{1,m}; \end{array} s^{-\rho}\right],\end{aligned} \tag{5.1}$$

where ${}_2\Psi_m[\cdot]$ *is the Fox-Wright function given by* (4.13) *and all involved complex powers of* (*for example*) *complex numbers are assumed to take on their principal values.*

Proof. Using the definition (4.14) of the Laplace transform and (4.11), we can obtain the result (5.1). In the course of the proof, the interchange of the order of integration and summation can be justified under the stated conditions. □

For later convenience, a special case of (5.1) when $\lambda = \beta_1$ and $\rho = \alpha_1$ is given in Lemma 5.2 below.

Lemma 5.2. *The following formula holds true for* $\min\{\Re(s), \Re(\alpha_1), \Re(\beta_1)\} > 0$:

$$\begin{aligned}\mathfrak{L}\left[t^{\beta_1-1} E_{\gamma,\kappa}\left\{(\alpha_1,\beta_1), \cdots, (\alpha_m,\beta_m); t^{\alpha_1}\right] : s\right\} \\ = \frac{s^{-\beta_1}}{\Gamma(\gamma)}\, {}_1\Psi_{m-1}\left[\begin{array}{c} (\gamma,\kappa); \\ \\ (\beta_j,\alpha_j)_{2,m}; \end{array} s^{-\alpha_1}\right],\end{aligned} \tag{5.2}$$

where all involved complex powers of (*for example*) *complex numbers are assumed to take on their principal values.*

Theorem 5.3. *Let* $c, \mathfrak{d}, \nu, \lambda, \rho \in \mathbb{R}^+$. *Also let* $\alpha_j, \beta_j, \gamma, \kappa \in \mathbb{C}$ *with*

$$\min\{\Re(\alpha_j), \Re(\beta_j)\} > 0 \qquad (j \in \{1, \cdots, m\}),$$

$$\Re(\kappa) > 0 \qquad \textit{and} \qquad \Re\left(\sum_{j=1}^{m} \alpha_j\right) > \max\{0, \Re(\kappa) - 1\}.$$

Then the solution of the following generalized fractional kinetic equation:

$$N(t) - N_0\, t^{\lambda-1}\, E_{\gamma,\kappa}\left[(\alpha_1,\beta_1),\cdots,(\alpha_m,\beta_m);\, \mathfrak{d}t^{\rho}\right] = -c^{\nu}\, {}_0D_t^{-\nu}\, N(t) \tag{5.3}$$

is given by

$$N(t) = \frac{N_0\, t^{\lambda-1}}{\Gamma(\gamma)} \sum_{r=0}^{\infty} (-c^{\nu}t^{\nu})^r\; {}_2\Psi_{m+1}\left[\begin{array}{r} (\gamma,\kappa),(\lambda,\rho); \\ (\nu r+\lambda,\rho),(\beta_j,\alpha_j)_{1,m}; \end{array}\; \mathfrak{d}t^{\rho}\right]. \tag{5.4}$$

Proof. Applying the Laplace transform (4.14) to the equation (5.3) and using the identity in Lemma 5.1, we obtain

$$\mathcal{N}(s) = \frac{N_0}{1+(c/s)^{\nu}} \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}}{\prod_{j=1}^{m}\Gamma(n\alpha_j+\beta_j)}\, \frac{\Gamma(\rho n+\lambda)}{s^{\rho n+\lambda}}\, \frac{\mathfrak{d}^n}{n!},$$

where, just as in the definition (4.14),

$$\mathcal{N}(s) := \mathcal{L}\{N(t) : s\}.$$

Using the geometric series:

$$\frac{1}{1+(c/s)^{\nu}} = \sum_{r=0}^{\infty}(-1)^r \left(\frac{c}{s}\right)^{\nu r} \qquad (|s|>c),$$

we find for $|p|>c$ that

$$\mathcal{N}(s) = N_0 \sum_{r=0}^{\infty}(-c^{\nu})^r \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}}{\prod_{j=1}^{m}\Gamma(n\alpha_j+\beta_j)}\, \frac{\Gamma(\rho n+\lambda)}{s^{\rho n+\nu r+\lambda}}\, \frac{\mathfrak{d}^n}{n!}. \tag{5.5}$$

Now, by inverting the Laplace transform on each side of (5.5) and using the following well-known identity:

$$\begin{aligned} &\mathcal{L}\{t^{\nu} : s\} = \frac{\Gamma(\nu+1)}{s^{\nu+1}} \\ &\Longleftrightarrow \mathcal{L}^{-1}\left(\frac{1}{s^{\nu+1}}\right) = \frac{t^{\nu}}{\Gamma(\nu+1)} \qquad \big(\Re(\nu)>-1;\ \Re(s)>0\big), \end{aligned} \tag{5.6}$$

we get

$$\begin{aligned} N(t) = N_0\, t^{\lambda-1} &\sum_{r=0}^{\infty}(-c^{\nu}t^{\nu})^r \\ &\cdot \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+\kappa n)\,\Gamma(\rho n+\lambda)}{\Gamma(\gamma)\,\Gamma(\rho n+\nu r+\lambda)\prod_{j=1}^{m}\Gamma(n\alpha_j+\beta_j)}\, \frac{(\mathfrak{d}t^{\rho})^n}{n!}, \end{aligned}$$

which, in view of the definition (4.13) of the Fox-Wright function, leads us easily to the right-hand side of (5.4). This completes the proof of Theorem 5.3. □

Theorem 5.4. *Let* $c, \mathfrak{d}, \nu \in \mathbb{R}^+$. *Also let* $\alpha_j, \beta_j, \gamma, \kappa \in \mathbb{C}$ *with*

$$\min\{\Re(\alpha_j), \Re(\beta_j) > 0 \qquad (j \in \{1, \cdots, m\}),$$

$$\Re(\kappa) > 0 \qquad \text{and} \qquad \Re\left(\sum_{j=1}^{m} \alpha_j\right) > \max\{0, \Re(\kappa) - 1\}.$$

Then the solution of the following generalized fractional kinetic equation:

$$N(t) - N_0\, t^{\beta_1 - 1}\, E_{\gamma,\kappa}\left[(\alpha_1, \beta_1), \cdots, (\alpha_m, \beta_m); \mathfrak{d}t^{\alpha_1}\right] = -c^{\nu}\, {}_0D_t^{-\nu}\, N(t) \tag{5.7}$$

is given by

$$N(t) = \frac{N_0\, t^{\beta_1 - 1}}{\Gamma(\gamma)} \sum_{r=0}^{\infty} (-c^{\nu} t^{\nu})^r\, {}_1\Psi_m \left[\begin{array}{r} (\gamma, \kappa); \\ (\nu r + \beta_1, \alpha_1), (\beta_j, \alpha_j)_{2,m}; \end{array} \mathfrak{d}t^{\alpha_1}\right]. \tag{5.8}$$

Proof. Proof of the result asserted by Theorem 5.4 runs parallel to that of Theorem 5.3. Here we use (5.2) instead of (5.1). The details are, therefore, being omitted. □

Remark 5.5. For $\kappa = \mathfrak{q} \in (0, 1) \cup \mathbb{N}$, the results in Theorem 5.3 and Theorem 5.4 reduce to those for the generalized fractional kinetic equation involving the generalized Mittag-Leffler function studied by Saxena *et al.* [79].

By setting $m = 1$ in (5.3), we get an interesting generalized fractional kinetic equation with its solution given by the following corollary.

Corollary 5.6. *Let* $c, \mathfrak{d}, \nu, \lambda, \rho \in \mathbb{R}^+$. *Also let* $\alpha, \beta, \gamma, \kappa \in \mathbb{C}$ *with*

$$\Re(\alpha) > \max\{0, \Re(\kappa) - 1\} \qquad \text{and} \qquad \min\{\Re(\beta), \Re(\kappa)\} > 0.$$

Then the solution of the following generalized fractional kinetic equation:

$$N(t) - N_0\, t^{\lambda - 1}\, E_{\alpha,\beta}^{\gamma,\kappa}\left[\mathfrak{d}t^{\rho}\right] = -c^{\nu}\, {}_0D_t^{-\nu}\, N(t) \tag{5.9}$$

is given by

$$N(t) = \frac{N_0\, t^{\lambda - 1}}{\Gamma(\gamma)} \sum_{r=0}^{\infty} (-c^{\nu} t^{\nu})^r\, {}_2\Psi_2 \left[\begin{array}{r} (\gamma, \kappa), (\lambda, \rho); \\ (\nu r + \lambda, \rho), (\beta, \alpha); \end{array} \mathfrak{d}t^{\rho}\right], \tag{5.10}$$

where $E_{\alpha,\beta}^{\gamma,\kappa}[z]$ *is the generalized Mittag-Leffler function defined by* (4.9).

