

## On the Representations of Finite Distributive Lattices

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ABSTRACT. A simple but elegant result of Rival states that every sublattice  $L$  of a finite distributive lattice  $\mathcal{P}$  can be constructed from  $\mathcal{P}$  by removing a particular family  $\mathcal{J}_L$  of its irreducible intervals.

Applying this in the case that  $\mathcal{P}$  is a product of a finite set  $\mathcal{C}$  of chains, we get a one-to-one correspondence  $L \mapsto \mathcal{D}_{\mathcal{P}}(L)$  between the sublattices of  $\mathcal{P}$  and the preorders spanned by a canonical sublattice  $\mathcal{C}^\circ$  of  $\mathcal{P}$ .

We then show that  $L$  is a tight sublattice of the product of chains  $\mathcal{P}$  if and only if  $\mathcal{D}_{\mathcal{P}}(L)$  is asymmetric. This yields a one-to-one correspondence between the tight sublattices of  $\mathcal{P}$  and the posets spanned by its poset  $J(\mathcal{P})$  of non-zero join-irreducible elements.

With this we recover and extend, among other classical results, the correspondence derived from results of Birkhoff and Dilworth, between the tight embeddings of a finite distributive lattice  $L$  into products of chains, and the chain decompositions of its poset  $J(L)$  of non-zero join-irreducible elements.

### 1. Introduction

All lattices considered in this paper are finite and distributive. For very basic notation, definitions, and concepts we refer the reader to [4]; many basic definitions are also given at the start of Section 2.

Classical results of Birkhoff and Dilworth, which we review in more detail in Section 2, show that any finite distributive lattice  $L$  can be embedded, as a sublattice, into a product of chains. They further yield a one-to-one correspondence (Corollary 2.5) between the tight such embeddings and chain decompositions of the poset  $J(L)$  of non-zero irreducible elements of  $L$ .

Our main goal in this paper is to reverse the point of view of this correspondence: instead of cataloguing the various embeddings of a particular lattice  $L$  into products

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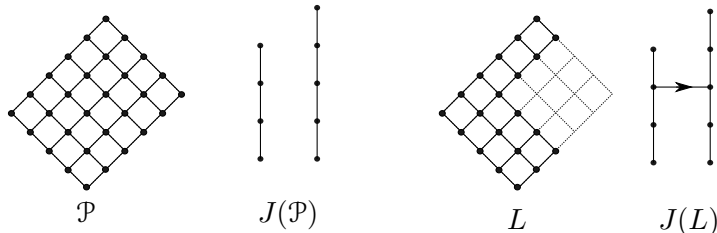


Figure 1: Product of chains  $\mathcal{P}$ , sublattice  $L$ , and their posets  $J(\mathcal{P})$  and  $J(L)$

of chains, we catalogue the various sublattices of a given product  $\mathcal{P}$  of chains.

Starting with the product  $\mathcal{P} = \prod_{C \in \mathcal{C}} C$  of a set  $\mathcal{C}$  of chains,  $J(\mathcal{P})$  is the parallel sum of the chains we get from the chains of  $\mathcal{C}$  by removing their minimum elements. One main idea is that, for a tight sublattice  $L$  of  $\mathcal{P}$ ,  $J(L)$  is isomorphic to an extension of  $J(\mathcal{P})$ . See Figure 1 for an example. If we consider only tight sublattices of  $\mathcal{P}$ , this is the whole picture: in Remark 4.4, we observe that  $L \mapsto J(L)$  gives a one-to-one correspondence between the tight sublattices of  $\mathcal{P}$  and the spanned poset extensions of  $J(\mathcal{P})$ —extensions with the same set of elements as  $J(\mathcal{P})$ .

But what if we consider non-tight embeddings/sublattices? One can see two examples of non-tight embeddings in the bottom half of Figure 2. As every lattice has a tight embedding into products of chains, non-tight embeddings are seldom considered. However, when we look to catalogue the sublattices of given product of chains, it is natural to consider them. In the paper [9], out of which this paper grew, we found it useful to consider non-tight sublattices of products of chains. We needed a characterisation of **all** sublattices of a product of chains.

In Section 3, we use a result of Rival [8] which characterises the sublattices of  $\mathcal{P}$ , to construct, for each sublattice of  $\mathcal{P}$ , a *preorder* (reflexive transitive relation)  $D_{\mathcal{P}}(L)$  which contains  $J(\mathcal{P})$ . In fact,  $D_{\mathcal{P}}(L)$  is a refinement of the poset  $\mathcal{C}^\circ$  that we get from  $J(\mathcal{P})$  by appending a new zero and unit.

