KYUNGPOOK Math. J. 60(2020), 53-69
https://doi.org/10.5666/KMJ.2020.60.1.53
pISSN 1225-6951 eISSN 0454-8124

Where Some Inert Minimal Ring Extensions of a Commutative Ring Come from

David Earl Dobbs
Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996-1320, USA
e-mail : dedobbs@comporium.net

Abstract. Let $(A, M) \subset (B, N)$ be commutative quasi-local rings. We consider the property that there exists a ring D such that $A \subseteq D \subset B$ and the extension $D \subset B$ is inert. Examples show that the number of such D may be any non-negative integer or infinite. The existence of such D does not imply $M \subseteq N$. Suppose henceforth that $M \subseteq N$. If the field extension $A/M \subseteq B/N$ is algebraic, the existence of such D does not imply that B is integral over A (except when B has Krull dimension 0). If $A/M \subseteq B/N$ is a minimal field extension, there exists a unique such D, necessarily given by $D = A + N$ (but it need not be the case that $N = MB$). The converse fails, even if $M = N$ and B/M is a finite field.

1. Introduction

All rings and algebras considered below are commutative and unital; all inclusions of rings, ring extensions and algebra/ring homomorphisms are unital. If A is a ring, then $\mathrm{Spec}(A)$ (resp., $\mathrm{Max}(A)$) denotes the set of all prime (resp., maximal) ideals of A and $\dim(A)$ denotes the Krull dimension of A. If $A \subseteq B$ are rings and $P \in \mathrm{Spec}(A)$, then $B_P := B_{A \setminus P}$. As usual, $|\mathcal{U}|$ denotes the cardinal number of a set $\mathcal{U}$; $\subset$ denotes proper inclusion; $\mathbb{F}_q$ denotes the finite field of cardinality q, for any prime-power q; and X, Y denote algebraically independent indeterminates over the ambient ring(s).

Let $A \subseteq B$ be rings. As usual, $[A, B]$ denotes the set of intermediate rings, $\{C \mid C$ is a ring such that $A \subseteq C \subseteq B\}$. If $A \neq B$, we say, following [16], that $A \subset B$ is a *minimal ring extension* if $[A, B] = \{A, B\}$; that is, if there is no ring C such that $A \subset C \subset B$. If $A \subset B$ is a minimal ring extension, it follows from [16, Théorème 2.2 (i) and Lemme 1.3] that there exists $M \in \mathrm{Max}(A)$ (called the *crucial*

Received June 12, 2019; revised January 7, 2020; accepted January 10, 2020.
2010 Mathematics Subject Classification: primary 13B99, secondary 13B21, 14A15.
Key words and phrases: commutative ring, ring extension, minimal ring extension, inert extension, maximal ideal, minimal field extension, quasi-local ring, integrality, pullback.

maximal ideal of $A \subset B$) such that the canonical injective ring homomorphism $A_M \to B_M$ can be viewed as a minimal ring extension while the canonical ring homomorphism $A_P \to B_P$ is an isomorphism for all prime ideals P of A other than M. An easy proof in [7] via globalization and a case analysis showed that, conversely, a minimal ring extension can be characterized as a ring extension for which there exists a crucial maximal ideal (in the above sense).

A minimal ring extension $A \subset B$ is either integrally closed (in the sense that A is integrally closed in B) or integral. We can mostly ignore the integrally closed minimal ring extensions here, but the interested reader can find additional material on that topic in [15, Section 3] and [2]. The first classification result on minimal ring extensions was due to Ferrand-Olivier [16, Lemme 1.2]: if k is a field, then a nonzero k-algebra B is a minimal ring extension of k (when we view $k \subseteq B$ via the injective structural map $k \to B$) if and only if B is k-algebra isomorphic to (exactly one of) a minimal field extension of k, $k \times k$ or $k[X]/(X^2)$. Now let $A \subset B$ be an integral ring extension, with its conductor $M := (A : B)$ $(:= \{b \in B \mid bB \subseteq A\})$. A standard homomorphism theorem shows that $A \subset B$ is a minimal ring extension if and only if $A/M \subset B/MB$ $(= B/M)$ is a minimal ring extension. In fact (cf. also [16, Lemme 1.2 and Proposition 4.1], [13, Lemma II.3]), the above-mentioned classification result of Ferrand-Olivier leads to the following trichotomy: $A \subset B$ is a (an integral) minimal ring extension if and only if $M \in \mathrm{Max}(A)$ and (exactly) one of the following three conditions holds: $A \subset B$ is said to be respectively *inert*, *decomposed*, or *ramified* if B/MB $(= B/M)$ is isomorphic, as an algebra over the field $k := A/M$, to a minimal field extension of k, $k \times k$, or $k[X]/(X^2)$. Notice that in this situation, where the minimal ring extension $A \subset B$ is integral, its conductor is M, which is also the crucial maximal ideal of $A \subset B$. An important result [14, Proposition 4.6] is that if $A \subset B$ is a (not necessarily integral) minimal ring extension with crucial maximal ideal M, then $A_M \subset B_M$ is a minimal ring extension and it is the same kind of minimal ring extension (that is, integrally closed, inert, decomposed or ramified) as $A \subset B$. For that reason, this paper will focus on ring extensions $A \subseteq B$ such that A is a quasi-local ring.

The above definition of an "inert extension" should not be confused with the use of this terminology in papers, such as [5], that follow the usage in a well-known paper of P. M. Cohn [4]. Remark 2.17 establishes that these two notions of an "inert extension" are logically independent.

Much is known about the possible existence of the various kinds of integral minimal ring extensions (that is, ramified, decomposed or inert). For instance, if A is any nonzero ring, then $A \subset B$ is ramified if one picks $M \in \mathrm{Max}(A)$ and takes B to be the idealization $A(+)A/M$. (The original textbook resource for idealizations is [30], with additional background on idealizations appearing in [22]; as usual, one views $A \subset A(+)A/M$ via the injective unital ring homomorphism $A \to A(+)A/M$, $a \mapsto (a, 0 + M)$.) The fact that $A \subset A(+)A/M$ is a minimal ring extension was shown in [6, Corollary 2.5], while the fact that this minimal ring extensions is ramified (and, hence, not inert) follows from [32, Lemma 2.1] (or from the fact that the maximal ideal of $A(+)A/M$ lying over M is $M(+)A/M$, which properly contains M). Similarly, if $M \in \text{Max}(A)$, then $A \subset A \times A/M$ is a decomposed extension (where one views this inclusion via the injective map $A \to A \times A/M$, $a \mapsto (a, a + M)$): see [15, page 805, lines 1-2]. In addition, generator-and-relations characterizations are known for ramified extensions and for decomposed extensions ([15, Proposition 2.12], see also [9, Lemma 2.1]).