In its *further* special case when $\kappa = \mathfrak{q} \in (0, 1) \cup \mathbb{N}$, Corollary 5.6 would reduce immediately to Corollary 5.7 below.

Corollary 5.7. *Let* $c, \mathfrak{d}, \nu, \lambda, \rho \in \mathbb{R}^+$. *Also let* $\alpha, \beta, \gamma \in \mathbb{C}$ *with* $\min\{\Re(\alpha), \Re(\beta)\} > 0$. *Suppose that* $\mathfrak{q} \in (0,1) \cup \mathbb{N}$. *Then the solution of the following generalized fractional kinetic equation*:

$$N(t) - N_0\, t^{\lambda-1}\, E_{\alpha,\beta}^{\gamma,\mathfrak{q}}[\mathfrak{d}t^\rho] = -c^\nu\, {}_0D_t^{-\nu}\, N(t) \tag{5.11}$$

is given by

$$N(t) = \frac{N_0\, t^{\lambda-1}}{\Gamma(\gamma)} \sum_{r=0}^{\infty} (-c^\nu t^\nu)^r\, {}_2\Psi_2 \left[\begin{array}{c} (\gamma, \mathfrak{q}), (\lambda, \rho); \\ (\nu r + \lambda, \rho), (\beta, \alpha); \end{array} \mathfrak{d}t^\rho \right], \tag{5.12}$$

where $E_{\alpha,\beta}^{\gamma,\mathfrak{q}}[z]$ *is the above-mentioned special case of the generalized Mittag-Leffler function in* (4.9) *when* $\kappa = \mathfrak{q} \in (0,1) \cup \mathbb{N}$.

Remark 5.8. The result asserted by Theorem 5.4 can also be suitably specialized to deduce solutions of certain generalized fractional kinetic equations analogous to those which are dealt with in Corollary 5.6 and Corollary 5.7.

6. Solution of Generalized Fractional Kinetic Equations by Using the Sumudu Transform

In this section we propose to investigate the solution of the generalized fractional kinetic equation involving the generalized Mittag-Leffler function (4.11) by applying the Sumudu transform technique. The following lemmas will be required in our derivations.

Lemma 6.1. *Let* $\min\{\Re(\lambda), \Re(\rho), \Re(u)\} > 0$. *Then the following Sumudu transform holds true*:

$$\begin{aligned} &\mathcal{S}\left[t^{\lambda-1}\, E_{\gamma,\kappa}\left[(\alpha_1,\beta_1), \cdots, (\alpha_m,\beta_m);\, t^\rho\right]; u\right] \\ &\qquad = \frac{u^{\lambda-1}}{\Gamma(\gamma)}\, {}_2\Psi_m \left[\begin{array}{c} (\gamma,\kappa), (\lambda,\rho); \\ (\beta_j, \alpha_j)_{1,m}; \end{array} u^\rho \right]. \end{aligned} \tag{6.1}$$

Proof. By using (4.11), we readily have

$$\begin{aligned} \mathfrak{S} &:= \mathcal{S}\left[t^{\lambda-1}\, E_{\gamma,\kappa}\left[(\alpha_1,\beta_1), \cdots, (\alpha_m,\beta_m);\, t^\rho\right]; u\right] \\ &= \int_0^\infty e^{-t} \sum_{n=0}^\infty \frac{(\gamma)_{\kappa n}}{\prod_{j=1}^m \Gamma(n\alpha_j + \beta_j)} \frac{(ut)^{\rho n + \lambda - 1}}{n!}\, dt \\ &= \sum_{n=0}^\infty \frac{(\gamma)_{\kappa n}}{\prod_{j=1}^m \Gamma(n\alpha_j + \beta_j)} \frac{u^{\rho n + \lambda - 1}}{n!} \int_0^\infty e^{-t}\, t^{\rho n + \lambda - 1}\, dt. \end{aligned} \tag{6.2}$$

This last integral in (6.2) can be evaluated by means of Euler's Gamma-function integral:

$$\int_0^\infty e^{-t}\, t^{z-1}\, dt = \Gamma(z) \qquad \big(\Re(z) > 0\big). \tag{6.3}$$

We thus find that

$$\begin{aligned}\mathfrak{S} &= \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}\, \Gamma(\rho n + \lambda)}{\prod\limits_{j=1}^{m} \Gamma(n\alpha_j + \beta_j)} \frac{u^{\rho n + \lambda - 1}}{n!} \\ &= \frac{u^{\lambda-1}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma + \kappa n)\, \Gamma(\rho n + \lambda)}{\prod\limits_{j=1}^{m} \Gamma(n\alpha_j + \beta_j)} \frac{u^{\rho n}}{n!},\end{aligned}$$

which, in view of (4.13), leads us to the right-hand side of (6.1). □

Remark 6.2. We find it to be convenient to record here a special case of (6.1) when $\lambda = \beta_1$ and $\rho = \alpha_1$ as Lemma 6.3 below.

Lemma 6.3. *Let* $\min\{\Re(\alpha_1), \Re(\beta_1), \Re(u)\} > 0$. *Then the following Sumudu transform holds true*:

$$\begin{aligned}&\mathcal{S}\left[t^{\beta_1-1}\, E_{\gamma,\kappa}\left[(\alpha_1,\beta_1), \cdots, (\alpha_m,\beta_m);\, t^{\alpha_1}\right]; u\right] \\ &\qquad = \frac{u^{\beta_1-1}}{\Gamma(\gamma)}\, {}_1\Psi_{m-1}\left[\begin{matrix}(\gamma,\kappa); \\ (\beta_j,\alpha_j)_{2,m};\end{matrix}\; u^{\alpha_1}\right].\end{aligned} \tag{6.4}$$

Theorem 6.4. *Let* $c, \mathfrak{d}, \nu, \lambda, \rho \in \mathbb{R}^+$ *and* $\Re(u) > 0$ *with* $|u| < c^{-1}$ $(c \neq \mathfrak{d})$. *Also let* $\alpha_j, \beta_j, \gamma, \kappa \in \mathbb{C}$ *with*

$$\min\{\Re(\alpha_j), \Re(\beta_j)\} > 0 \qquad (j \in \{1, \cdots, m\}),$$

$$\Re(\kappa) > 0 \qquad \text{and} \qquad \Re\left(\sum_{j=1}^{m} \alpha_j\right) > \max\left[0, \Re(\kappa) - 1\right].$$

Then the solution of the following generalized fractional kinetic equation:

$$N(t) - N_0\, t^{\lambda-1}\, E_{\gamma,\kappa}\left[(\alpha_1,\beta_1), \cdots, (\alpha_m,\beta_m);\, \mathfrak{d}t^{\rho}\right] = -c^{\nu}\, {}_0D_t^{-\nu}\, N(t) \tag{6.5}$$

is given by

$$N(t) = \frac{N_0\, t^{\lambda-2}}{\Gamma(\gamma)} \sum_{r=0}^{\infty} \left(-c^{\nu} t^{\nu}\right)^r\, {}_2\Psi_{m+1}\left[\begin{matrix}(\gamma,\kappa), (\lambda,\rho); \\ (\nu r + \lambda - 1, \rho), (\beta_j,\alpha_j)_{1,m};\end{matrix}\; \mathfrak{d}t^{\rho}\right]. \tag{6.6}$$

Proof. Taking the Sumudu transform on both sides of (6.5) and using Lemma 6.1 and (4.20), we find that

$$\mathfrak{N}(u) = N_0 \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}\, \Gamma(\rho n + \lambda)}{\prod_{j=1}^{m} \Gamma(n\alpha_j + \beta_j)} \frac{\mathfrak{d}^n u^{\rho n + \lambda - 1}}{n!} - c^\nu u^\nu\, \mathfrak{N}(u), \tag{6.7}$$

where

$$\mathfrak{N}(u) := \mathcal{S}[N(t); u]. \tag{6.8}$$

Equivalently, we can write (6.7) as follows:

$$\mathfrak{N}(u) = N_0 \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}\, \Gamma(\rho n + \lambda)}{\prod_{j=1}^{m} \Gamma(n\alpha_j + \beta_j)} \frac{u^{\rho n + \lambda - 1}}{1 + c^\nu u^\nu} \frac{\mathfrak{d}^n}{n!}. \tag{6.9}$$