This preorder is an analogue of  $J(L)$  in the sense that our main theorem, Theorem 4.1 is an analogue of Theorem 2.1, representing  $L$  as the appropriately defined lattice of ‘downsets’ of  $D_{\mathcal{P}}(L)$ . Using this, we prove Corollary 4.2 in Section 4, and so achieve our main goal of cataloguing the sublattices of  $\mathcal{P}$ , by giving a correspondence between them and the so-called spanned preorder extensions of  $\mathcal{C}^\circ$ . Results similar to those in this section can be found with a different presentation in the recent article [7] of Retakh and Saks.

In Section 5, we return to the point of view of embeddings of a given lattice, and extend Birkhoff’s classical correspondence between the tight embeddings of a lattice  $L$  into products of chains and the chain decompositions of  $J(L)$ . Birkhoff’s correspondence is categorical, and so is our extension, extending the correspondence

between morphisms, but we avoid talk of categories. We get, in Theorem 5.3, a one-to-one correspondence between embeddings of a lattice  $L$  into products of chains, and ‘pointed chain covers’ of an extension  $J^\circ(L)$  of  $J(L)$ . Similar results can be found with a different presentation in the recent article [7].

In [5], Koh used a clever construction to show that any distributive lattice  $L$  can be represented as the lattice  $\mathcal{A}_m(P)$  of maximal antichains of some poset  $P$ . We finish off, in Section 6, by showing that his construction arises naturally from our point of view, and extend it to determine, in Corollary 6.12, **all** posets  $P$  such that  $L \cong \mathcal{A}_m(P)$  for a given  $L$ .

## 2. Notation and Background

A lattice embedding  $e : L \rightarrow L'$  is a  $\{0, 1\}$ -embedding if it preserves the zero and unit elements:  $e(0_L) = 0_{L'}$  and  $e(1_L) = 1_{L'}$ . It is *tight* if it is a  $\{0, 1\}$ -embedding and preserves covers:  $x \prec y$  implies that  $e(x) \prec e(y)$ . An element  $x$  of a lattice  $L$  is *join-irreducible* if  $x = a \vee b$  implies that  $x = a$  or  $x = b$ . Meet-irreducible elements are defined analogously. The set of non-zero join-irreducible elements of  $L$  is denoted  $J(L)$ . It induces a subposet of  $L$  which is also denoted by  $J(L)$ . For a subset  $S$  of a lattice  $L$ , we let  $\bigvee S = \bigvee_{x \in S} x$  be the join of the elements of  $S$ . We often write  $\bigvee_L S$  to specify that the join takes place in  $L$ . A subset  $S$  of a poset is a *downset* or *ideal* if  $x \in S$  and  $y \leq x$  implies  $y \in S$ . The minimum downset containing an element  $x$  is denoted  $\langle x \rangle$ . A *chain*  $C$  of length  $n$  in a poset  $P$  is subposet isomorphic to the linear order  $Z_{n+1}$  on the  $n + 1$  elements  $\{0, 1, \dots, n\}$ . A *chain decomposition* of a poset  $P$  is a partition of its elements into a family  $\mathcal{C}$  of chains  $C_1, \dots, C_d$ . For a family  $\mathcal{C} = \{C_1, \dots, C_d\}$  of disjoint chains, the product  $\prod \mathcal{C} := \prod_{i=1}^d C_i$  consists of all  $d$ -tuples  $x = (x_1, \dots, x_d)$  where  $x_i \in C_i$  for each  $i \in [d]$ . It is ordered by  $x \leq y$  if  $x_i \leq y_i$  for each  $i$ .

In his famous representation theorem in [1], Birkhoff showed that  $L \cong \mathcal{D}(J(L))$  where  $\mathcal{D}(P)$ , for a poset  $P$ , is the family of downsets of  $P$  ordered by inclusion. Specifically, he showed the following.

**Theorem 2.1.**([1]) *Let  $L$  be a finite distributive lattice. The map*

$$\mathbf{S} : L \rightarrow \mathcal{D}(J(L)) : x \mapsto \langle x \rangle \cap J(L)$$

*is a lattice isomorphism. Its inverse is  $S \mapsto \bigvee_L S$ .*

Observing that a downset in  $\mathcal{D}(P)$  is join-irreducible if and only if it has a unique maximal element, one can easily show that  $J(\mathcal{D}(P)) \rightarrow P : \langle x \rangle \mapsto x$  is an isomorphism. Thus  $P \cong Q$  if  $\mathcal{D}(P) \cong \mathcal{D}(Q)$ . This immediately yields a one-to-one correspondence,  $L \mapsto J(L)$ , between finite distributive lattices and finite posets.