Existence results for inert extensions are not as widely available as for ramified or decomposed extensions. Indeed, some rings A do not have any inert extensions. For instance, if A is a special principal ideal ring (in the sense of [33, page 245]) but not a field (for instance, $\mathbb{Z}/4\mathbb{Z}$), then there does not exist a ring B such that $A \subset B$ is inert [8, Proposition 8]. At the other extreme, [12, Example 4.4] has recently shown that there is no finite absolute upper bound on the number of A-algebra isomorphism classes represented by an inert extension of A as A varies over the class of finite (quasi-)local rings that are not fields. On the other hand, for any finite local ring A that is not a field, the number of A-algebra isomorphism classes that can be represented by an inert extension of A must be finite [10, Theorem 2.3 (a)]. The situation is also complicated if the base ring is a field k, for the number of k-algebra isomorphism classes represented by an inert extension of k (that is, by a minimal field extension of k) is 0 if k is algebraically closed, 1 if k is a real closed field, and denumerably infinite if k is a finite field (cf. [11, Proposition 2.4]).

However, some clarity (if not a complete pattern for the existence of inert extensions) was provided by the following result [11, Theorem 2.5 (b)]. If (A, M) is a finite local ring, then there exist denumerably many pairwise non-isomorphic finite ring extensions B of A such that (for each such B) there exists a ring A^* such that $A^* \in [A, B]$ and $A^* \subset B$ is an inert extension, so that necessarily, B is a separable A^*-algebra (in the sense of [24]) and each such B and its associated A^* are finite local rings with the same maximal ideal; it can be arranged that (one such) A^* is $A + MB$, in which case the maximal ideal that A^* shares with B is MB; and if A is not a field, it can also be arranged (for each of the above B) that $A^* \subset B$ is not a Galois ring extension (in the sense of [3]). Starting in the next paragraph, we will explain how **our main purpose here** is to generalize [11, Theorem 2.5 (b)] and to indicate limitations on further generalizations of it. But first, we wish to indicate an important offshoot of (all three parts of) [11, Theorem 2.5]. As this result showed the existence of finite local rings $A^* \subset B$ such that B is a separable A^*-algebra and $A^* \subset B$ is not a Galois ring extension, it served to reveal that [29, Corollary XV.3] is incorrect. As explained in [11, Remark 2.6 (a)-(c)] (with some follow-up in [10]), that error is one of a number of errors in [29] that stem from a homological error in [18]. As that error was identified in [11] and discussed further in [10], there will be no need to mention that matter further here.

Recall that [11, Theorem 2.5 (b)] established that each finite local ring A is the base ring of some chain of rings $A \subseteq D \subset B$ such that the extension $D \subset B$ is inert. Although it was also shown in [11] that each such A has infinitely many (pairwise non-isomorphic) such B, it was not noticed that if both of the local rings A and B are given and the ambient conditons in [11, Theorem 2.5 (b)] hold, then D is uniquely determined. This fact will be a special case of our main result, Theorem

2.10. Along the way, we ask the following two general questions. Given an inclusion of distinct quasi-local rings $(A, M) \subset (B, N)$, how many rings $D \in [A, B]$ such that $D \subset B$ is inert should one expect? Is there a natural sufficient condition for a unique such D to exist that has as a byproduct the above-mentioned uniqueness of $D = A^*$ (given A and B) in the context of [11, Theorem 2.5 (b)]? As the summary in the next two paragraphs explains, the first question is answered in Examples 2.1 – 2.3. Those examples, along with subsequent examples and results, help to direct a focusing process that leads to an answer to the second question in Theorem 2.10, which is followed by a pair of applications and additional examples and results to further indicate the theoretical possibilities.

We begin by showing in Example 2.1 that there exists an integral (non-minimal) ring extension $(A, M) \subset (B, N)$ of quasi-local rings such that there is no ring $D \in [A, B]$ for which $D \subset B$ is inert. Considering different quasi-local data $(A, M) \subset (B, N)$ in Example 2.2, we find that (by allowing B not to be integral over A), the new data produce infinitely many $D \in [A, B]$ such that $D \subset B$ is inert. As the data in both Example 2.1 and Example 2.2 satisfy $M = N$, we impose that condition, along with the integrality of $A \subset B$, on another pair of quasi-local rings $(A, M) \subset (B, N)$ in Example 2.3, with the result that the number of rings $D \in [A, B]$ such that $D \subset B$ is inert can be any preassigned positive integer. Looking at matters from a different perspective, we then ask the following question. If $(A, M) \subset (B, N)$ are quasi-local rings such that $M \subseteq N$ and the induced extension of residue fields $A/M \subseteq B/N$ is algebraic, must it be the case that $A \subset B$ is integral? Example 2.5 and Proposition 2.6 combine to show that the answer to this question is in the affirmative if and only if $\dim(B) = 0$.

Example 2.8 complicates the focusing process by producing quasi-local rings $(A, M) \subset (B, N)$ for which $M \not\subseteq N$ and there does exist $D \in [A, B]$ such that $D \subset B$ is inert. Such data (for which one cannot view A/M as a subfield of B/N) is so contrary to the spirit of the motivating result from [11] that a new theoretical tool needs to be emphasized. That tool is the pullback (which is, of course, always implicit in studying inert extensions and actually is used at various points throughout this paper). By collecting many related facts in Proposition 2.9, we can then proceed to our main result, Theorem 2.10: if $(A, M) \subset (B, N)$ are quasi-local rings with $M \subseteq N$, such that $A/M \subset B/N$ is a minimal field extension, then $D = A + N = A/M \times_{B/N} B$ is the unique ring $D \in [A, B]$ such that $D \subset B$ is inert. The special case of this result where $A \subset B$ is assumed integral (so that $M \subseteq N$ automatically) is given in Corollary 2.11, while the special case of Corollary 2.11 where B is a finite ring (so that $A \subset B$ is automatically integral) is given in Corollary 2.12. Then Example 2.13 shows that the converse of Theorem 2.10 fails. Indeed, in any positive Krull dimension n, one can find an integral extension $(A, M) \subset (B, N)$ of quasi-local n-dimensional rings such that $M = N$, there exists a unique $D \in [A, B]$ such that $D \subset B$ is inert, and $A/M \subset B/M$ is a finite-dimensional, but not minimal, field extension. Lastly, Example 2.15 (resp., Remark 2.16) gives one more way in which the general setting considered here goes beyond the context from [11, Theorem 2.5 (b)]: there exist data satisfying the conditions in

Theorem 2.10 such that $M = 0$ (resp., $M \neq 0$), the ring extension $A \subset B$ is integral and $MB \subset N$. It is noteworthy that the data in Example 2.15 feature the smallest possible values of $|A|$ and $|B|$ for any set of data with the asserted properties.

Further information about minimal ring extensions can be found in some of the above-mentioned references, as well as in [28] and [31]. Any unexplained material is standard, as in [20] and [26].