Using the binomial series expansion of $(1 + c^\nu u^\nu)^{-1}$ in (6.9) and inverting the Sumudu transform on both sides of the resulting equation, we get

$$N(t) = N_0 \sum_{r=0}^{\infty} (-c^\nu)^r \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}\, \Gamma(\rho n + \lambda)}{\prod_{j=1}^{m} \Gamma(n\alpha_j + \beta_j)} \frac{\mathfrak{d}^n}{n!} \mathcal{S}^{-1}\left\{u^{\rho n + \nu r + \lambda - 1}\right\}.$$

Finally, we make use of the following formula:

$$\mathcal{S}^{-1}\{u^\nu\} = \frac{t^{\nu-1}}{\Gamma(\nu)} \qquad \big(\min\{\Re(\nu), \Re(u)\} > 0\big).$$

After some simplification, we thus find that

$$\begin{aligned} N(t) = N_0\, t^{\lambda-2} &\sum_{r=0}^{\infty} (-c^\nu t^\nu)^r \\ &\cdot \sum_{n=0}^{\infty} \frac{\Gamma(\gamma + \kappa n)\, \Gamma(\rho n + \lambda)}{\Gamma(\gamma)\, \Gamma(\rho n + \nu r + \lambda - 1) \prod_{j=1}^{m} \Gamma(n\alpha_j + \beta_j)} \frac{\mathfrak{d}^n t^{\rho n}}{n!}, \end{aligned}$$

which, in view of (4.13), leads us to the right-hand side of (6.6). This complete the proof of Theorem 6.4. □

Theorem 6.5. *Let* $c, \mathfrak{d}, \nu \in \mathbb{R}^+$ *and* $\Re(u) > 0$ *with* $|u| < c^{-1}$ $(c \neq \mathfrak{d})$*. Also let* $\alpha_j, \beta_j, \gamma, \kappa \in \mathbb{C}$ *with*

$$\min\{\Re(\alpha_j), \Re(\beta_j)\} > 0 \qquad (j \in \{1, \cdots, m\}),$$

$$\Re(\kappa) > 0 \qquad \textit{and} \qquad \Re\left(\sum_{j=1}^{m} \alpha_j\right) > \max\{0, \Re(\kappa) - 1\}.$$

Then the solution of the following generalized fractional kinetic equation:

$$N(t) - N_0\, t^{\beta_1 - 1} E_{\gamma,\kappa}\left[(\alpha_1, \beta_1), \cdots, (\alpha_m, \beta_m); \mathfrak{d} t^{\alpha_1}\right] = -c^{\nu}\, {}_0D_t^{-\nu} N(t) \tag{6.10}$$

is given by

$$N(t) = \frac{N_0\, t^{\beta_1 - 2}}{\Gamma(\gamma)} \sum_{r=0}^{\infty} \left(-c^{\nu} t^{\nu}\right)^r {}_1\Psi_m \left[\begin{array}{r} (\gamma, \kappa); \\ (\nu r + \beta_1 - 1, \alpha_1), (\beta_j, \alpha_j)_{2,m}; \end{array} \mathfrak{d} t^{\rho} \right]. \tag{6.11}$$

Proof. Our demonstration of Theorem 6.5 would run parallel to that of Theorem 6.4. Here, in this case, we use (6.4) instead of (6.1). We, therefore, omit the details involved. □

Upon setting $m = 1$ in Theorem 6.4, we can deduce the following simpler result.

Corollary 6.6. *Let* $c, \mathfrak{d}, \nu, \lambda, \rho \in \mathbb{R}^+$ *and* $\Re(u) > 0$ *with* $|u| < c^{-1}$ $(c \neq \mathfrak{d})$. *Also let* $\alpha, \beta, \gamma, \kappa \in \mathbb{C}$ *with*

$$\Re(\alpha) > \max\{0, \Re(\kappa) - 1\} \qquad \textit{and} \qquad \min\{\Re(\beta), \Re(\kappa)\} > 0.$$

Then the solution of the following generalized fractional kinetic equation:

$$N(t) - N_0\, t^{\lambda - 1} E_{\alpha,\beta}^{\gamma,\kappa}\left[\mathfrak{d} t^{\rho}\right] = -c^{\nu}\, {}_0D_t^{-\nu} N(t) \tag{6.12}$$

is given by

$$N(t) = \frac{N_0\, t^{\lambda - 2}}{\Gamma(\gamma)} \sum_{r=0}^{\infty} \left(-c^{\nu} t^{\nu}\right)^{\nu r} {}_2\Psi_2 \left[\begin{array}{r} (\gamma, \kappa), (\lambda, \rho); \\ (\nu r + \lambda - 1, \rho), (\beta, \alpha); \end{array} \mathfrak{d} t^{\rho} \right]. \tag{6.13}$$

If we set $m = 1$ *and* $\kappa = \mathfrak{q} \in (0,1) \cup \mathbb{N}$ in Theorem 6.4, we are led easily to Corollary 6.7 below, which would follow also as a *further* special case of Corollary 6.6 when $\kappa = \mathfrak{q} \in (0,1) \cup \mathbb{N}$.

Corollary 6.7. *Let* $c, \mathfrak{d}, \nu, \lambda, \rho \in \mathbb{R}^+$ *and* $\Re(u) > 0$ *with* $|u| < c^{-1}$ $(c \neq \mathfrak{d})$. *Also let* $\alpha, \beta, \gamma \in \mathbb{C}$ *with*

$$\min\{\Re(\alpha), \Re(\beta)\} > 0 \qquad \textit{and} \qquad \mathfrak{q} \in (0,1) \cup \mathbb{N}.$$

Then the solution of the following generalized fractional kinetic equation:

$$N(t) - N_0\, t^{\lambda - 1} E_{\alpha,\beta}^{\gamma,\mathfrak{q}}\left[\mathfrak{d} t^{\rho}\right] = -c^{\nu}\, {}_0D_t^{-\nu} N(t) \tag{6.14}$$

is given by

$$N(t) = \frac{N_0\, t^{\lambda-2}}{\Gamma(\gamma)} \sum_{r=0}^{\infty} \left(-c^{\nu} t^{\nu}\right)^r {}_2\Psi_2 \left[\begin{array}{r} (\gamma, \mathfrak{q}), (\lambda, \rho); \\ (\nu r + \lambda - 1, \rho), (\beta, \alpha); \end{array} \mathfrak{d} t^{\rho} \right]. \tag{6.15}$$

We conclude this section by remarking that the results presented here are general enough to yield, as their special cases, solutions of a number of known or new fractional kinetic equations involving such other special functions as (for example) those considered by Haubold and Mathai [26] and Saxena *et al.* [74, 75, 78]. Moreover, in our investigation here, our choice to make use of the Sumudu transform instead of the classical Laplace transform is prompted by the various problems considered here and also by the fact that the closed-form results derived here happen to be remarkably simpler (see also [23, 86]).

7. Fractional Differintegral Operators Based Upon the Cauchy-Goursat Integral Formula

Operators of fractional differintegrals (that is, fractional derivatives and fractional integrals), which are based essentially upon the familiar Cauchy-Goursat integral formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz \qquad (n \in \mathbb{N}_0),$$

were considered by (among others) Sonin in 1869, Letnikov in 1868 onwards, and Laurent in 1884. Here, as usual, the function $f(z)$ is analytic everywhere within and on a simple closed contour C in the complex z-plane, taken in the positive (counter-clockwise) direction and z_0 is any point interior to the contour C. In recent years, many authors have demonstrated the usefulness of fractional calculus operators (based upon the above-mentioned Cauchy-Goursat integral formula) in obtaining particular solutions of numerous families of homogeneous (as well as nonhomogeneous) linear ordinary and partial differential equations which are associated, for example, with many of the following *celebrated* equations as well as their close relatives:

I. The Gauss Equation:

$$z(1-z)\frac{d^2w}{dz^2} + [\gamma - (\alpha+\beta+1)z]\frac{dw}{dz} - \alpha\beta w = 0 \tag{7.1}$$

II. The Kummer Equation:

$$z\frac{d^2w}{dz^2} + (\gamma - z)\frac{dw}{dz} - \alpha w = 0 \tag{7.2}$$

III. The Euler Equation:

$$z^2 \frac{d^2w}{dz^2} + z\frac{dw}{dz} - \rho^2 w = 0 \tag{7.3}$$

IV. The Coulomb Equation:

$$z\frac{d^2w}{dz^2} + (2\lambda - z)\frac{dw}{dz} + (\mu - \lambda)\, w = 0 \tag{7.4}$$

V. The Laguerre-Sonin Equation:

$$z\frac{d^2w}{dz^2} + (\alpha + 1 - z)\frac{dw}{dz} + \lambda w = 0 \tag{7.5}$$

VI. The Chebyshev Equation:

$$\left(1 - z^2\right)\frac{d^2w}{dz^2} - z\frac{dw}{dz} + \lambda^2 w = 0 \tag{7.6}$$

VII. The Weber-Hermite Equation:

$$\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + (\lambda - 1)\, w = 0 \tag{7.7}$$

Numerous earlier contributions on fractional calculus along the aforementioned lines are reproduced, with proper credits, in the works of Nishimoto (*cf.* [54, 55]). Moreover, a rather systematic analysis (including interconnections) of many of the results involving (homogeneous or nonhomogeneous) linear differential equations associated with (for example) the Gauss hypergeometric equation (7.1) can be found in the works of Nishimoto *et al.* [61, 62] and the recent contribution on this subject by Wang *et al.* [106] (see also some other recent applications considered by Lin *et al.* [42] and Prieto *et al.* [65]).