For a chain decomposition  $\mathcal{C}$  of a poset let  $\mathcal{C}_0$  be the family of chains we get from the chains in  $\mathcal{C}$  by adding a new minimum element to each. In [2], Dilworth proved the following embedding theorem.

**Theorem 2.2.**([2]) *For any chain decomposition  $\mathcal{C}$  of a poset  $P$  the map  $S \mapsto \bigvee_{\mathcal{P}} S$  is an embedding of  $\mathcal{D}(P)$  into  $\mathcal{P} = \prod \mathcal{C}_0$ .*

With Theorem 2.1, this immediately gives the following.

**Corollary 2.3.** *For any decomposition  $\mathcal{C}$  of  $J(L)$  into chains, the map*

$$e_{\mathcal{C}} : L \rightarrow \prod \mathcal{C}_0 : x \mapsto \bigvee_{\mathcal{P}} \mathbf{S}(x)$$

*is an embedding of  $L$  into  $\mathcal{P} = \prod \mathcal{C}_0$ .*

In [6], Larson makes explicit a converse to Corollary 2.3, showing essentially the following.

**Theorem 2.4.**([6]) *The embeddings  $e_{\mathcal{C}}$  of Corollary 2.3 are tight, and for every tight embedding  $e$  there is a chain decomposition  $\mathcal{C}$  of  $J(L)$  such that  $e = e_{\mathcal{C}}$ .*

It is a trivial observation that different chain decompositions of  $J(L)$  yield different embeddings of  $L$  into products of chains, so with Corollary 2.3, this yields the following correspondence.

**Corollary 2.5.** *Let  $L$  be a finite distributive lattice. The map  $\mathcal{C} \mapsto e_{\mathcal{C}}$  is a one-to-one correspondence between the chain decompositions of  $J(L)$  and the tight embeddings of  $L$  into products of chains.*

### 3. The Rival Construction $D_{\mathcal{P}}(L)$

Recalling a result of Rival in the first subsection, we use it in the second subsection to construct the preorder  $D_{\mathcal{P}}(L)$  and the appropriate notion of a downset lattice. We then give an immediate restatement of Theorem 2.1 in terms of our new definitions. This will be a special case of our main theorem, which will be put off until the next section. In the third subsection, we look at some examples to try to give intuition to the construction. In the fourth subsection, we clean up some technical details of the construction.

#### 3.1. Rival's Theorem

An *irreducible interval* of a lattice  $L$  is the set  $[\alpha, \beta] = \{x \in L \mid \alpha \leq x \leq \beta\}$  for any join-irreducible element  $\alpha$  of  $L$  and any meet-irreducible element  $\beta$ . For a set  $\mathcal{J}$  of irreducible intervals of  $L$  we let  $\bigcup \mathcal{J} = \bigcup_{I \in \mathcal{J}} I$ . The set  $\mathcal{J}$  is *closed* if  $I \subseteq \bigcup \mathcal{J}$  implies  $I \in \mathcal{J}$ . The key to our results is the following theorem of Rival.

**Theorem 3.1.**([8]) *If  $L$  is a sublattice of a finite distributive lattice  $L'$ , then*

$$L = L' \setminus \bigcup \mathcal{J}$$

*for some (closed) family  $\mathcal{J}$  of irreducible intervals of  $L'$ .*

#### 3.2. Setup and The Definition of $D_{\mathcal{P}}(L)$

For the rest of the paper  $\mathcal{P}$  is always a product  $\mathcal{P} = \prod \mathcal{C}$  of finite chains  $\mathcal{C} = \{C_1, \dots, C_d\}$ . We will denote the elements of a chain  $C_i$  by integers, so that  $\alpha + 1$

will denote the cover of an element  $\alpha$  in  $C_i$ ; but we still refer to the maximum element of  $C_i$  as  $1_{C_i}$ . The subscript differentiates it from the element  $\mathbf{1}$  that is occasionally used in examples to label the first or second element of a chain. We let  $\mathbf{0} = (0_{C_1}, \dots, 0_{C_d})$  and  $\mathbf{1} = (1_{C_1}, \dots, 1_{C_d})$  denote the zero and unit of  $\mathcal{P}$ . When  $L$  is a sublattice of  $\mathcal{P}$ , Theorem 3.1 yields a unique closed family  $\mathcal{J}_L$  such that  $L = \mathcal{P} \setminus \bigcup \mathcal{J}_L$ . The join-irreducible elements of  $\mathcal{P}$  are clearly of the form