2. Results

Recall from [11, Theorem 2.5 (b)] that if (A, M) is any finite (quasi-) local ring which is not a field, then A is a subring of some finite local ring (B, N) such that there exists $D \in [A, B]$ with the property that the ring extension $D \subset B$ is inert. (The ring D that was constructed in the cited result was denoted there by A^*. This D had the additional properties that $N = MB$, the canonical A-algebra homomorphism of residue fields $A/M \to D/N$ was an isomorphism of fields, and the canonical map of residue fields $A/M \to B/N$ could be viewed as a minimal field extension; these additional properties will each play a role later in this section.) One should not expect the existence of such a ring D in general if both of the quasi-local rings $(A, M) \subset (B, N)$ are specified in advance.

The easiest kind of example illustrating this fact is provided by considering any nonzero quasi-local (finite, if you wish) ring (A, M) and taking B to be any quasi-local ring such that $A \subset B$ is a minimal ring extension which is not inert; that is, by taking $A \subset B$ to be any ramified (minimal ring) extension. For instance, take B to be the idealization $A(+)A/M$. (Suitable background on idealizations can be found in [22]; as usual, one views $A \subset A(+)A/M$ via the injective unital ring homomorphism $A \to A(+)A/M$, $a \mapsto (a, 0 + M)$.) The fact that $A \subset A(+)A/M$ is a minimal ring extension is a special case of [6, Corollary 2.5], while the fact that this minimal ring extensions is ramified (and, hence, not inert) follows from [32, Lemma 2.1] (or from the fact that the maximal ideal of $A(+)A/M$ is $M(+)A/M$, which properly contains M). A less trivial kind of example is provided in Example 2.1, where the data also satisfy $M = N$.

The proof of Example 2.1 will use the following well known facts. Let F be a field, with algebraic closure $\overline{F}$. Then F is an ordered field (in the sense of [23, Definition 1, page 270]) if and only if F is a totally real field (in the sense of [23, Definition 2, page 271]): cf. [23, page 271 and Corollary 2, page 274]. Also, F is a real closed field (in the sense of [23, Definition 3, page 273], that is, a totally real field such that no algebraic proper field extension of F is a totally real field) if and only if F is not algebraically closed and $[\overline{F} : F] < \infty$.

Example 2.1. Let K be any field which is neither an ordered field nor an algebraically closed field (for instance, take K to be any finite field). Let L be any algebraic closure of K; let (B, M) be any valuation domain which is not a field but is of the form $B = L + M$ (for instance, take $B := L[[X]]$ where X is an analytic indeterminate over L); and put $A := K + M$. Then $(A, M) \subset (B, M)$ is an integral, but not minimal, ring extension involving quasi-local rings A and B, such that there does not exist $D \in [A, B]$ with the property that the ring extension $D \subset B$ is inert.

Proof of Example 2.1. By definition, M is the maximal ideal of B. Note that A arises from the classical "$(D+M)$-construction". Therefore, since K is a field, it follows from [19, Theorem A (c), (d), pages 560-561] that M is the unique maximal ideal of A, that is, (A, M) is quasi-local. Similarly, since $K \subset L$ is an integral extension, it follows from [19, Theorem A (b), page 560] that $A \subset B$ is also an integral extension. (Alternatively, since we can identify $B/M = L$ and $A/M = K$, we can view A as the pullback $K \times_L B$, and so the two preceding conclusions can also be obtained from more general pullback-theoretic results, namely, [17, Corollary 1.5 (1), (5)].)

Next, since M is a common ideal of A and B, it follows from a standard homomorphism theorem that the assignment $E \mapsto E/M$ gives a bijection $[A, B] \to [A/M, B/M] = [K, L]$ (cf. [13, Lemma II.3]), which is actually an order-isomorphism when $[A, B]$ and $[A/M, B/M]$ are each regarded as posets under inclusion. The assertion that $A \subset B$ is not a minimal ring extension is therefore equivalent to the assertion that $K \subset L$ is not a minimal ring extension; that is (since L is algebraic over K), to the assertion that $K \subset L$ is not a minimal field extension.

It will suffice to prove that $F \subset L$ is not a minimal field extension if F is a field in $[K, L]$. Indeed, the case $F = K$ would show, in view of the above reasoning, that $A \subset B$ is not a minimal ring extension. In addition, it would also follow that there does not exist $D \in [A, B]$ such that $D \subset B$ is an inert extension (for otherwise, we would have $D/M \in [A/M, B/M] = [K, L]$ and $D/M \subset B/M = L$ would be a minimal field extension).

Suppose the assertion fails. Pick some $F \in [K, L]$ such that $F \subset L$ is a minimal field extension. Then $F \subset L$ is a finite-dimensional field extension. Since L is an algebraic closure of F, it follows from the facts that were recalled above that F is a real closed field, hence a totally real field, hence an ordered field. As every subfield of an ordered field is an ordered field [23, page 271], it then follows that K is an ordered field, the desired contradiction. The proof is complete. □

The behavior of the data in Example 2.2 will be very different from that in Example 2.1, even though we will require that $M = N$. Specifically, the transition from the previous example to the next example involves going from data for which there are no rings D with the desired behavior to data supporting infinitely many such rings D. That infinitude will ultimately be seen to stem from the infinitude of the set of prime numbers.

Example 2.2. Let K be a field, X an indeterminate over K, and p any odd prime number. Then $K(X^p) \subset K(X)$ is a minimal field extension. It follows that $A := K$ and $B := K(X)$ are (quasi-)local rings with the same maximal ideal $M := 0$ and there exist infinitely many rings $D \in [A, B]$ such that $D \subset B$ is an inert (minimal ring) extension. In addition, the induced extension of residue fields $A/M \subset B/M$ is not a minimal field extension; in fact, this field extension is not even algebraic.

Proof of Example 2.2. Since $A/M \subset B/M$ can be identified with the (purely) transcendental field extension $K \subset K(X)$ and $K(X) = K(X^p)(X)$, it is enough to prove that the polynomial $Y^p - X^p$ is irreducible in the polynomial ring $K(X^p)[Y]$. (Indeed, it would then follow that $[K(X) : K(X^p)] = p$, so that $K(X^p) \in [A, B]$ is such that $K(X^p) \subset B$ is a minimal field extension, that is, an inert extension; and that $K(X^{p_1}) \neq K(X^{p_2})$ whenever p_1 and p_2 are distinct odd prime numbers.) In view of a classic irreducibility result in field theory (which holds even if $p = 2$: see [27, Theorem 16, page 221] or [25, Theorem 51, page 62]), it therefore suffices to show that there does not exist $\xi \in K(X^p)[Y]$ such that $X^p = \xi^p$.