In the cases of (ordinary as well as partial) differential equations of *higher* orders, which have stemmed naturally from the Gauss hypergeometric equation (7.1) and its many relatives and extensions, including some of the above-listed linear differential equations (7.2) to (7.7), there have been several seemingly independent attempts to present a remarkably large number of scattered results in a unified manner. We choose to furnish here the generalizations (and unification) proposed in one of the latest works on this subject by Tu *et al.* [99] in which references to many earlier related works can be found. We find it to be convenient to begin by recalling the following definition of a *fractional differintegral* (that is, *fractional derivative* and *fractional integral*) of $f(z)$ of order $\nu \in \mathbb{R}$.

Definition 7.1.([54, 55, 94]) If the function $f(z)$ is analytic and has no branch point inside and on $\mathcal{C}$, where

$$\mathcal{C} := \left\{\mathcal{C}^-, \mathcal{C}^+\right\}, \tag{7.8}$$

$\mathcal{C}^-$ is a contour along the cut joining the points z and $-\infty + i\mathfrak{J}(z)$, which starts from the point at $-\infty$, encircles the point z once counter-clockwise, and returns to the point at $-\infty$, $\mathcal{C}^+$ is a contour along the cut joining the points z and $\infty + i\mathfrak{J}(z)$, which starts from the point at ∞, encircles the point z once counter-clockwise, and returns to the point at ∞,

$$f_\nu(z) = {}_{\mathcal{C}}f_\nu(z) := \frac{\Gamma(\nu+1)}{2\pi i}\int_{\mathcal{C}} \frac{f(\zeta)\,d\zeta}{(\zeta - z)^{\nu+1}} \tag{7.9}$$

$$\big(\nu \in \mathbb{R} \setminus \mathbb{Z}^-;\ \mathbb{Z}^- := \{-1,-2,-3,\cdots\}\big)$$

and

$$f_{-n}(z) := \lim_{\nu \to -n} \{f_\nu(z)\} \qquad (n \in \mathbb{N}), \tag{7.10}$$

where $\zeta \neq z$,

$$-\pi \leqq \arg(\zeta - z) \leqq \pi \qquad \text{for} \qquad \mathcal{C}^-, \tag{7.11}$$

and

$$0 \leqq \arg(\zeta - z) \leqq 2\pi \qquad \text{for} \qquad \mathcal{C}^+, \tag{7.12}$$

then $f_\nu(z)$ $(\nu > 0)$ is said to be the *fractional derivative of* $f(z)$ *of order* ν and $f_\nu(z)$ $(\nu < 0)$ is said to be the *fractional integral of* $f(z)$ *of order* $-\nu$, provided that

$$|f_\nu(z)| < \infty \qquad (\nu \in \mathbb{R}). \tag{7.13}$$

Throughout the remainder of this section, we shall simply write f_ν for $f_\nu(z)$ whenever the argument of the differintegrated function f is clearly understood by the surrounding context. Moreover, in case f is a many-valued function, we shall tacitly consider the *principal value* of f in this investigation.

Each of the following general results is capable of yielding particular solutions of many simpler families of linear ordinary fractional differintegral equations (*cf.* Tu *et al.* [99]) including (for example) the classical differential equations listed above [*cf.* (7.1) to (7.7)].

Theorem 7.2. *Let* $P(z;p)$ *and* $Q(z;q)$ *be polynomials in* z *of degrees* p *and* q*, respectively, defined by*

$$\begin{aligned} P(z;p) &:= \sum_{k=0}^{p} a_k\, z^{p-k} \\ &= a_0 \prod_{j=1}^{p} (z - z_j) \qquad (a_0 \neq 0;\ p \in \mathbb{N}) \end{aligned} \tag{7.14}$$

and

$$Q(z;q) := \sum_{k=0}^{q} b_k\ z^{q-k} \qquad (b_0 \neq 0;\ q \in \mathbb{N}). \tag{7.15}$$

Suppose also that $f_{-\nu}\,(\neq 0)$ *exists for a given function* f. *Then the following non-homogeneous linear ordinary fractional differintegral equation:*

$$\begin{aligned} P(z;p)\,\phi_\mu(z) + \left[\sum_{k=1}^{p}\binom{\nu}{k}P_k(z;p) + \sum_{k=1}^{q}\binom{\nu}{k-1}Q_{k-1}(z;q)\right]\phi_{\mu-k}(z) \\ + \binom{\nu}{q}q!\ b_0\ \phi_{\mu-q-1}(z) = f(z) \end{aligned} \tag{7.16}$$

$$(\mu,\nu \in \mathbb{R};\ p,q \in \mathbb{N})$$

has a particular solution of the form:

$$\phi(z) = \left(\left(\frac{f_{-\nu}(z)}{P(z;p)}e^{H(z;p,q)}\right)_{-1}\ e^{-H(z;p,q)}\right)_{\nu-\mu+1} \tag{7.17}$$

$$(z \in \mathbb{C} \setminus \{z_1, \cdots, z_p\}),$$

where, for convenience,

$$H(z;p,q) := \int^{z} \frac{Q(\zeta;q)}{P(\zeta;p)}d\zeta \qquad (z \in \mathbb{C} \setminus \{z_1, \cdots, z_p\}), \tag{7.18}$$

provided that the second member of (7.17) *exists.*

Theorem 7.3. *Under the various relevant hypotheses of Theorem 7.2, the following homogeneous linear ordinary fractional differintegral equation:*

$$\begin{aligned} P(z;p)\,\phi_\mu(z) + \left[\sum_{k=1}^{p}\binom{\nu}{k}P_k(z;p) + \sum_{k=1}^{q}\binom{\nu}{k-1}Q_{k-1}(z;q)\right]\phi_{\mu-k}(z) \\ + \binom{\nu}{q}q!\ b_0\ \phi_{\mu-q-1}(z) = 0 \end{aligned} \tag{7.19}$$

$$(\mu,\nu \in \mathbb{R};\ p,q \in \mathbb{N})$$

has solutions of the form:

$$\phi(z) = K\left(e^{-H(z;p,q)}\right)_{\nu-\mu+1}, \tag{7.20}$$

where K is an arbitrary constant and $H(z;p,q)$ is given by (7.18)*, it being provided that the second member of* (7.20) *exists.*

Next, for a function $u = u(z,t)$ of two *independent* variables z and t, we find it to be convenient to use the following notation:

$$\frac{\partial^{\mu+\nu} u}{\partial z^{\mu} \partial t^{\nu}}$$

in order to abbreviate the partial fractional differintegral of $u(z,t)$ of order μ with respect to z *and* of order ν with respect to t $(\mu, \nu \in \mathbb{R})$. We now state the following general result (see, for details, Tu *et al.* [99]).