$$\alpha_i = (\alpha)_i := (0_{C_1}, 0_{C_2}, \dots, 0_{C_{i-1}}, \alpha, 0_{C_{i+1}}, \dots, 0_{C_d})$$

for  $i \in [d]$  and  $\alpha \in C_i$  and the meet-irreducible elements are

$$\beta^j = (\beta)^j := (1_{C_1}, 1_{C_2}, \dots, 1_{C_{j-1}}, \beta, 1_{C_{j+1}}, \dots, 1_{C_d})$$

for  $j \in [d]$  and  $\beta \in C_j$ . We generally use the notation  $\alpha_i$  and  $\beta^j$  but add the parentheses when needed for clarity, in particular, we do this when  $\alpha$  has or  $\beta$  has a subscript. The irreducible intervals of  $\mathcal{P}$  are exactly the intervals  $[\alpha_i, \beta^j]$  for  $i, j \in [d]$ ,  $\alpha \in C_i$ , and  $\beta \in C_j$ , where  $\alpha \leq \beta$  if  $i = j$ . Though an empty set is not technically an interval of a poset, it will be useful to consider the empty set  $[\alpha_i, \beta^i]$ , when  $\alpha > \beta$ , as an irreducible interval of  $\mathcal{P}$ .

Given  $\mathcal{C}$  and  $\mathcal{P} = \prod \mathcal{C}$ , let  $\mathcal{C}^\circ = J^\circ(\mathcal{P})$  be the subposet of  $\mathcal{P}$  induced by  $J(\mathcal{P}) \cup \{\mathbf{0}, \mathbf{1}\}$ . Occasionally, in arguments, we will let  $(\alpha + 1)_i$  refer to  $\mathbf{1}$  when  $\alpha = 1_{C_i}$ . This is consistent with viewing  $\mathbf{1}$  as a new unit of  $C_i$ . Dually, we will occasionally let  $(\beta - 1)^j$  refer to  $\mathbf{0}$  when  $\beta = 0_{C_j}$ . These shortcuts will help us avoid having to deal separately with the extremal cases of  $\alpha$  and  $\beta$  in arguments.

Recall that a poset, and in particular  $\mathcal{C}^\circ$ , is a reflexive relation. We refer to a reflexive relation simply as a *reflexive digraph*, and view posets as reflexive digraphs, saying  $(x, y)$  is an *arc*, or writing  $x \rightarrow y$ , to mean  $x \leq y$ . A *spanned extension* of  $\mathcal{C}^\circ$  is any digraph on the elements of  $\mathcal{C}^\circ$ , which contains it. A *preorder* is a transitive reflexive digraph.

The following is our extension of  $J(L)$ , we chose to label it with a  $D$  rather than a  $J$  to emphasise the fact that it is a digraph, and not necessarily a poset.

**Definition 3.2.** ( $D_{\mathcal{P}}(\mathcal{J})$  and  $D_{\mathcal{P}}(L)$ ) For any family  $\mathcal{C} = \{C_1, \dots, C_d\}$  of chains and any family of irreducible intervals  $\mathcal{J}$  of  $\mathcal{P} = \prod \mathcal{C}$ , let  $D_{\mathcal{P}}(\mathcal{J})$  be the spanned extension of  $\mathcal{C}^\circ$  that we get from  $\mathcal{C}^\circ$  by letting  $(\beta + 1)_j \rightarrow \alpha_i$  for each  $[\alpha_i, \beta^j] \in \mathcal{J}$ . For a sublattice  $L$  of  $\mathcal{P}$ , let  $D_{\mathcal{P}}(L) = D_{\mathcal{P}}(\mathcal{J}_L)$ .

An  $(x, y)$ -*path*  $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_\ell$  in a digraph  $D$  is a sequence of elements  $x = v_1, v_2, \dots, v_\ell = y$  such that  $v_i \rightarrow v_{i+1}$  for each  $i$ . The following is a straightforward extension, to digraphs, of the definition of downset of a poset.