Suppose, on the contrary, that such an element ξ exists. We have $\xi = g/h$ for some nonzero elements $g, h \in K(X^p)[Y]$. By finding a common denominator, we have nonzero polynomials $g_1, g_2 \in K[X^p][Y]$ and $f \in K[X^p]$ such that $g = g_1/f$ and $h = g_2/f$. Then $\xi = g_1/g_2$ and so $X^p g_2^p = \xi^p g_2^p = (g_1^p/g_2^p)g_2^p = g_1^p$. Let deg denote "degree in X" for polynomials in K[Y][X]. As $g_1, g_2 \in K[Y][X^p] \subseteq K[Y][X]$, we get $m := \deg(g_1) \in p\mathbb{Z}$ and $n := \deg(g_2) \in p\mathbb{Z}$. Consequently,

$$p + pn = \deg(X^p) + \deg(g_2^p) = \deg(X^p g_2^p) = \deg(g_1^p) = pm.$$

Dividing by p gives $1 + n = m$, whence $1 = m - n \in p\mathbb{Z} + p\mathbb{Z} = p\mathbb{Z}$, the desired contradiction. The proof is complete. □

In response to the surfeit of rings D with the desired behavior that was found in Example 2.2, we next examine some data such that $M = N$ and the ring extension $A \subset B$ is integral (so that, in particular, the extension of residue fields $A/M \subset B/M$ is algebraic). The upshot in Example 2.3 is that data of this kind can support any nonzero finite number of rings D with the desired behavior. Thus, for such data, there is no finite absolute upper bound on the number of rings $D \in [A, B]$ such that the ring extension $D \subset B$ is inert.

Example 2.3. Fix a positive integer n and a prime number p. Let $p_1, \ldots, p_n$ be n pairwise distinct prime numbers. Put $d := \prod_{j=1}^n p_j$ and $L := \mathbb{F}_{p^d}$. For all $i = 1, \ldots, n$, put $e_i := d/p_i$ and $K_i := \mathbb{F}_{p^{e_i}}$. With X an indeterminate over L, let $x := X + (X^2) \in L[X]/(X^2)$. Then $x^2 = 0 \neq x$, $B := L[X]/(X^2)$ can be written additively as $L \oplus Lx$ so that B is a (quasi-)local ring with maximal ideal $M := Lx$, $A := \mathbb{F}_p + M$ is a (quasi-)local subring of B that also has maximal ideal M, and there are exactly n pairwise distinct rings $D \in [A, B]$ such that $D \subset B$ is an inert (minimal ring) extension (namely, the n rings $D_i := K_i + M$ for $i = 1, \ldots, n$). In addition, the induced extension of residue fields $A/M \subset B/M$ is algebraic, in fact, finite-dimensional. However, $A/M \subset B/M$ is a minimal field extension if and only if $n = 1$.

Proof of Example 2.3. Since $A = \mathbb{F}_p \oplus M$ and $B = L \oplus M$ as abelian groups, the assertions that A and B are each (quasi-)local rings with the same maximal ideal M follow from the fact that in any ring, the sum of a unit and a nilpotent element is a unit.

Next, as in the proof of Example 2.1, the assignment $E \mapsto E/M$ gives an order-isomorphism $[A, B] \to [A/M, B/M]$ of posets under inclusion. Also, since B is a finite ring, the ring extension $A \subset B$ is integral. (For one well known proof of this standard fact, see [29, Theorem XIII.1].) If $E \in [A, B]$, then $M = M \cap E \in \mathrm{Spec}(E)$ and in fact, it follows from the integrality of the extension $E \subseteq B$ that E is a quasi-local ring with unique maximal ideal M (cf. [1, Corollary 5.8]). Thus, if $D \in [A, B]$, then $D \subset B$ is an inert extension if and only if $D/M \subset B/M$ is a minimal field extension. Observe that the field extension $A/M \subset B/M$ can be identified with $\mathbb{F}_p \subset L = \mathbb{F}_{p^d}$. Recall the result from the classical Galois theory of finite fields that a proper field extension $k_1 \subset k_2$ of finite fields is a minimal field extension if and only if $[k_2 : k_1]$ is a prime number. It follows that if $D \in [A, B]$, then $D/M \subset B/M$ is a minimal field extension if and only if $D/M = \mathbb{F}_{p^{d/s}}$ for some prime number s; that is, if and only if $D/M = \mathbb{F}_{p^{e_i}}$ $(= K_i)$ for some $i = 1, \ldots, n$; that is, if and only if $D = K_i + M$ $(= D_i)$ for some $i = 1, \ldots, n$. Also, if $1 \leq i_1 < i_2 \leq n$, then $K_{i_1} \neq K_{i_2}$ since $[K_{i_1} : \mathbb{F}_p] = e_{i_1} \neq e_{i_2} = [K_{i_2} : \mathbb{F}_p]$; that is, $D_{i_1}/M \neq D_{i_2}/M$, whence $D_{i_1} \neq D_{i_2}$. Thus, the D_i are precisely the n pairwise distinct rings $D \in [A, B]$ such that $D \subset B$ is inert.

Since the field extension $A/M \subset B/M$ can be identified with $\mathbb{F}_p \subset L = \mathbb{F}_{p^d}$, we have $[B/M : A/M] = d < \infty$. In particular, this field extension is finite-dimensional and, hence, algebraic. Moreover, it follows the above-mentioned result from the Galois theory of finite fields that $A/M \subset B/M$ is a minimal field extension if and only if d is a prime number; that is, if and only if $n = 1$. The proof is complete. $\square$

Remark 2.4. In the context of Example 2.3, $D_i = K_i + M$ can be regarded as the pullback (in the category of commutative rings) $D_i = D_i/M \times_{B/M} B = K_i \times_L B$. In particular, by the final assertion in the statement of Example 2.3, $A \subset B$ is an inert extension $\Leftrightarrow$ $A/M \subset B/M$ is a minimal field extension $\Leftrightarrow$ $n = 1$ $\Leftrightarrow$ $A = \mathbb{F}_p + M = \mathbb{F}_p \times_{B/M} B = K_1 + N = K_1 \times_L B$ and $B = L + N = \mathbb{F}_{p^{p_1}} + N$. Our main result, Theorem 2.10, will give a sufficient condition for the existence of a unique D with the suitable behavior. The D that will be found in Theorem 2.10 will admit an explicit description as a sum/pullback somewhat in the style of Example 2.3. However, the sufficient condition that is assumed in Theorem 2.10 will, more in the style of the context for [11, Theorem 2.5 (b)] than that of Example 2.3, not require that $M = N$.

Consider quasi-local rings $(A, M) \subset (B, N)$. Theorem 2.10 will, in the spirit of [11, Theorem 2.5 (b)], give a sufficient condition that there exists a unique ring $D \in [A, B]$ such that $D \subset B$ is an inert extension. Motivated by the specific data and hypotheses in [11, Theorem 2.5 (a)-(c)], one may expect that any such sufficient condition should feature a role for the field extension $A/M \subseteq B/N$. However, for this to be a well-defined ring extension, one must have $M \subseteq N$. (Indeed, if (A, M) is a quasi-local ring and (C, P) is a quasi-local A-algebra, there is a unique A-algebra homomorphism $A \to C/P$, necessarily given by $a \mapsto a \cdot 1_C + P$ where 1_C is the multiplicative identity element of C, and so there exists an A-algebra homomorphism $A/M \to C/P$ if and only if $h(M) = 0$; that is, if and only if $M \cdot 1_C \subseteq P$.) Note that $M = N$ in Examples 2.1 – 2.3, while the ring extension $A \subset B$ was integral in Examples 2.1 and 2.3 but not in Example 2.2. In view of the data in those examples, it seems natural to ask whether the specification that $M \subseteq N$, together with the requirement that the field extension $A/M \subseteq B/N$ is algebraic, entails that the ring extension $A \subset B$ is integral. Example 2.5 will answer this question in the negative, but Proposition 2.6 will provide a positive answer for the special case where B is of (Krull) dimension 0.