Theorem 7.4. *Let the polynomials $P(z;p)$ and $Q(z;q)$ be defined by* (7.14) *and* (7.15)*, respectively. Suppose also that the function $H(z;p,q)$ is given by* (7.18)*. Then the following partial fractional differintegral equation:*

$$\begin{aligned} P(z;p)\frac{\partial^{\mu} u}{\partial z^{\mu}} &+ \left[\sum_{k=1}^{p-1}\binom{\nu}{k}P_k(z;p) + \sum_{k=1}^{q}\binom{\nu}{k-1}Q_{k-1}(z;q-1)\right]\frac{\partial^{\mu-k} u}{\partial z^{\mu-k}} \\ &+\gamma\frac{\partial^{\mu-p} u}{\partial z^{\mu-p}} = \alpha\frac{\partial^{\mu-p+2} u}{\partial z^{\mu-p}\partial t^2} + \beta\frac{\partial^{\mu-p+1} u}{\partial z^{\mu-p}\partial t} \end{aligned} \tag{7.21}$$

$$(\mu, \nu \in \mathbb{R};\ p, q \in \mathbb{N})$$

has solutions of the form:

$$u(z,t) = \begin{cases} K_1\left(e^{-H(z;p,q-1)}\right)_{\nu-\mu+1} e^{\xi t} & (\alpha \neq 0) \\ K_2\left(e^{-H(z;p,q-1)}\right)_{\nu-\mu+1} e^{\eta t} & (\alpha = 0;\ \beta \neq 0), \end{cases} \tag{7.22}$$

where K_1 and K_2 are arbitrary constants, α, β, and γ are given constants, and (for convenience)

$$\xi := \frac{-\beta \pm \sqrt{\beta^2 + 4(\gamma-\delta)\alpha}}{2\alpha} \quad (\alpha \neq 0) \quad \text{and} \quad \eta := \frac{\gamma-\delta}{\beta} \quad (\alpha = 0;\ \beta \neq 0), \tag{7.23}$$

with

$$\delta := \binom{\nu}{p} p!\, a_0, \tag{7.24}$$

provided that the second member of (7.22) *exists in each case.*

We conclude this section by remarking further that either or both of the polynomials $P(z;p)$ and $Q(z;q)$, involved in Theorem 7.2 to Theorem 7.4, can be of degree 0 as well. Thus, in the definitions (7.14) and (7.15) (as also in Theorem 7.2 to

Theorem 7.4, $\mathbb{N}$ may easily be replaced (if and where needed) by $\mathbb{N}_0$. Furthermore, it is fairly straightforward to see how each of these general theorems can be suitably specialized to yield numerous simpler results scattered throughout the ever-growing literature on fractional calculus.

8. Applications Involving a Class of Non-Fuchsian Differential Equations

In this section, we aim at applying Theorem 7.2 in order to find (explicit) particular solutions of the following general class of non-Fuchsian differential equations with six parameters:

$$\left(1+\frac{l}{z}\right)\frac{d^2\varphi}{dz^2}+\left[\alpha+\frac{\beta}{z}\left(1+\frac{l}{z}\right)\right]\frac{d\varphi}{dz}+\left[\gamma+\frac{\delta}{z}+\frac{\varepsilon}{z^2}\left(1+\frac{l}{z}\right)\right]\varphi(z) = f(z) \qquad (z\in\mathbb{C}\setminus\{0,-1\}), \tag{8.1}$$

where f is a given function and the parameters α, β, γ, δ, ε, and l are unrestricted, in general. Indeed, if we make use of the transformation:

$$\varphi(z) = z^{\rho}\, e^{\lambda z}\, \phi(z), \tag{8.2}$$

constrain the various parameters involved in (5.1) *and* (5.2) so that

$$\rho = -\frac{1}{2}\beta = \frac{-1\pm\sqrt{1+4\varepsilon}}{2} \qquad \text{and} \qquad \lambda = \frac{-\alpha\pm\sqrt{\alpha^2-4\gamma}}{2}, \tag{8.3}$$

then Theorem 7.2 would eventually imply that the nonhomogeneous linear ordinary differential equation (8.1) has a particular solution in the following form:

$$\varphi(z) = z^{\rho}e^{\lambda z}\phi(z) = z^{\rho}e^{\lambda z}\Bigg(\Big(\big(z^{1-\rho}e^{-\lambda z}f(z)\big)_{-\nu}\cdot(z+l)^{-\nu-\alpha l-1}\cdot e^{(2\lambda+\alpha)z}\Big)_{-1} \cdot (z+l)^{\nu+\alpha l}\cdot e^{-(2\lambda+\alpha)z}\Bigg)_{\nu-1} \qquad (z\in\mathbb{C}\setminus\{0,-l\};\ \nu\in\mathbb{R}), \tag{8.4}$$

and (by Theorem 7.3) the corresponding homogeneous linear ordinary differential equation:

$$\left(1+\frac{l}{z}\right)\frac{d^2\varphi}{dz^2}+\left[\alpha+\frac{\beta}{z}\left(1+\frac{l}{z}\right)\right]\frac{d\varphi}{dz}+\left[\gamma+\frac{\delta}{z}+\frac{\varepsilon}{z^2}\left(1+\frac{l}{z}\right)\right]\varphi(z) = 0 \qquad (z\in\mathbb{C}\setminus\{0,-l\}), \tag{8.5}$$

has solutions given by

$$\varphi(z) = z^{\rho}e^{\lambda z}\phi(z) = Kz^{\rho}e^{\lambda z}\left((z+l)^{\nu+\alpha l}\cdot e^{-(2\lambda+\alpha)z}\right)_{\nu-1} \qquad (z\in\mathbb{C}\setminus\{0,-l\};\ \nu\in\mathbb{R}), \tag{8.6}$$

where K is an arbitrary constant, the parameters ρ and λ are given (as before) by (8.3), and

$$\nu = \frac{\lambda^2 l + \rho\alpha + \delta}{2\lambda + \alpha}.$$

For various special choices for the *free* parameters occurring in (8.1) and (8.5), one can apply the results of this section to many known non-Fuchsian differential equations. These include (for example) a special limit (*confluent*) case of the Gauss hypergeometric equation (7.1), referred to as the *Whittaker equation* (see, for example, [110, p. 337, Equation 16.1 (B)]; see also [18, Vol. I, p. 248, Equation 6.1 (4)]), the so-called *Fukuhara equation* (*cf.* [21]; see also [57]), the *Tricomi equation* (*cf.* [98, p. 7, Equation 1.2 (1)]; see also [18, Vol. I, p. 251, Equation 6.2 (13)]), the familiar *Bessel equation* (*cf.* [108]), and so on. For a systematic investigation of these and many other closely-related differential equations (including, for example, many of the familiar differential equations list at the beginning of Section 4 here), we refer the interested reader to the recent works of Nishimoto *et al.* [56, 57, 58, 59, 60, 61, 62], Salinas de Romero *et al.* [69, 70], Galué [22], Lin *et al.* [43, 44, 45, 46, 47, 48], Tu *et al.* [99, 100, 101, 102], and Wang *et al.* [106, 107].

9. The Classical Gauss and Jacobi Differential Equations Revisited

The main purpose of this section (and Section 10 below) is to follow rather closely and analogously the investigations in (for example) [39, 46, 90, 106, 107] of solutions of some general families of second-order linear ordinary differential equations, which are associated with the familiar *Bessel differential equation* of general order ν (*cf.* [18, Vol. II, Chapter 7]; see also [108] and [110, Chapter 17]):

$$z^2 \frac{d^2w}{dz^2} + z \frac{dw}{dz} + \left(z^2 - \nu^2\right) w = 0, \tag{9.1}$$

which is named after Friedrich Wilheim Bessel (1784-1846). More precisely, just as in the earlier works [44, 90] (see also [40, 41]), which dealt *systematically* with *Legendre's differential equation* (*cf.* [18, Vol. I, p. 121, Equation 3.2(1)]; see also [110, Chapter 15]):

$$\left(1 - z^2\right) \frac{d^2w}{dz^2} + -2z \frac{dw}{dz} + \left(\nu(\nu+1) - \frac{\mu^2}{1 - z^2}\right) w = 0, \tag{9.2}$$

we aim here in this section at demonstrating how the underlying simple fractional-calculus approach to the solutions of the classical differential equations (9.1) and (9.2) would lead us analogously to several interesting consequences including (for example) an *alternative* investigation of solutions of the following *two-parameter* family of second-order ordinary differential equations (see also [106]):

$$z(1-z) \frac{d^2w}{dz^2} + [(\rho - 2\lambda)z + \lambda + \sigma] \frac{dw}{dz} + \lambda(\rho - \lambda + 1) w = 0, \tag{9.3}$$

We begin by setting

(9.4)
$$\mu = 2,\ \nu \mapsto \lambda,\ p-1 = q = 1,\ a_0 = -1,\ a_1 = 1,\ a_2 = 0,\ b_0 = \rho, \quad \text{and} \quad b_1 = \sigma$$
$$(\rho \neq 0;\ \lambda \in \mathbb{R})$$

in Theorem 7.2. We can thus deduce the following application of Theorem 7.2 relevant to the linear ordinary differential equation (9.3).

