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Quasi-reversibility of the Ring of 2×2 Matrices over an Arbitrary Field

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Abstract. A ring R is *quasi-reversible* if $0 \neq ab \in I(R)$ for $a, b \in R$ implies $ba \in I(R)$, where $I(R)$ is the set of all idempotents in R. In this short paper, we prove that the ring of 2×2 matrices over an arbitrary field is quasi-reversible, which is an answer to the question given by Da Woon Jung et al. in [Bull. Korean Math. Soc., 56(4) (2019) 993-1006].

1. Introduction

Let R be a ring. Use $I(R)$ to denote the set of all idempotents in R and $I(R)' = I(R) \setminus \{0\}$. Let $Mat_n(R)$ be $n \times n$ matrix ring over R. Following Da Woon Jung et. al. [1] a ring R is *quasi-reversible* provided that if $ab \in I(R)'$ for $a, b \in R$, then $ba \in I(R)$.

Theorem 1.1.([1, Theorem 1.8]) *$Mat_2(\mathbb{Z}_2)$ is quasi-reversible.*

2. Main Result

Now, we propose an answer to Question 1 stated in Da Woon Jung et. al. [1].

Question 2.1. Let K be a field. Is $Mat_2(K)$ quasi-reversible?

Theorem 2.2. *Let K be a filed. Then $Mat_2(K)$ is quasi-reversible.*

Proof. Let K be a field and $A, B \in Mat_2(K)$ such that $AB \in I(Mat_2(K))'$. If A is invertible, then

$$BA = A^{-1}(AB)A = A^{-1}(ABAB)A = BABA.$$

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Hence $BA \in I(Mat_2(K))$. Similarly, if B is invertible, then BA is idempotent. It remains to consider the case that A and B are non-invertible. To this end we prove the following claims:

Claim 1. If $M = (m_{ij})$ is a singular 2×2 matrix, then $M^2 = tr(M)M$ where, $tr(M) = m_{11} + m_{22}$.

Proof of Claim 1. Since $|M| = 0$, it follows that $m_{11}m_{22} = m_{12}m_{21}$. Consequently, we have

$$\begin{aligned} M^2 &= \begin{pmatrix} m_{11}^2 + m_{12}m_{21} & m_{11}m_{12} + m_{12}m_{22} \\ m_{11}m_{21} + m_{21}m_{12} & m_{12}m_{21} + m_{22}^2 \end{pmatrix} \\ &= \begin{pmatrix} m_{11}^2 + m_{11}m_{22} & m_{12}(m_{11} + m_{22}) \\ m_{21}(m_{11} + m_{22}) & m_{11}m_{22} + m_{22}^2 \end{pmatrix} \\ &= (m_{11} + m_{22})M \\ &= tr(M)M. \end{aligned}$$

Claim 2. If $M = (m_{ij})$ is a non-zero singular and idempotent 2×2 matrix, then $tr(M) = 1$.

Proof of Claim 2. By Claim 1, we have $0 \neq M = M^2 = tr(M)M$ thus $(tr(M) - 1)M = 0$ and hence $tr(M) = 1$.

Now, Let A and B be two non-invertible matrices such that $AB \in I(Mat_2(K))'$. Then, Claim 2 concludes $tr(AB) = 1$. Thus, by Claim 1 we deduce that

$$(BA)^2 = tr(BA)BA = tr(AB)BA = BA.$$

Therefore, BA is idempotent and the proof is completed. □

References

[1] D. W. Jung, C. I. Lee, Y. Lee, S. Park, S. J. Ryu, H. J. Sung and S. J. Yun, *On reversibility related to idempotents*, Bull. Korean Math. Soc., **56(4)**(2019), 993–1006.