Example 2.5.

(a) Let $K \subset L$ be a minimal field extension, and let (B, N) be any valuation domain which is not a field but is of the form $B = L + N$. Put $A := K$. Then $(A, M) \subset (B, N)$ are quasi-local rings such that $M := 0 \subset N$, the induced extension of residue fields $A/M \subseteq B/N$ is algebraic, $D := K + N \in [A, B]$ is such that $D \subset B$ is an inert (minimal ring) extension, and the ring extension $A \subset B$ is not integral.

(b) If $1 \leq n \leq \infty$ is preassigned and the minimal field extension $K \subset L$ is given, then there exists a valuation domain B as in (a) such that $\dim(B) = n$ (and all the assertions in (a) hold).

Proof of Example 2.5. Given $1 \leq n \leq \infty$ and a minimal field extension $K \subset L$, it is well known that there exists a valuation domain (B, N) such that $B = L + N$ and $\dim(B) = n$ (cf. [20, Corollary 18.5]). It therefore suffices to prove (a). As in the third sentence of the proof of Example 2.1, we get that (B, N) and (D, N) are quasi-local rings. Next, since the field extension $A/M \subseteq B/N$ can be identified with $K \subset L$, the assertion that $A/M \subseteq B/N$ is algebraic follows from the fact that any minimal field extension is algebraic. Next, consider the ring extension $(D, N) \subset (B, N)$. As the induced residue field extension $D/N \subseteq B/N$ can be identified with the (necessarily algebraic, hence integral) minimal field extension $K \subset L$, it follows (cf. [17, Corollary 1.5 (5)]) that $D \subset B$ is integral and inert.

It remains only to show that the ring extension $A \subset B$ is not integral. Suppose that the assertion fails. Then each element of N ($\subset B$) is integral over A. As L is integral (algebraic) over $K = A$ and integrality is preserved by sums (cf. [26, Theorem 13]), it follows that the integral domain $B = L + N$ is integral over the field $A = K$. As B is not a field, we have the desired contradiction (cf. [1, Proposition 5.7]), which completes the proof. □

Proposition 2.6. *Let $(A, M) \subset (B, N)$ be quasi-local rings such that $M \subseteq N$, the induced extension of residue fields $A/M \subseteq B/N$ is algebraic, and* $\dim(B) = 0$. *Then the ring extension $A \subset B$ is integral.*

Proof. Using the canonical isomorphism $(A + N)/N \cong A/(N \cap A) = A/M$, we see that $A + N$ is the pullback $A/M \times_{B/N} B$. As $A/M \subseteq B/N$ is algebraic (integral), it follows, by applying [17, Corollary 1.5 (5)] to this pullback description, that the ring extension $A + N \subseteq B$ is integral. Thus, by the transitivity of "is an integral ring extension" (cf. [26, Theorem 40]), it will suffice to prove that $A + N$ is integral over A. Since integrality preserves sums (cf. [26, Theorem 13]), it will be enough to show that each element of N is integral over A. Thus, it will suffice to show that each element of N is nilpotent. This, in turn, holds, since N is the nil-radical of B (cf. [1, Proposition 1.8]), the point being that the hypotheses ensure that N is the only prime ideal of B. The proof is complete. □

Remark 2.7.

(a) By Example 2.5 (b), the conclusion of Proposition 2.6 would fail if we delete the hypothesis that $\dim(B) = 0$. (In detail, the proof of Example 2.5 shows that $(A, M) = (K, 0) \subset (L + N, N) = (B, N)$ are quasi-local rings such that $M = 0 \subseteq N$; that the extension $A/M \subseteq B/N$ identifies with the minimal, hence algebraic, field extension $K \subset L$; and that the extension $A \subset B$ is not integral. Of course, $\dim(B) \neq 0$ since B is a quasi-local integral domain whose maximal ideal N is nonzero, the point being that B is not a field.)

(b) For an example of data satisfying the hypothesis of Proposition 2.6, one could take B as any finite (quasi-)local ring having a proper subring A. For such data, B may or may not be a field: consider taking (A, B) to be $(\mathbb{F}_2, \mathbb{F}_4)$ or $(\mathbb{F}_2, \mathbb{F}_4[X]/(X^2))$.

By tweaking the data from Example 2.5, we next present data that behave very differently, in the sense that a ring D with the desired behavior exists although $M \not\subseteq N$.

Example 2.8. Let (A, M) be a valuation domain but not a field, let K be the quotient field of A, and suppose that there exists a minimal field extension $K \subset L$. (For instance, take $A := \mathbb{Z}_{2\mathbb{Z}}$, with $K = \mathbb{Q}$, and $L := \mathbb{Q}(\sqrt{2})$.) Choose (B, N) to be any valuation domain which is not a field but is of the form $B = L + N$. Put $D := K + N$. Then $(A, M) \subset (B, N)$ are quasi-local rings such that $M \not\subseteq N$ (and so there does not exist an A-algebra homomorphism $A/M \to B/N$) and $D \in [A, B]$ is such that $D \subset B$ is an inert (minimal ring) extension.

Proof of Example 2.8. As in the proofs of Examples 2.1 and 2.5, we get that (D, N) is quasi-local. As recalled in the first sentence of the proof of Example 2.5, it is possible to choose B as stipulated. Also, A and B are each quasi-local. As $N \cap A \subseteq N \cap K = 0$ and $M \neq 0$, we have that $M \not\subseteq N$. Hence, by the discussion prior to Example 2.5, there does not exist an A-algebra homomorphism $A/M \to B/N$. (In particular, we cannot view A/M as a subfield of B/N.)

It remains only to show that $D \subset B$ is inert. As the induced residue field extension $D/N \subseteq B/N$ can be identified with the (necessarily algebraic, hence integral) minimal field extension $K \subset L$, it follows (cf. [17, Corollary 1.5 (5)]) that $D \subset B$ is integral and inert. The proof is complete. □

It is surely apparent from several of the above proofs that pullback constructions are relevant to a more general study of inert extensions. Parts (a) and (b) of Proposition 2.9 are a couple of straightforward results in this same vein. Proposition 2.9(c) is a related result that figures in the proof of our main result. Proposition

2.9 (c) is stated at a level of generality that, in our opinion, should be recorded. Although we did not pause to note the elementary result Proposition 2.9 (a) (iii) earlier, it could have been used to shorten the next-to-last paragraph of the proof of Example 2.5 and the end of the proof of Example 2.8. Indeed, if Proposition 2.9 (a) (iii) had been available there, we would have not needed to show first that the ring extension $D \subset B$ is integral before concluding that this extension is inert.