Theorem 9.1. *If the given function f satisfies the constraint* (7.13) *and $f_{-\lambda} \neq 0$, then the following nonhomogeneous linear ordinary differential equation:*

$$z(1-z)\,\frac{d^2\phi}{dz^2} + [(\rho - 2\lambda)z + \lambda + \sigma]\,\frac{d\phi}{dz} + \lambda\,(\rho - \lambda + 1)\,\phi = f\,(z) \tag{9.5}$$
$$(z \in \mathbb{C} \setminus \{0,1\};\ \rho \neq 0;\ \lambda \in \mathbb{R})$$

has a particular solution of the form:

$$\phi\,(z) = \left(\left(f_{-\lambda}\,(z) \cdot z^{\sigma-1} \cdot (1-z)^{-\rho-\sigma-1}\right)_{-1} \cdot z^{-\sigma} \cdot (1-z)^{\rho+\sigma}\right)_{\lambda-1} \tag{9.6}$$
$$(z \in \mathbb{C} \setminus \{0,1\};\ \rho \neq 0;\ \lambda \in \mathbb{R}),$$

provided that the second member of 9.6 exists.

Furthermore, the following homogeneous linear ordinary differential equation:

$$z(1-z)\,\frac{d^2\phi}{dz^2} + [(\rho - 2\lambda)z + \lambda + \sigma]\,\frac{d\phi}{dz} + \lambda\,(\rho - \lambda + 1)\,\phi = 0 \tag{9.7}$$
$$(z \in \mathbb{C} \setminus \{0,1\};\ \rho \neq 0;\ \lambda \in \mathbb{R})$$

has solutions of the form:

$$\phi\,(z) = K\left(z^{-\sigma} \cdot (1-z)^{\rho+\sigma}\right)_{\lambda-1} \qquad (z \in \mathbb{C} \setminus \{0,1\};\ \rho \neq 0;\ \lambda \in \mathbb{R}), \tag{9.8}$$

where K is an arbitrary constant, it being provided that the second member of (9.8) *exists.*

Remark 9.2. If we consider the case when $|z| < 1$, by making use of the familiar binomial expansion, we find from the assertion (9.8) of Theorem 9.1 that

$$\phi(z) = K\,\sum_{n=0}^{\infty}(-1)^n \binom{\rho+\sigma}{n}\left(z^{n-\sigma}\right)_{\lambda-1} \qquad (|z| < 1). \tag{9.9}$$

Thus, in view of the following well-exploited fractional differintegral formula:

$$(z^\lambda)_\nu = e^{-i\pi\nu}\,\frac{\Gamma(\nu-\lambda)}{\Gamma(-\lambda)}\,z^{\lambda-\nu} \tag{9.10}$$
$$\left(\nu\in\mathbb{R};\ z\in\mathbb{C};\ \left|\frac{\Gamma(\nu-\lambda)}{\Gamma(-\lambda)}\right|<\infty\right).$$

we readily obtain

$$\begin{aligned}\phi(z) &= K\,e^{-i\pi(\lambda-1)}\,\frac{\Gamma(\lambda+\sigma-1)}{\Gamma(\sigma)}\,z^{1-\lambda-\sigma}\\ &\quad\cdot\ {}_2F_1\,(-\rho-\sigma,1-\sigma;2-\lambda-\sigma;z) \qquad (|z|<1)\end{aligned} \tag{9.11}$$

in terms of the Gauss hypergeometric function ${}_2F_1$ (see [18, Vol. I, Chapter 2]).

Remark 9.3. If we consider the case when $|z|>1$, by appropriately applying the familiar binomial expansion once again, we find from the assertion (9.8) of Theorem 9.1 that

$$\phi(z) = K\,e^{-i\pi(\rho+\sigma)}\sum_{n=0}^{\infty}(-1)^n\binom{\rho+\sigma}{n}\left(z^{\rho-n}\right)_{\lambda-1} \qquad (|z|>1). \tag{9.12}$$

Thus, in view of the fractional differintegral formula (9.10), we find the following *explicit* solution of the differential equation (9.7) when $|z|>1$:

$$\begin{aligned}\phi(z) &= K\,e^{-i\pi(\lambda+\rho+\sigma-1)}\,\frac{\Gamma(\lambda-\rho-1)}{\Gamma(-\rho)}\,z^{\rho-\lambda+1}\\ &\quad\cdot\ {}_2F_1\left(-\rho-\sigma,\lambda-\rho-1;-\rho;\frac{1}{z}\right) \qquad (|z|>1),\end{aligned} \tag{9.13}$$

in terms of the Gauss hypergeometric function ${}_2F_1$ (see [18, Vol. I, Chapter 2]).

10. A Family of Unified Alternative Solutions Resulting from Theorem 6.5

We now propose to develop *alternative* solutions of several classical differential equations of mathematical physics in a *unified* manner by suitably applying the assertions of Theorem 9.1, Remark 9.2, and Remark 9.3.

I. Gauss's Differential Equation [see also Equation (7.1)]**:**

$$z(1-z)\frac{d^2\varphi}{dz^2}+[\gamma-(\alpha+\beta+1)z]\,\frac{d\varphi}{dz}-\alpha\beta\varphi=0, \tag{10.1}$$

which possesses the following well-known power-series solution relative to the regular singular point $z=0$ (see, for example, [30, p. 162]):

$$\varphi^{(1)}(z) = {}_2F_1\,(\alpha,\beta;\gamma;z) \qquad (|z|<1). \tag{10.2}$$

Furthermore, upon setting

$$\lambda = \alpha, \quad \rho = \alpha - \beta - 1 \qquad \text{and} \qquad \sigma = \gamma - \alpha$$

in (9.11), we obtain the following explicit solution of (10.1):

$$\varphi^{(2)}(z) = z^{1-\gamma} \; {}_2F_1\left(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; z\right) \qquad (|z| < 1). \tag{10.3}$$

Thus, by combining the *linearly independent* solutions $\varphi^{(1)}(z)$ and $\varphi^{(2)}(z)$, we find the following well-known *general* solution of the Gauss differential equation (10.1) by means of fractional calculus:

$$\begin{aligned}\varphi(z) &= K_1\, \varphi^{(1)}(z) + K_2\, \varphi^{(2)}(z)\\ &= K_1\, {}_2F_1\left(\alpha, \beta; \gamma; z\right) + K_2\, z^{1-\gamma} \; {}_2F_1\left(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; z\right) \qquad (|z| < 1),\end{aligned} \tag{10.4}$$

where K_1 and K_2 are arbitrary constants, it being understood that each member of (10.4) exists.

Alternatively, if we set

$$\lambda = \beta, \quad \rho = \beta - \alpha - 1 \qquad \text{and} \qquad \sigma = \gamma - \beta$$

in (9.13), then we obtain the following explicit solution of (10.1) [30, p. 162]:

$$\varphi^{(3)}(z) = z^{-\alpha} \; {}_2F_1\left(\alpha, \alpha - \gamma + 1; \alpha - \beta + 1; \frac{1}{z}\right) \qquad (|z| > 1). \tag{10.5}$$

If, on the other hand, we choose to set

$$\lambda = \alpha, \quad \rho = \alpha - \beta - 1 \qquad \text{and} \qquad \sigma = \gamma - \alpha$$

in (9.12), then we obtain the following *further* explicit solution of (10.1) [30, p. 162]:

$$\varphi^{(4)}(z) = z^{-\beta} \; {}_2F_1\left(\beta - \gamma + 1, \beta; \beta - \alpha + 1; \frac{1}{z}\right) \qquad (|z| > 1), \tag{10.6}$$

which does indeed follow also from (10.5) upon interchanging the rôles of the parameters α and β. Thus, if we combine the solutions $\varphi^{(3)}(z)$ and $\varphi^{(4)}(z)$ appropriate to the point at infinity, we find the following *general* solution of the Gauss differential equation (10.1) by means of fractional calculus:

$$\begin{aligned}\varphi(z) &= K_1^*\, \varphi^{(3)}(z) + K_2^*\, \varphi^{(4)}(z)\\ &= K_1^*\, z^{-\alpha} \; {}_2F_1\left(\alpha, \alpha - \gamma + 1; \alpha - \beta + 1; \frac{1}{z}\right)\\ &\qquad + K_2^*\, z^{-\beta} \; {}_2F_1\left(\beta - \gamma + 1, \beta; \beta - \alpha + 1; \frac{1}{z}\right) \qquad (|z| > 1),\end{aligned} \tag{10.7}$$

where K_1^* and K_2^* are arbitrary constants, it being understood that each member of (10.7) exists.

