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On Generalized FI-extending Modules

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Abstract. A module M is called *FI-extending* if every fully invariant submodule of M is essential in a direct summand of M. In this work, we define a module M to be *generalized FI-extending (GFI-extending)* if for any fully invariant submodule N of M, there exists a direct summand D of M such that $N \leq D$ and that D/N is singular. The classes of FI-extending modules and singular modules are properly contained in the class of GFI-extending modules. We first develop basic properties of this newly defined class of modules in the general module setting. Then, the GFI-extending property is shown to carry over to matrix rings. Finally, we show that the class of GFI-extending modules is closed under direct sums but not under direct summands. However, it is proved that direct summands are GFI-extending under certain restrictions.

1. Introduction

In recent years, the theory of extending modules and rings has come to play an important role in the theory of rings and modules. A module M is called an *extending* (or a *CS*) module if every submodule of M is essential in a direct summand. Although this generalization of injectivity is extremely useful, it does not satisfy some important properties. For example, direct sums of extending modules are not necessarily extending, and full or upper triangular matrix rings over right extending rings are not necessarily right extending. Much work has been done on finding necessary and sufficient conditions to ensure that the extending property is preserved under various extensions (cf., [5]). There have also been numerous generalizations of the extending modules including the following: (1) M is a C_{11}*-module* [11] if each submodule of M has a complement that is a direct summand of M; and (2) M is *FI-extending* [2] if every fully invariant submodule is essential in a direct summand of M. We refer the reader to [12] for different kind of generalizations.

In [2], the following statements were proved: (1) Any direct sum of FI-extending

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modules is FI-extending; (2) A ring R is right FI-extending if and only if the upper triangular matrix ring is so; and (3) The FI-extending property of a ring R carries over to the full matrix ring $M_n(R)$, $n \geq 1$.

In this work, we determine a generalization of the FI-extending module which is not only preserved under various extensions including direct sums and matrix constructions. We define a module M to be *generalized FI-extending (GFI-extending)* if for any fully invariant submodule N of M, there exists a direct summand D of M such that $N \leq D$ and that D/N is singular. In Section 2, we first provide basic results and properties of GFI-extending modules. In addition, we show that the GFI-extending property of a ring R carries over to the full matrix ring $M_n(R)$, $n \geq 1$. The focus in Section 3 is on direct sums and summands of GFI-extending modules.

Throughout this paper, all rings are associative with unity and all modules are unital right modules. Recall from [2], a submodule N of M is called *fully invariant* if $f(N) \subseteq N$ for all $f \in End_R(M)$. Many distinguished submodules of a module are fully invariant (e.g., the socle, the Jacobson radical, the singular module, etc.). Recall from [7], a module M is called *singular* if $Z(M) = M$, where $Z(M) = \{m \in M : mI = 0$ for some essential right ideal I of $R\}$, and called *non-singular* if $Z(M) = 0$. We use R to denote such a ring and M to denote a right R-module. If $X \subseteq M$, then $X \leq M$, $E(M)$, $Z(M)$, and $End_R(M)$ to denote that X is a submodule of M, the injective hull of M, the singular submodule of M, and the ring of endomorphisms of M, respectively. For R, $M_n(R)$ symbolizes the full ring of n-by-n matrices over R. The right annihilator of $m \in M$ is denoted by $r(m) = \{r \in R : mr = 0\}$. Other terminology and notation can be found in [1, 4, 5, 6, 8, 12].

2. GFI-extending Modules

In this section, we begin with the definition of the main concept of this paper. Then, we study relationships between the extending, FI-extending and GFI-extending modules. We also consider connections between the singular and GFI-extending modules.

Definition 2.1. Let M be an R-module. M is said to be a *GFI-extending module* if for any $N \trianglelefteq M$, there exists a direct summand D of M such that $N \leq D$ and that D/N is singular.

Proposition 2.2. *Let M be a R-module. Consider the following statements:*

(i) *M is extending,*

(ii) *M is FI-extending,*

(iii) *M is GFI-extending.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). In general, the converses to these implications do not hold.