Proposition 2.9.

(a) *Let I be a common ideal of rings $A \subseteq B$. Then:*

(i) *A is (isomorphic to) the pullback $A/I \times_{B/I} B$.*

(ii) *$A \subset B$ is a minimal ring extension if and only if $A/I \subset B/I$ is a minimal ring extension.*

(iii) *$A \subseteq B$ is an integral ring extension if and only if $A/I \subseteq B/I$ is an integral ring extension.*

(b) *Let $A \subseteq B$ be quasi-local rings with the same maximal ideal M. Then the following five conditions are equivalent:*

(1) *The ring extension $A \subset B$ is inert;*

(2) *The ring extension $A \subset B$ is integral and inert;*

(3) *$A \subset B$ is an integral minimal ring extension;*

(4) *$A \subset B$ is a minimal ring extension;*

(5) *$A/M \subset B/M$ is a minimal field extension.*

(c) *Let (B, M) be a quasi-local ring, let $\pi : B \to B/M$ be the canonical surjection, let k be a proper subfield of B/M, and let $A := k \times_{B/M} B$. Let $j : A \to B$ be the injective ring homomorphism and let $p : A \to k$ be the surjective ring homomorphism that are canonically associated with the pullback definition of A. (So, if we view $A \subseteq B$, then j is the inclusion map and $p = \pi|_A$.) Put $\mathcal{M} := j^{-1}(M)$. Then:*

(i) *A is a quasi-local ring with maximal ideal $\mathcal{M}$, and $j(A)$ is a quasi-local ring with maximal ideal M.*

(ii) *$j(A) \subset B$ is an inert extension (necessarily with crucial maximal ideal M) if and only if $k \subset B/M$ is a minimal field extension.*

Proof. (a) Let $\pi : B \to B/I$ denote the canonical surjection. To prove (i), it suffices to observe that $\pi^{-1}(A/I) = A$.
(ii) As in the proof of Example 2.1, the assignment $E \mapsto E/I$ gives an order-isomorphism $[A, B] \to [A/I, B/I]$ of posets under inclusion (cf. [13, Lemma II.3]). The assertion follows at once.
(iii) One could simply apply [17, Corollary 1.5 (5)] to the pullback description from (i). Alternatively, the concrete nature of this pullback allows for the following

straightforward calculational arguments. For the "only if" assertion, if $b \in B$, apply π to an integrality equation for b over A; for the "if" assertion, argue as in the proof of [21, Lemma 4.6], *mutatis mutandis.*

(b) (1) $\Rightarrow$ (2): By definition, every inert (minimal) ring extension is integral.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4): Trivial.

(4) $\Rightarrow$ (1): As A is quasi-local, the crucial maximal ideal of the minimal ring extension $A \subset B$ must be M. Then, since $MB = M \neq B$, it follows from [16, Théorème 2.2 (ii)] that $A \subset B$ is an integral extension. It cannot be decomposed, since any decomposed extension of A would have two distinct maximal ideals lying over (its crucial maximal ideal) M. It cannot be ramified, as any ramified extension of A would have a maximal ideal that properly contains (its crucial maximal ideal) M. Therefore, by the process of elimination, $A \subset B$ must be inert.

(1) $\Rightarrow$ (5): This follows from the definition of an inert extension.

(5) $\Rightarrow$ (1): Assume (5). Then $A/M \subset B/M$ is an integral extension and so, by (a) (iii), $A \subset B$ must be an integral extension. As in the proof of (a) (ii), any ring in $[A/M, B/M]$ is of the form E/M with $E \in [A, B]$ and $M = M \cap E \in \text{Spec}(E)$. Then, by integrality (specifically, by [1, Proposition 5.7]), the integral domain E/M must be a field. Therefore, $A/M \subset B/M$ is a minimal ring extension. Hence, by (a) (ii), $A \subset B$ is a minimal ring extension. As (5) was assumed, (1) now follows from the definition of an inert extension.

(c) (i) By applying [17, Theorem 1.4] to the pullback that defined A, we see that the structural map i induces a homeomorphism $\text{Spec}(B) \to \text{Spec}(A)$ in the Zariski topology, thus giving an order-isomorphism $\text{Spec}(B) \to \text{Spec}(A)$ of posets under inclusion. It follows that A is a quasi-local ring with unique maximal ideal $j^{-1}(M) = \mathcal{M}$. Then by a standard homomorphism theorem, $j(A)$ is quasi-local with unique maximal ideal $j(\mathcal{M}) = M$.

(ii) By inspecting commutative diagrams, it is easy to see that $\ker(p) = j^{-1}(M) = \mathcal{M}$, $\ker(\pi|_{j(A)}) = M$, and j induces an isomorphism of fields $A/\mathcal{M} \to j(A)/M = k$. Then, since M is the common unique maximal ideal of $j(A)$ and B, it follows from the equivalences (5) $\Leftrightarrow$ (1) $\Leftrightarrow$ (2) in (b) that $j(A) \subset B$ is a (necessarily integral) inert extension if and only if $k \subset B/M$ is a minimal field extension. Moreover, it follows that when these equivalent conditions hold, the crucial maximal ideal of $j(A) \subset B$ must be the unique maximal ideal of $j(A)$, namely, M. The proof is complete. $\square$

We next present our main result.

Theorem 2.10. *Let $(A, M) \subset (B, N)$ be quasi-local rings such that $M \subseteq N$. View $A/M \subseteq B/N$ by means of the induced* (*injective unique A-algebra*) *homomorphism $A/M \to B/N$, and suppose that $A/M \subseteq B/N$ is a minimal field extension* (*and hence algebraic*). *Then there exists a unique $D \in [A, B]$ such that $D \subset B$ is an inert* (*minimal ring*) *extension. Moreover, this ring D is $A + N$; that is, D is the pullback $D = A/M \times_{B/N} B$.*

Proof. Consider the ring $A + N \in [A, B]$. As at the beginning of the proof of

Proposition 2.6, we have the isomorphism $(A+N)/N \cong A/(N \cap A) = A/M$ and $A+N$ is the pullback $A/M \times_{B/N} B$. Hence, by Proposition 2.9(c)(i), $A+N$ is a quasi-local ring with maximal ideal N. Then, since $(A+N)/N \subseteq B/N$ is a minimal field extension, it follows from Proposition 2.9(b) [(5) $\Rightarrow$ (2)] (cf. also Proposition 2.9(c)(ii)) that $A+N \subset B$ is a (necessarily integral) inert extension.