**Definition 3.3.** A subset  $S \subseteq D$  of vertices of a digraph  $D$  is a *downset* (or an *initial set*) if for all  $y$  in  $S$  and all  $(x, y)$ -paths in  $D$ ,  $x$  is also in  $S$ . A downset set  $S$  of  $D$  is a *non-trivial downset* if it is a non-empty proper subset of  $D$ . For a vertex  $x$  of  $D$  let  $\langle x \rangle$  be the smallest downset of  $D$  containing  $x$ .

**Definition 3.4.** ( $\mathcal{D}(D)$  and  $\mathcal{D}_{\text{nt}}(D)$ ) For any digraph  $D$ , let  $\mathcal{D}(D)$  be the family of downsets of  $D$  ordered by inclusion, and  $\mathcal{D}_{\text{nt}}(D)$  be the subfamily of non-trivial downsets.

As with posets, it is clear for digraphs that unions and intersections of downsets are downsets, so  $\mathcal{D}_{\text{nt}}(D)$  is indeed a poset for any digraph  $D$ . In our main theorem we will see that  $\mathcal{D}_{\text{nt}}(D)$  is, in fact, the lattice  $L$  when  $D$  is the spanned extension  $D_{\mathcal{P}}(L)$  of  $\mathcal{C}^\circ$ .

For a poset  $P$  let  $P^\vee$  be the poset we get from  $P$  by adding a new minimum element and let  $P^\circ$  be the poset we get from  $P^\vee$  by adding a new maximum element. It has been observed in several places, (see, for example, [6]), that

$$(3.1) \quad \mathcal{D}_{\text{nt}}(P^\circ) \cong \mathcal{D}(P).$$

Indeed, the isomorphism is the map that drops the appended unit. As  $\mathcal{D}(P) \cong \mathcal{D}(P')$  implies  $P \cong P'$  we have

$$(3.2) \quad \mathcal{D}_{\text{nt}}(P^\circ) \cong \mathcal{D}_{\text{nt}}(Q^\circ) \Rightarrow P^\circ \cong Q^\circ.$$

With (3.1), the following base case of our main theorem is immediate from Theorem 2.1. It will also be clear from the top example in Figure 2 of the next subsection.

**Lemma 3.5.** *Where  $\mathcal{P}$  is a product of chains, the map*

$$\mathbf{T} : \mathcal{P} \rightarrow \mathcal{D}_{\text{nt}}(\mathcal{C}^\circ) : x \mapsto \langle x \rangle \cap \mathcal{C}^\vee$$

*is an isomorphism. Its inverse is  $S \mapsto \bigvee_{\mathcal{P}} S$ .*

*Proof.* Indeed  $\mathcal{P}$  is a lattice, so we may apply Theorem 2.1, and then (3.1), to get the isomorphism

$$\mathcal{P} \cong \mathcal{D}(J(\mathcal{P})) \cong \mathcal{D}_{\text{nt}}(J^\circ(\mathcal{P})) = \mathcal{D}_{\text{nt}}(\mathcal{C}^\circ). \quad \square$$

Before we prove our main theorem, that  $\mathbf{T}$  is an isomorphism from  $L$  to  $\mathcal{D}_{\text{nt}}(D_{\mathcal{P}}(L))$  for **any** sublattice of  $\mathcal{P}$ , we will need, in Proposition 3.7 to show that because  $J_L$  is closed,  $D_{\mathcal{P}}(L)$  is a preorder.

First though, let's explain our definition of  $D_{\mathcal{P}}(L)$  with a couple of examples.

### 3.3. Some Examples

A directed graph that extends a poset can be depicted by adding arcs to the Hasse diagram of the poset, though direction must be made explicit on these arcs, as they need not all go up. As the digraphs  $D_{\mathcal{P}}(L)$  that we depict thus will be preorders, so transitive, we maintain the convention of not drawing the transitively implied arcs.

The top half of Figure 2 shows the same lattices  $\mathcal{P} = Z_5 \times Z_6$  and  $L_1 = \mathcal{P} \setminus [3_2, 2^1]$  as are shown in Figure 1, but now, instead of  $J(\mathcal{P})$  and  $J(L_1)$ , it shows with them the digraph  $\mathcal{C}^\circ$  and its spanned extension  $D_{\mathcal{P}}(L_1)$ .