Lastly, since any solution of the Gauss differential equation (10.1) is linearly expressible in terms of two linearly independent solutions (see, for example, [30, p. 168]), it is not difficult to deduce from the above observations that (see, for example, [18, Vol. I, p. 108, Equation 2.10 (2)])

$$\begin{aligned} {}_2F_1(\alpha,\beta;\gamma;z) &= A\,(-z)^{-\alpha}\ {}_2F_1\left(\alpha,\alpha-\gamma+1;\alpha-\beta+1;\frac{1}{z}\right) \\ &\quad + B\,(-z)^{-\beta}\ {}_2F_1\left(\beta-\gamma+1,\beta;\beta-\alpha+1;\frac{1}{z}\right) \end{aligned} \tag{10.8}$$

$$\big(|z|>1;\ \ |\arg(-z)| \leqq \pi-\epsilon;\ \ 0<\epsilon<\pi\big),$$

where, for convenience, the coefficients A and B are given by

$$A := \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} \qquad \text{and} \qquad B := \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}. \tag{10.9}$$

The analytic continuation formula (10.8) is usually derived by the calculus of residues and the Mellin-Barnes contour integral representation for the Gauss hypergeometric function occurring on its left-hand side (see, for details, [18, Vol. I, p. 62, Section 2.1.4]). Moreover, it is easily seen from this analytic continuation formula (10.8) that *asymptotically*, for *large* $|z|$, we have

$${}_2F_1(\alpha,\beta;\gamma;z) \sim A\,(-z)^{-\alpha} + B\,(-z)^{-\beta} \tag{10.10}$$

$$\big(|z|\to\infty;\ \ |\arg(-z)| \leqq \pi-\epsilon;\ \ 0<\epsilon<\pi\big),$$

where the coefficients A and B are given (as before) by (10.9).

II. Jacobi's Differential Equation:

$$(1-z^2)\frac{d^2\Theta}{dz^2} + [\beta-\alpha-(\alpha+\beta+2)z]\,\frac{d\Theta}{dz} + \nu(\nu+\alpha+\beta+1)\Theta = 0, \tag{10.11}$$

which, in its *special* case when $\nu = n \in \mathbb{N}_0$, would reduce to the relatively more familiar differential equation satisfied by the *classical* Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ given explicitly by

$$\begin{aligned} P_n^{(\alpha,\beta)}(z) &= \sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{n+\beta}{n-k}\left(\frac{z-1}{2}\right)^{n-k}\left(\frac{z+1}{2}\right)^{k} \\ &= \binom{n+\alpha}{n}\,{}_2F_1\left(-n,n+\alpha+\beta+1;\alpha+1;\frac{1-z}{2}\right). \end{aligned} \tag{7.12}$$

Indeed, upon setting

$$z \longmapsto 1-2z, \qquad \frac{d}{dz} \longmapsto -\frac{1}{2}\,\frac{d}{dz}, \qquad \frac{d^2}{dz^2} \longmapsto \frac{1}{4}\,\frac{d^2}{dz^2} \qquad \text{and} \qquad \Theta \longmapsto \Phi,$$

Jacobi's differential equation (10.11) assumes the following form:

$$z(1-z)\frac{d^2\Phi}{dz^2}+[\alpha+1-(\alpha+\beta+2)z]\,\frac{d\Phi}{dz}+\nu(\nu+\alpha+\beta+1)\Phi=0. \tag{10.12}$$

Clearly, we have

$$\Theta(1-2z)=\Phi(z) \qquad \text{and} \qquad \Theta(z)=\Phi\left(\frac{1-z}{2}\right). \tag{10.13}$$

By setting

$$\lambda=\nu+\alpha+\beta+1, \qquad \rho=2\nu+\alpha+\beta \qquad \text{and} \qquad \sigma=-\nu-\beta$$

in (9.11) and (9.13), or (*alternatively*) by *directly* applying the hypergeometric solutions given by (10.2), (10.3), (10.5) and (10.6), we obtain the following explicit solutions of (10.12):

$$\Phi^{(1)}(z)={}_2F_1\left(-\nu,\nu+\alpha+\beta+1;\alpha+1;z\right) \qquad (|z|<1), \tag{10.14}$$

$$\Phi^{(2)}(z)=z^{-\alpha}\;{}_2F_1\left(-\nu-\alpha,\nu+\beta+1;1-\alpha;z\right) \qquad (|z|<1), \tag{10.15}$$

$$\Phi^{(3)}(z)=z^{\nu}\;{}_2F_1\left(-\nu,-\nu-\alpha;-2\nu-\alpha-\beta;\frac{1}{z}\right) \qquad (|z|>1) \tag{10.16}$$

and

$$\begin{aligned}\Phi^{(4)}(z)&=z^{-\nu-\alpha-\beta-1}\\ &\cdot\;{}_2F_1\left(\nu+\beta+1,\nu+\alpha+\beta+1;2\nu+\alpha+\beta+2;\frac{1}{z}\right) \qquad (|z|>1).\end{aligned} \tag{10.17}$$

Thus, if we make use of the relationships given by (10.13) in our observations (10.14) to (10.17), we are led fairly easily to the following explicit solutions of the *general* Jacobi differential equation (10.11):

$$\Theta^{(1)}(z)={}_2F_1\left(-\nu,\nu+\alpha+\beta+1;\alpha+1;\frac{1-z}{2}\right) \qquad (|1-z|<2), \tag{10.18}$$

(10.19)

$$\Theta^{(2)}(z)=(1-z)^{-\alpha}\;{}_2F_1\left(-\nu-\alpha,\nu+\beta+1;1-\alpha;\frac{1-z}{2}\right) \qquad (|1-z|<2),$$

(10.20)
$$\Theta^{(3)}(z) = (1-z)^{\nu} \ {}_2F_1\left(-\nu, -\nu-\alpha; -2\nu-\alpha-\beta; \frac{2}{1-z}\right) \qquad (|1-z| > 2)$$

and

$$\begin{aligned}\Theta^{(4)}(z) &= (1-z)^{-\nu-\alpha-\beta-1} \\ &\cdot \ {}_2F_1\left(\nu+\beta+1, \nu+\alpha+\beta+1; 2\nu+\alpha+\beta+2; \frac{2}{1-z}\right) \qquad (|1-z| > 2).\end{aligned} \tag{10.21}$$

Remark 10.1. The solution $\Theta^{(1)}(z)$ given by (10.18) can indeed be rewritten in terms of the *classical* Jacobi function $P_\nu^{(\alpha,\beta)}(z)$ $(\nu \in \mathbb{C})$ defined by

$$\begin{aligned}P_\nu^{(\alpha,\beta)}(z) &:= \sum_{k=0}^{\infty} \binom{\nu+\alpha}{k}\binom{\nu+\beta}{\nu-k}\left(\frac{z-1}{2}\right)^{\nu-k}\left(\frac{z+1}{2}\right)^{k} \\ &= \binom{\nu+\alpha}{\nu} \ {}_2F_1\left(-\nu, \nu+\alpha+\beta+1; \alpha+1; \frac{1-z}{2}\right) \qquad (\nu \in \mathbb{C}).\end{aligned} \tag{10.22}$$

Remark 10.2. In view of the familiar Euler transformation (see, for example, [18, Vol. I, p. 64, Equation 2.1.4 (23)]):

$$\begin{gathered}{}_2F_1(\alpha, \beta; \gamma; z) = (1-z)^{\gamma-\alpha-\beta} \ {}_2F_1(\gamma-\alpha, \gamma-\beta; \gamma; z) \\ \big(|\arg(1-z)| \leqq \pi-\epsilon; \quad 0 < \epsilon < \pi\big),\end{gathered} \tag{10.23}$$

we can rewrite the solution $\Theta^{(4)}(z)$ given by (10.21) in the following *equivalent* form:
(10.24)
$$\begin{gathered}\Theta^{(4)}(z) = \frac{2^{\nu+\alpha+\beta+1}\, e^{i\pi\beta}}{(1-z)^{\nu+\alpha+1}(1+z)^{\beta}} \ {}_2F_1\left(\nu+1, \nu+\alpha+1; 2\nu+\alpha+\beta+2; \frac{2}{1-z}\right) \\ (|1-z| > 2; \quad \nu \in \mathbb{C}),\end{gathered}$$

which obviously is expressible in terms of the *Jacobi function of the second kind* defined by (*cf.*, *e.g.*, [18, Vol. II, p. 170, Equation 10.8 (18)])

$$\begin{aligned}Q_\nu^{(\alpha,\beta)}(z) &:= \frac{2^{\nu+\alpha+\beta}\, \Gamma(\nu+\alpha+1)\Gamma(\nu+\beta+1)}{(z-1)^{\nu+\alpha+1}(z+1)^{\beta}\, \Gamma(2\nu+\alpha+\beta+2)} \\ &\cdot \ {}_2F_1\left(\nu+1, \nu+\alpha+1; 2\nu+\alpha+\beta+2; \frac{2}{1-z}\right)\end{aligned} \tag{10.25}$$

$$(|1-z| > 2; \;\; \nu \in \mathbb{C}).$$

In concluding this section, we observe that such general results as Theorems 7.2, 7.3 and 7.4 and their various companions (proven by Tu *et al.* [99]) can be applied similarly in order to derive explicit solutions of many other interesting families of ordinary and partial differential equations.