It remains only to prove that if some $D \in [A,B]$ is such that $D \subset B$ is inert, then $D = A+N$. As D and B necessarily have the same unique maximal ideal, $N \subset D$, and so $A+N \subseteq D \subset B$. Since we showed above that $A+N \subset B$ is inert, hence a minimal ring extension, it must be that $A+N = D$.

The proof is complete. □

We next isolate two special cases of Theorem 2.10

Corollary 2.11. *Let $(A,M) \subset (B,N)$ be an integral extension of quasi-local rings. (Then, necessarily, $M \subseteq N$.) View $A/M \subseteq B/N$ by means of the induced (injective unique A-algebra) homomorphism $A/M \to B/N$, and suppose that $A/M \subseteq B/N$ is a minimal field extension (and hence algebraic). Then there exists a unique $D \in [A,B]$ such that $D \subset B$ is an inert (minimal ring) extension. Moreover, this ring D is $A+N$; that is, D is the pullback $D = A/M \times_{B/N} B$.*

Proof. In view of Theorem 2.10, it suffices to show that $M \subseteq N$. This, in turn, is a consequence of integrality (cf. [1, Corollary 5.8]). □

Corollary 2.12. *Let $(A,M) \subset (B,N)$ be finite (quasi-)local rings. (Then, necessarily, $M \subseteq N$.) View $A/M \subseteq B/N$ by means of the induced (injective unique A-algebra) homomorphism $A/M \to B/N$, and suppose that $A/M \subseteq B/N$ is a minimal field extension (and hence algebraic). Then there exists a unique $D \in [A,B]$ such that $D \subset B$ is an inert (minimal ring) extension. Moreover, this ring D is $A+N$; that is, D is the pullback $D = A/M \times_{B/N} B$.*

Proof. As any ring extension involving finite rings is integral, an application of Corollary 2.11 completes the proof. □

It is natural to ask if the converse of Theorem 2.10 is valid. Despite expectations that may have been raised by [11, Theorem 2.5 (b)], Example 2.13 provides a negative answer to this question.

Example 2.13. Let $1 \leq n \leq \infty$. Then there exists an integral extension $(A,M) \subset (B,M)$ of quasi-local rings with the same maximal ideal for which there exists a unique ring $D \in [A,B]$ such that $D \subset B$ is an inert (minimal ring) extension, $\dim(A) = \dim(B) = n$, and the induced (algebraic) field extension $A/M \subseteq B/M$ is finite-dimensional but not a minimal field extension. Furthermore, it can also be arranged that B/M is a finite field. One way to construct such data is the following. Let p be a prime number, put $K := \mathbb{F}_p$ and $L := \mathbb{F}_{p^{(p^2)}}$, take (B,M) to be any n-dimensional valuation domain which is not a field but is of the form $B = L+M$, and put $A := K+M$. For this data set, with $F := \mathbb{F}_{p^p}$, the assertion holds for $D := F+M$.

Proof of Example 2.13. It is well known that there exists a valuation domain

(B, M) such that $B = L + M$ and $\dim(B) = n$ (cf. [20, Corollary 18.5]). As the field extension $A/M \subseteq B/M$ identifies with the (finite-dimensional, hence algebraic) field extension $K \subset L$, it follows (cf. [17, Corollary 1.5 (5)]) that $A \subset B$ is integral. Hence, by [26, Theorem 48] (or by [20, Exercise 12 (4), page 203]), $\dim(A) = \dim(B) = \dim(F + M)$ $(= n)$. Moreover, $A, F + M$ and B are each quasi-local rings having maximal ideal M (cf. [19, Theorem A (c), (d), pages 560-561]). Next, as in the proof of Example 2.1, the assignment $E \mapsto E/M$ gives an order-isomorphism $[A, B] \to [A/M, B/M] = [K, L]$ of posets under inclusion. In particular, each ring in $[A, B]$ is expressible as the sum of a uniquely determined field in $[K, L]$ and M. Since the classical Galois theory of finite fields ensures that $[K, L]$ is linearly ordered (in fact, we have chosen the data so that $[K, L] = \{K, F, L\}$), the only field $k \in [K, L]$ such that $k \subset L$ is a minimal field extension is $k = F$. Therefore, by Proposition 2.9(b), the only ring $R \in [A, B]$ such that $R \subset B$ is inert is $R = k + M = F + M = D$. The proof is complete. □

Remark 2.14. In contrast to the relationship between Example 2.5 and Proposition 2.6, it should be noted that the case where $\dim(B) = 0$ does not behave differently from the case $\dim(B) > 0$ for the question that was studied in Example 2.13. To see this, take A, B and D to be, respectively, the fields K, L and F from Example 2.13.

Recall that [11, Theorem 2.5(b)] gave a chain $(A, M) \subseteq (A^*, N) \subset (B, N)$ (whose steps are necessarily integral extensions) of (quasi-)local finite rings such that $0 \neq M \subseteq MB = N$, $A^* \subset B$ is inert and $A/M \subset B/N$ is a minimal field extension. It follows from Theorem 2.10 that A^* is uniquely determined by these conditions. This naturally leads to the question of whether the conditions in Theorem 2.10 force N to equal MB (at least in case $A \subset B$ is integral). We next show that this question has a negative answer (when viewed in the general setting of Theorem 2.10). In fact, Example 2.15 identifies the smallest possible data $(A, M) \subset (B, N)$ illustrating that negative answer.

Example 2.15. There exist finite (quasi-) local rings $(A, M) \subset (B, N)$ such that (necessarily $A \subset B$ is an integral ring extension and $M \subseteq N$, and the induced extension of residue fields is such that) $A/M \subset B/N$ is a minimal field extension, there exists a unique $D \in [A, B]$ such that $D \subset B$ is an inert (minimal ring) extension, and $MB \subset N$. Furthermore, it can also be arranged that A is a finite field. One way to construct such data is the following. Take $A := K := \mathbb{F}_2$ and $B := \mathbb{F}_4[X]/(X^2)$, where X is an indeterminate over $L := \mathbb{F}_4$.

Proof of Example 2.15. Note that $x := X + (X^2) \in B$ satisfies $x^2 = 0 \neq x$, the unique maximal ideal of B is $N := Lx$, the unique maximal ideal of A is $M := 0 \subset N$, and the additive structure of B is given by $B = L \oplus N$. As L is algebraic (integral) over $K = A$ and $N^2 = 0$, it is clear that $A \subset B$ is integral. Also, the field extension $A/M \subseteq B/N$ can be identified with the minimal field extension $K \subset L$. By Theorem 2.10, the only ring $D \in [A, B]$ such that $D \subset B$ is inert is given by $D = A + N = \mathbb{F}_2 + \mathbb{F}_4 x$. Finally, $MB = 0 \subset N$ since $x \neq 0$. □

Remark 2.16. It may be of interest to illustrate Theorem 2.10 by constructing data that is somewhat in the spirit of Example 2.15 but also satisfies $M \neq 0$ (and $MB \subset N$). One way to do so is the following. Take any minimal field extension $K \subset L$, let (B, N) be any DVR of the form $B = L + N$ (for instance, $L[[X]]$, where X is an analytic indeterminate over L and $N = XB$), and put $A := K + N^2$. Then A is a quasi-local ring with maximal ideal $M := N^2$, the extension of residue fields $A/M \subset B/N$ can be identified with the minimal field extension $K \subset L$, the ring extension $A \subset B$ is easily seen to be integral, the unique ring $D \in [A, B]$ such that $D \subset B$ is inert is $D = A + N = K + N$, $M \neq 0$, and $MB = N^2(L + N) = N^2 + N^3 = N^2 \subset N$, as desired.