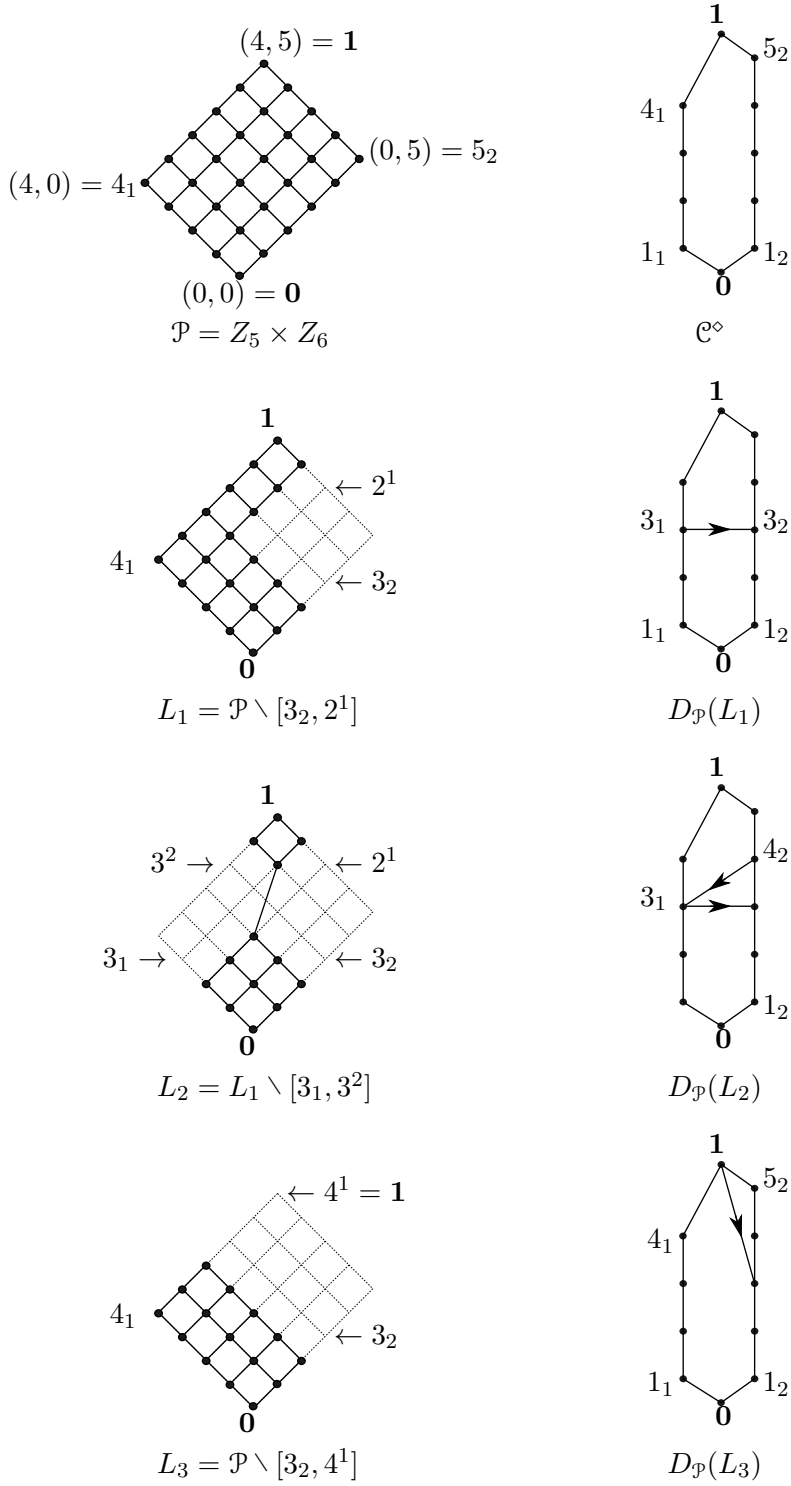


Figure 2: Sublattices of  $\mathcal{P} = Z_5 \times Z_6$  and their digraphs  $D_{\mathcal{P}}$

The construction of  $D_{\mathcal{P}}(L)$  arises from observing what we must do to  $\mathcal{C}^\circ$  so that  $\mathbf{T}$  remains an isomorphism when we remove some irreducible interval  $[\alpha_i, \beta^j]$  from  $\mathcal{P}$ . Referring to the second example in Figure 2, when we remove  $[3_2, 2^1]$  from  $\mathcal{P}$ , we see that we should add the  $3_2 \geq 3_1$  to  $\mathcal{C}^\circ$  to maintain the isomorphism. Indeed, this ensures that, for example, the set  $\mathbf{T}((2, 3)) = \{2_1, 1_1, 0, 1_2, 2_2, 3_2\}$  is no longer a downset. In general it ‘kills’ any downset  $S$  containing  $3_2$  but not  $3_1$ . These are exactly the downsets  $\mathbf{T}(x)$  for  $x \in [3_2, 2^1]$ .