11. Further Miscellaneous Applications of Fractional Calculus

For the purpose of those in the audience who are interested in pursuing investigations on the subject of fractional calculus, we give here references to *some* of the other applications of fractional calculus operators in the mathematical sciences, which are *not* mentioned in the preceding sections.

- **(i)** Theory of Generating Functions of Orthogonal Polynomials and Special Functions (see, for details, [91]);
- **(ii)** Geometric Function Theory (especially the Theory of Analytic, Univalent, and Multivalent Functions) (see, for details, [92, 93]);
- **(iii)** Integral Equations (see, for details, [25, 83, 84]);
- **(iv)** Integral Transforms (see, for details, [32, 50]);
- **(v)** Generalized Functions (see, for details, [50]);
- **(vi)** Theory of Potentials (see, for details, [66]).

12. Other Recent Developments

In the past several decades, various real-world issues have been modeled in many areas by using some very powerful tools. One of these tools is fractional calculus. Several important definitions have been introduced for fractional-order derivatives, including: the Riemann-Liouville, the Grünwald-Letnikov, the Liouville-Caputo, the Caputo-Fabrizio and the
Atangana-Baleanu fractional-order derivatives (see, for example, [7, 12, 14, 31, 64, 112]).

By using the fundamental relations of the Riemann-Liouville fractional integral, the Riemann-Liouville fractional derivative was constructed, which involves the convolution of a given function and a power-law kernel (see, for details, [31, 64]). The Liouville-Caputo (LC) fractional derivative involves the convolution of the local derivative of a given function with a power-law function [13]. Recently, Caputo and Fabrizio [12] and Atangana and Baleanu [7] proposed some interesting fractional-order derivatives based upon the exponential decay law which is a generalized power-law function (see [1, 3, 4, 5, 6, 8]). The Caputo-Fabrizio (CFC) fractional-order derivative as well as the Atangana-Baleanu (ABC) fractional-order derivative allow us to describe complex physical problems that follow, at the same time, the power law and the exponential decay law (see, for details, [1, 3, 4, 5, 6, 8]).

In a noteworthy earlier investigation, Srivastava and Saad [95] investigated the model of the gas dynamics equation (GDE) by extending it to some new models involving the time-fractional gas dynamics equation (TFGDE) with the Liouville-Caputo (LC), Caputo-Fabrizio (CFC) and Atangana-Baleanu (ABC) time-fractional derivatives. They employed the Homotopy Analysis Transform Method (HATM) in order to calculate the approximate solutions of TFGDE by using LC, CFC and ABC in the Liouville-Caputo sense and studied the convergence analysis of HATM by finding the interval of convergence through the h-curves. Srivastava and Saad [95] also showed the effectiveness and accuracy of this method (HATM) by comparing the approximate solutions based upon the LC, CFC and ABC time-fractional derivatives.

Given the *homogeneous* time-fractional gas dynamics equation (TFGDE) as follows:

$$\frac{\partial^\alpha \psi}{\partial \tau^\alpha} + \psi \, \frac{\partial \psi}{\partial \varsigma} - \psi(1-\psi) = 0, \tag{12.1}$$

where

$$(\varsigma, \tau) \in (0,\infty) \times (0, \tau_0) \quad \text{and} \quad 0 < \alpha \leqq 1. \tag{12.2}$$

Srivastava and Saad [95] used the HATM (see, for example, [38, 67]) in order to solve the LC, CFC and ABC analogues of the TFGDE (12.1). They obtained these analogous equations by replacing the time-fractional derivative $\dfrac{\partial^\alpha \psi}{\partial \tau^\alpha}$ in the TFGDE (12.1) by

$$ {}^{\mathrm{LC}}_{\ 0}D^\alpha_\tau \psi, \quad {}^{\mathrm{CFC}}_{\ \ 0}D^\alpha_\tau \psi \qquad \text{and} \qquad {}^{\mathrm{ABC}}_{\ \ 0}D^\alpha_\tau \psi, $$

successively, where the order α of the time-fractional derivatives is constrained by

$$n-1 < \alpha \leqq n \qquad (n \in \mathbb{N} := \{1,2,3,\cdots\}).$$

The corresponding LC, CFC and ABC time-fractional analogues of the TFGDE (12.1) are given by

$${}^{\mathrm{LC}}_{\ 0}D^\alpha_\tau \psi + \psi \, \frac{\partial \psi}{\partial \varsigma} - \psi(1-\psi) = 0 \qquad (0 < \alpha \leqq 1;\ \varsigma \in \mathbb{R};\ \tau > 0), \tag{12.3}$$

$${}^{\mathrm{CFC}}_{\ \ 0}D^\alpha_\tau \psi + \psi \, \frac{\partial \psi}{\partial \varsigma} - \psi(1-\psi) = 0 \qquad (0 < \alpha \leqq 1;\ \varsigma \in \mathbb{R};\ \tau > 0) \tag{12.4}$$

$${}^{\mathrm{ABC}}_{\ \ 0}D^\alpha_\tau \psi + \psi \, \frac{\partial \psi}{\partial \varsigma} - \psi(1-\psi) = 0 \qquad (0 < \alpha \leqq 1;\ \varsigma \in \mathbb{R};\ \tau > 0), \tag{12.5}$$

respectively. Here

$$ {}^{\mathrm{LC}}_{\ \ 0}D^{\alpha}_{\tau} \qquad \text{and} \qquad {}^{\mathrm{CFC}}_{\ \ \ 0}D^{\alpha}_{\tau} $$

denote the time-fractional derivatives of order α for a suitably defined function $\mathfrak{f}(\tau)$, which are defined, respectively, by

$$ {}^{\mathrm{LC}}_{\ \ 0}D^{\alpha}_{\tau}\big(\mathfrak{f}(\tau)\big) = J^{m-\alpha}D^{m}\big(\mathfrak{f}(\tau)\big) = \frac{1}{\Gamma(m-\alpha)}\int_0^{\tau}(\tau-t)^{m-\alpha-1}\,\mathfrak{f}^{(m)}(t)\,dt $$

$$ (m-1<\alpha \leqq m;\ m\in\mathbb{N};\ \mathfrak{f}\in C^m_{\mu};\ \mu \geqq -1) $$

and

$$ {}^{\mathrm{CFC}}_{\ \ \ 0}D^{\alpha}_{\tau}\big(\mathfrak{f}(\tau)\big) = \frac{M(\alpha)}{1-\alpha}\int_0^{\tau}\exp\left(-\frac{\alpha(\tau-t)}{1-\alpha}\right)D\big(\mathfrak{f}(t)\big)\,dt $$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$ and ${}^{\mathrm{ABC}}_{\ \ \ 0}D^{\alpha}_{\tau}\big(\mathfrak{f}(\tau)\big)$ is known as the ABC time-fractional derivative of order α in the Liouville-Caputo sense given, for a suitably defined function $\mathfrak{f}(\tau)$, by

$$ {}^{\mathrm{ABC}}_{\ \ \ 0}D^{\alpha}_{\tau}\big(\mathfrak{f}(\tau)\big) = \frac{M(\alpha)}{1-\alpha}\int_0^{\tau}E_{\alpha}\left(-\frac{\alpha(\tau-t)}{1-\alpha}\right)D\big(\mathfrak{f}(t)\big)\,dt, $$

where

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)} $$

is the Mittag-Leffler function and $M(\alpha)$ is a normalization function with the same properties as in the Liouville-Caputo (LC) and the Caputo-Fabrizio (CFC) cases. For the details of this and other closely-related investigations, the interested reader should see the work by Srivastava and Saad [95].

In the bibliography of this presentation, we have chosen to include a remarkably large number of recently-published books, monographs and edited volumes (as well as journal articles) dealing with the extensively-investigated subject of fractional calculus and its widespread applications. Indeed, judging by the on-going contributions to the theory and applications of *Fractional Calculus and Its Applications*, which are continuing to appear in some of the leading journals of mathematical, physical, statistical and engineering sciences, the importance of the subject-matter dealt with in this survey-cum-expository article cannot be over-emphasized.

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