In addition to the meaning given above (and in the already-cited references) to the terminology of an "inert extension," several papers in the literature ascribe quite a different meaning to this terminology. As this other usage derives from a paper of P. M. Cohn [4], it will be referred to here as a "C-inert extension" to avoid confusion. By definition, a ring extension $A \subseteq B$ is said to be C-*inert* if the factorizations of an element of B are the same in B as in A. We close with a remark that briefly addresses the referee's suggestion that we include a comparison of the two notions of an inert extension.

Remark 2.17. Although the two above-mentioned usages of "inert extension" are each motivated by factorization results in classical algebraic number theory, neither of these usages implies the other. To see this, note first that if X and Y are algebraically independent indeterminates over the field $\mathbb{R}$ of real numbers, then the ring extension $\mathbb{R}[X^2 + Y^2] \subset \mathbb{R}[X, Y]$ is C-inert by [5, Example 2.3 (1)], but it is not an inert extension in our sense (as considerations of transcendence degree over $\mathbb{R}$ show that $\mathbb{R}[X, Y]$ is not an integral ring extension of $\mathbb{R}[X^2 + Y^2]$). On the other hand, if $A \subset B$ is an inert extension (in our sense) and B is an integral domain, then the integrality of the extension ensures that $A \subset B$ is not C-inert; to see this, use the comment of Costa [5, page 495] that the (C-)inert variant of a result on retracts [5, Corollary 1.4] is also valid (the point being that A is not algebraically closed in B).

References

[1] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, Reading (MA), 1969.

[2] J.-P. Cahen, D. E. Dobbs and T. G. Lucas, *Characterizing minimal ring extensions*, Rocky Mountain J. Math., **41**(2011), 1081–1125.

[3] S. U. Chase, D. K. Harrison and A. Rosenberg, *Galois theory and Galois cohomology of commutative rings*, Mem. Amer. Math. Soc., **52**(1965), 15–33.

[4] P. M. Cohn, *Bezout rings and their subrings*, Proc. Cambridge Philos. Soc., **64**(1968), 251–264.

[5] D. L. Costa, *Retracts of polynomial rings*, J. Algebra, **44**(1977), 492–502.

[6] D. E. Dobbs, *Every commutative ring has a minimal ring extension*, Comm. Algebra, **34**(2006), 3875–3881.

[7] D. E. Dobbs, *Recent progress on minimal ring extensions and related concepts*, Int. J. Math. Game Theory Algebra, **18(3)**(2009), 187–212.

[8] D. E. Dobbs, *On the commutative rings with at most two proper subrings*, Int. J. Math. Math. Sci., (2016), Art. ID 6912360, 13 pages.

[9] D. E. Dobbs, *Certain towers of ramified minimal ring extensions of commutative rings*, Comm. Algebra, **46(8)**(2018), 3461–3495.

[10] D. E. Dobbs, *Characterizing finite fields via minimal ring extensions*, Comm. Algebra, **47**(2019), 4945-–4957.

[11] D. E. Dobbs, *A minimal ring extension of a large finite local prime ring is probably ramified*, J. Algebra Appl., **19**(2020), 2050015, 27 pp.

[12] D. E. Dobbs, *On the nature and number of isomorphism classes of the minimal ring extensions of a finite commutative ring*, submitted for publication.

[13] D. E. Dobbs, B. Mullins, G. Picavet, and M. Picavet-L'Hermitte, *On the FIP property for extensions of commutative rings*, Comm. Algebra, **33**(2005), 3091–3119.

[14] D. E. Dobbs, G. Picavet, M. Picavet-L'Hermitte and J. Shapiro, *On intersections and composites of minimal ring extensions*, JP J. Algebra, Number Theory Appl., **26**(2012), 103–158.

[15] D. E. Dobbs and J. Shapiro, *A classification of the minimal ring extensions of certain commutative rings*, J. Algebra, **308**(2007), 800–821.

[16] D. Ferrand and J.-P. Olivier, *Homomorphismes minimaux d'anneaux*, J. Algebra, **16**(1970), 461–471.

[17] M. Fontana, *Topologically defined classes of commutative rings*, Ann. Mat. Pura Appl., **123**(1980), 331–355.

[18] G. L. Ganske and B. R. McDonald, *Finite local rings*, Rocky Mountain J. Math., **3(4)**(1973), 521–540.

[19] R. W. Gilmer, *Multiplicative ideal theory*, Queen's Papers in Pure Appl. Math. **12**, Queen's Univ., Kingston, Ont., 1968.

[20] R. Gilmer, *Multiplicative ideal theory*, Marcel Dekker, New York, 1972.

[21] B. Greenberg, *Coherence in cartesian squares*, J. Algebra, **50**(1978), 12–25.

[22] J. A. Huckaba, *Commutative rings with zero divisors*, Marcel Dekker, New York, 1988.

[23] N. Jacobson, *Lectures in abstract algebra III: theory of fields and Galois theory*, Van Nostrand, Princeton-Toronto-London-New York, 1964.

[24] G. J. Janusz, *Separable algebras over commutative rings*, Trans. Amer. Math. Soc., **122**(1966), 461–479.

[25] I. Kaplansky, *Fields and rings*, Univ. Chicago Press, Chicago, 1969.

[26] I. Kaplansky, *Commutative rings*, rev. ed., Univ. Chicago Press, Chicago, 1974.

[27] S. Lang, *Algebra*, Addison-Wesley, Reading (MA), 1965.

[28] T. G. Lucas, *Minimal integral ring extensions*, J. Commut. Algebra, **3(1)**(2011), 47–81.

[29] B. R. McDonald, *Finite rings with identity*, Marcel Dekker, New York, 1974.

[30] M. Nagata, *Local rings*, Wiley-Interscience, New York, 1962.

[31] G. Picavet and M. Picavet-L'Hermitte, *About minimal morphisms*, Multiplicative Ideal Theory in Commutative Algebra, 369–386, Springer, New York, 2006.

[32] G. Picavet and M. Picavet-L'Hermitte, *Modules with finitely many submodules*, Int. Electron. J. Algebra, **19**(2016), 119–131.

[33] O. Zariski and P. Samuel, *Commutative algebra I*, Van Nostrand, Princeton-Toronto-London, 1958.