The third example in the figure shows a non-tight sublattice  $L_2$  of  $\mathcal{P}$ , observe that the corresponding extension  $D_{\mathcal{P}}(L_2)$  of  $\mathcal{C}^\circ$  is not a poset. The fourth example exhibits the necessity of 0 and 1 in  $D_{\mathcal{P}}$ .

### 3.4. The Closure of $\mathcal{J}$ and Transitivity of $D_{\mathcal{P}}(\mathcal{J})$

In this subsection we show that  $D_{\mathcal{P}}(\mathcal{J})$  is closed if and only if  $\mathcal{J}$  is transitive. It is a pretty intuitive notion, but we prove it via a technical lemma that we be quite useful later.

Recall that we included the empty intervals of the form  $[\alpha_i, \beta^i]$ , where  $\alpha > \beta$ , as irreducible intervals of  $\mathcal{P} = \prod \mathcal{C}$ . By definition then, these are thus contained in any closed family  $\mathcal{J}$  of irreducible intervals. This is consistent with  $D_{\mathcal{P}}(\mathcal{J})$  containing  $\mathcal{C}^\circ$ . In particular, we assume for any sublattice  $L$  of  $\mathcal{P}$  that the closed family  $\mathcal{J}_L$  such that  $L = \mathcal{P} \setminus \bigcup \mathcal{J}_L$  provided by Theorem 3.1, contains these elements.

Now our technical lemma.

**Lemma 3.6.** *Let  $D = D_{\mathcal{P}}(\mathcal{J})$  for some sublattice  $L = \mathcal{P} \setminus \bigcup \mathcal{J}$  of  $\mathcal{P} = \prod \mathcal{C}$ . The following are equivalent for  $\alpha \in C_i \in \mathcal{C}$  and  $\beta \in C_j \in \mathcal{C}$ .*

- (i)  $[\alpha_i, \beta^j] \subseteq \bigcup \mathcal{J}$ .
- (ii) For all  $x \in L$ ,  $x_i \geq \alpha \Rightarrow x_j > \beta$ .
- (iii) There is a  $((\beta + 1)_j, \alpha_i)$ -path in  $D$ .

*Proof.* The equivalence of (i) and (ii) is immediate as both are equivalent to the statement that there are no elements  $x \in L$  with  $\alpha \leq x_i$  and  $x_j \leq \beta$ . So we show the equivalence of (ii) and (iii).

On the one hand, consider an  $((\beta + 1)_{s_\ell}, (\alpha_1)_{s_1})$ -path in  $D$ . As  $\mathcal{J}$  is assumed to contain all empty intervals, we can write this as

$$(\beta + 1)_{s_\ell} = (\alpha_\ell)_{s_\ell} \rightarrow \cdots \rightarrow (\alpha_2)_{s_2} \rightarrow (\alpha_1)_{s_1}.$$

By the definition of  $D$  this means that  $[(\alpha_i)_{s_i}, (\alpha_{i+1} - 1)^{s_{i+1}}] \in \mathcal{J}$  for each  $i$ . Let  $x \in L$  be such that  $x_{s_1} \geq \alpha_1$ . By the equivalence of (i) and (ii) this implies  $x_{s_2} > \alpha_2 - 1$ , which in turn implies  $x_{s_3} > \alpha_3 - 1$ , etc., until we get that  $x_{s_\ell} > \alpha_\ell - 1 = \beta$ , as needed.

On the other hand, assume that there is no such path from  $(\alpha_\ell)_{s_\ell}$  to  $(\alpha_1)_{s_1}$ . We will find  $x$  in  $L$  with  $x_{s_1} \geq \alpha_1$  and  $x_{s_\ell} < \alpha_\ell = \beta + 1$ . Indeed, for  $k \in [d]$  let  $x_k$  be the maximum value for which there is a path in  $D$  from  $(x_k)_k$  to  $(\alpha_1)_{s_1}$ . If no such path exists, let  $x_k$  be 0. Clearly  $x_{s_1} \geq \alpha_1$ , and by the assumption that there



is no path from  $(\alpha_\ell)_{s_\ell}$  to  $(\alpha_1)_{s_1}$  we have that  $x_{s_\ell} < \alpha_\ell$ . We have just to show that  $x$  is in  $L$ .