Proof. (i) ⇒ (ii) and (ii) ⇒ (iii). These implications are clear.

Let R be the ring of 2-by-2 upper triangular matrices over the integers. Then, R_R is FI-extending module by [11, Theorem 2.4] and [3, Proposition 1.2]. However, the submodule of R_R generated by $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 2 \end{smallmatrix}\right]$ is uniform but not essential in a direct summand of R_R. Hence, R_R is not extending. Thus, (ii) $\not\Rightarrow$ (i).

Finally, let M be a singular (not FI-extending) R-module with a unique composition series $M \supset U \supset V \supset 0$. From [9], $M \oplus (U/V)$ is GFI-extending module. Since U/V is a simple module and M is not FI-extending, $M \oplus (U/V)$ is not FI-extending module. As a special example we can provide the following. Let $R = \mathbb{Z}_2[x_1, x_2]$ be the commutative polynomial ring with the indeterminants x_1, x_2 over the field $\mathbb{Z}_2$ and I is the ideal of R generated by x_1^2, x_2^2, x_1x_2. Then R/I is singular R module which is not FI-extending. Hence R/I is a GFI-extending module which is not FI-extending. Thus, (iii) $\not\Rightarrow$ (ii). □

Lemma 2.3. *Let M be an R-module. If M is singular, then M is GFI-extending.*

Proof. This proof is routine. □

In general, the reverse implication of the above result does not hold. For example, let $\mathbb{Z}$ be the ring of all integers. Then, $\mathbb{Z}$ is extending. Hence, it is FI-extending. By Proposition 2.2, $\mathbb{Z}$ is GFI-extending as a right $\mathbb{Z}$-module, but it is nonsingular.

The following lemma gives the equivalence of the FI-extending and GFI-extending modules.

Lemma 2.4. *Let M be a nonsingular or projective module. Then, M is GFI-extending if and only if M is FI-extending.*

Proof. Let M be a nonsingular or projective module. For all fully invariant submodule N of M, M/N is singular if and only if N is essential submodule of M. □

The following proposition is a consequence of Definition 2.1 and gives a characterization of the GFI-extending modules by fully invariant submodules.

Proposition 2.5. *The followings are equivalent for a module M.*

(i) *M is GFI-extending.*

(ii) *For any fully invariant submodule N of M, M has a decomposition $M = K \oplus K'$ such that $N \leq K$ and that $M/(K' + N)$ is singular.*

(iii) *For any fully invariant submodule N of M, M/N has a decomposition $M/N = K/N \oplus K'/N$ such that K is a direct summand of M and that M/K' is singular.*

(iv) *For any fully invariant submodule N of M, there is a direct summand K of M such that for any $x \in K$ there is an essential right ideal I of R such that $xI \leq N$.*

Proof. The proof is straightforward. □

Lemma 2.6. *Let M be a module and N a fully invariant submodule of M. If M is GFI-extending, then N is GFI-extending.*

Proof. Assume M is a GFI-extending module. Let L be a fully invariant submodule of N. By [2, Lemma 1.1], L is a fully invariant submodule of M. Hence, there exists a direct summand K of M such that $L \leq K$ and K/L is singular. Now, $M = K \oplus K'$ for some submodule K' of M. So, $N = (N \cap K) \oplus (N \cap K')$ by [2, Lemma 1.1]. Since $L \leq N \cap K$ and $(N \cap K)/L \leq K/L$, $(N \cap K)/L$ is singular. Thus, N is a GFI-extending module. □

Proposition 2.7. *Let M be a GFI-extending module and N be a fully invariant submodule of M. If D is fully invariant direct summand of M such that $(D+N)/D$ is nonsingular, then $D \cap N$ is direct summand of N.*

Proof. Let $Y = D \cap N$. Then, by [2, Lemma 1.1], Y is a fully invariant submodule of M. Since $End(N)_R \subseteq End(M)_R$, Y is a fully invariant submodule of N. By Lemma 2.6, N is GFI-extending. Then, there exists a direct summand K of N such that K/Y is singular. If $K \neq Y$, then $D \neq D+K$. Hence, there exists $d+k \in D+K$ such that $d + k \notin D$. So $k \neq 0$. Moreover, there exist a right ideal I_R of R_R such that $I_R \leq_e R_R$. Then, $kI \subseteq Y$. So, $(d+k)I \subseteq D$ and $(D+K)/D$ is singular. Since $(D+K)/D \leq (D+N)/D$ and $(D+N)/D$ is nonsingular, then $D+K = D$. This is a contradiction. So, $K = Y$. Thus, $D \cap N$ is a direct summand of N. □

The next result establishes connections between a GFI-extending module and its injective hull.

Proposition 2.8. *Assume that $Z(R_R) = 0$. Let M be a module. Then, M is GFI-extending if and only if for each fully invariant submodule S of M there exists $e^2 = e \in End_R(E(M))$, such that $S \leq e(E(M))$, $e(E(M))/S$ is singular and $e(M) \leq M$.*

Proof. ($\Rightarrow$). Assume that M is GFI-extending and S is a fully invariant submodule of M. There exists a direct summand X of M such that $S \leq X$ and X/S is singular. Now, $M = X \oplus Y$ for some submodule Y of M. Hence $E(M) = E(X) \oplus E(Y)$. Let $e : E(M) \to E(X)$ be the projection endomorphism. Then, $e(M) \leq M$ and $X/S \leq E(X)/S = e(E(M))/S$. Since $Z(R_R) = 0$, then $e(E(M))/S$ is singular.

($\Leftarrow$). Let S be a fully invariant submodule of M. Then, $e(E(M))/S$ is singular. Now, $S \leq M \cap e(E(M)) = e(M) \leq M \leq E(M)$ gives that $e(M) \leq e(E(M))$ and $e(M)/S \leq e(E(M))/S$. Thus, $e(M)/S$ is singular. But $e(M)$ is a direct summand of M. Hence, M is GFI-extending. □

In the next lemma and theorem, we prove that the GFI-extending property of a ring R carries over to the full matrix ring $M_n(R)$ $(n > 1)$.

Lemma 2.9. *Let X be a right ideal of R and $e^2 = e \in R$ such that eR/X is singular. Thus, $dM_n(R)/M_n(X)$ is singular where d is the diagonal n-by-n matrix with e in all the diagonal positions.*

Proof. The proof is a clear consequence of [2, Lemma 2.2] because eR is projective

R-module and $dM_n(R)$ is a projective $M_n(R)$-module. So X is essential submodule in R_R (resp. $M_n(X)$ is essential submodule in $dM_n(R)$) if and only if eR/X (resp. $dM_n(R)/M_n(X)$) is singular. □

Theorem 2.10. *If R is right GFI-extending, then $M_n(R)$ is right GFI-extending for all positive integers n.*

Proof. Let L be a fully invariant submodule of $M_n(R)$. There exists a fully invariant submodule X of R such that $L = M_n(X)$. Also, there exists $e = e^2 \in R$ such that eR/X is singular. By Lemma 2.9, $dM_n(R)/M_n(X)$ is singular where d is the diagonal n-by-n matrix with e in all the diagonal positions. Hence, $M_n(R)$ is right GFI-extending. □

3. Direct Sum of GFI-Extending Modules

In [2], it was proved that a direct sum of FI-extending modules is FI-extending. It is also known that a direct sum of singular modules is singular [7]. In this section, we show that GFI-extending property is closed under direct sums.

Theorem 3.1. *Direct sums of modules with the GFI-extending property have again the GFI-extending property.*

Proof. Suppose the modules A_i $(i \in I)$ have the GFI-extending property. If X is a fully invariant submodule of the direct sum $M = \oplus_{i\in I} A_i$, then $X = \oplus_{i\in I}(X \cap A_i)$ by [2, Lemma 1.1]. Clearly, $X \cap A_i$ is fully invariant in A_i for each $i \in I$. So, there exists a direct summand H_i of A_i such that $X \cap A_i \leq H_i$ and $H_i/(X \cap A_i)$ is singular. Then, $H = \oplus_{i\in I} H_i$ is a direct summand of M such that $X \leq H$ and H/X is singular. It follows that M is a GFI-extending module. □

Corollary 3.2. *If M is a direct sum of extending (e.g., uniform) modules, then M is GFI-extending.*

Corollary 3.3. *Let M be a $\mathbb{Z}$-module (i.e., an Abelian group). If M satisfies any of the following conditions, then M is a GFI-extending $\mathbb{Z}$-module.*

(i) *M is finitely generated.*

(ii) *M is of bounded order (i.e., $nM = 0$, for some positive integer n).*

(iii) *M is divisible.*

Proof. (i) Every finitely generated Abelian group is a direct sum of uniform $\mathbb{Z}$-modules.
(ii) This part is from [10, p.262].
(iii) If M is divisible, then it is an extending $\mathbb{Z}$-module. Moreover, by Proposition 2.2, M is a GFI-extending $\mathbb{Z}$-module. □

The following corollaries are direct consequences of Theorem 3.1.

Corollary 3.4. *Let $M = M_1 \oplus M_2$ with M_1 being singular and M_2 semi-simple. Then, M is GFI-extending.*

Proof. The proof is clear by Theorem 3.1. □

Corollary 3.5. *Let $M = M_1 \oplus M_2$ with M_1 being GFI-extending and M_2 semi-simple. Suppose that for any fully invariant submodule N of M, $N \cap M_1$ is a direct summand of N. Then, M is GFI-extending.*

Proof. Let N be any fully invariant submodule of M. As in Corollary 3.4, $N + M_1$ is a direct summand of M. By hypothesis, $N = (N \cap M_1) \oplus K$ for some submodule K of N. Since N is a fully invariant submodule of M, $N \cap M_1$ is a fully invariant submodule of M_1. Since M_1 is GFI-extending, there exists a direct summand T of M_1 such that $T/(N \cap M_1)$ is singular. But, $N = (N \cap M_1) \oplus K + M_1 = M_1 \oplus K$. So, $(T+K)/N = (T+K)/[(N \cap M_1) \oplus K] \cong T/(N \cap M_1) \oplus K/K$ is singular. Since $T \oplus K$ is a direct summand of $N + M_1$ and hence a direct summand of M, M is GFI-extending. □

In general, it is not known whether a direct summand of a GFI-extending module is also GFI-extending. However, in the following theorem, we prove that this implication is true under some restrictions.

Theorem 3.6. *If the module $M = B \oplus C$ has the GFI-extending property and B is a fully invariant direct summand, then both B and C have the GFI-extending property.*

Proof. By Lemma 2.6, B has the GFI-extending property. To conclude that C has the GFI-extending property, pick a fully invariant submodule F of C, and apply the GFI-extending property of M to its fully invariant submodule $B \oplus F$. We infer that a direct summand H of M contains $B \oplus F$ such that $H/(B \oplus F)$ is singular. Thus, $H = B \oplus (H \cap C)$, where $H \cap C$ is direct summand of C with $(H \cap C)/F$ is singular. □

Proposition 3.7. *Let R be a nonsingular ring and M be a GFI-extending module. Then, $M = Z(M) \oplus T$ for some nonsingular submodule T of M and T is $Z(M)$-injective.*

Proof. If $Z(M) = 0$ or $Z(M) = M$, it is trivial. Suppose $0 \leq Z(M) \leq M$. $Z(M)$ is a fully invariant submodule of M. Since M is GFI-extending, there are direct summands K, T of M such that $M = K \oplus T$, $Z(M) \leq K$ and that $K/Z(M)$ is singular. So, K is singular. Since $Z(M) = Z(K) \oplus Z(T) = K \oplus Z(T)$, so $Z(M) = K$ and T is nonsingular. Hence, for any submodule N of $Z(M)$, $Hom_R(N,T) = 0$, T is $Z(M)$-injective. □

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