If  $x$  is not in  $L$ , then it is in some  $[(\alpha)_i, (\beta)^j] \in \mathcal{J}$ . As  $[(\alpha)_i, (\beta)^j] \in \mathcal{J}$ , we have that  $(\beta + 1)_j \rightarrow (\alpha)_i$  is in  $D$ . As  $x \in [(\alpha)_i, (\beta)^j]$ , we have that  $\alpha \leq x_i$  and  $x_j \leq \beta$ . Now by the choice of  $x$ ,  $\alpha \leq x_i$  implies that there is an path from  $(\alpha)_i$  to  $(\alpha_1)_{s_1}$ , so the arc  $((\beta + 1)_j, (\alpha)_i)$  gives us a path from  $(\beta_j + 1)_{s_j}$  to  $(\alpha_1)_{s_1}$ . But then  $\beta_j + 1 \leq x_j$  which contradicts  $x_j \leq \beta_j$ .  $\square$

As  $\mathcal{J}_L$  is closed, the following proposition allows us to assume that the digraph  $D_{\mathcal{P}}(L)$ , for any sublattice  $L$  of  $\mathcal{P}$ , is a preorder.

**Proposition 3.7.** *Let  $D = D_{\mathcal{P}}(\mathcal{J})$  for some sublattice  $\mathcal{P} \setminus \bigcup \mathcal{J}$  of  $\mathcal{P}$ . Then  $\mathcal{J}$  is closed if and only if  $D$  is transitive.*

*Proof.* On the one hand, assume that  $\mathcal{J}$  is closed. Then we can replace (i) of Lemma 3.6 with  $[\alpha_i, \beta^j] \in \mathcal{J}$ . Now that  $\gamma_k \rightarrow \beta_j$  and  $\beta_j \rightarrow \alpha_i$  in  $D$  gives, by Lemma 3.6, that  $[\alpha_i, \gamma^k] \in \mathcal{J}$  which means  $\gamma_k \rightarrow \alpha_i$  in  $D$ . So  $D$  is transitive.

On the other hand, assume that  $\mathcal{J}$  is not closed. Then there is some  $[\alpha_i, \beta^j]$  in  $\bigcup \mathcal{J}$  but not in  $\mathcal{J}$ . So there is a  $(\alpha_i, \beta^j)$ -path in  $D$ , while  $(\alpha_i, \beta^j)$  is not in  $D$ .  $\square$

## 4. Main Results

In this section we give our main results corresponding the sublattices of product  $\mathcal{P} = \prod \mathcal{C}$  of chains to the spanned preorder extensions of  $\mathcal{C}^*$ . In the first subsection, we give our main theorem, extending Birkhoff's representation theorem from  $J(L)$  to the more general construction  $D_{\mathcal{P}}(L)$ . In the second subsection, we explicitly state the one-to-one correspondence. In the final subsection, we classify the sublattices of  $\mathcal{P}$  in terms of the properties of the corresponding preorder extensions.

### 4.1. Main Theorem

Generalising Theorem 2.1 we have the following.

**Theorem 4.1.** *Let  $L$  be a sublattice of a product  $\mathcal{P} = \prod \mathcal{C}$  of chains, and let  $D = D_{\mathcal{P}}(L)$ . The map*

$$\mathbf{T} : L \rightarrow \mathcal{D}_{\text{nt}}(D) : x \mapsto \langle x \rangle \cap \mathcal{C}^\circ$$

*is a lattice isomorphism. Its inverse is  $S \mapsto \bigvee_{\mathcal{P}} S$ .*

*Proof.* As it simplifies induction, we prove the slightly more general result that  $\mathbf{T}$  is an isomorphism for  $D = D_{\mathcal{P}}(\mathcal{J})$  where  $\mathcal{J}$  is any family of irreducible intervals of  $\mathcal{P}$  such that  $L = \mathcal{P} \setminus \bigcup \mathcal{J}$ . In the case that  $\mathcal{J}$  is empty, or contains only empty families, we have  $L = \mathcal{P}$  and  $D = \mathcal{C}^\circ$ , so the isomorphism is given in Lemma 3.5.

In the general case, we first observe that  $\mathcal{D}_{\text{nt}}(D)$  is a subposet of  $\mathcal{P} = \mathcal{D}_{\text{nt}}(\mathcal{C}^\circ)$ . Indeed, as  $D$  is a spanned extension of  $\mathcal{C}^\circ$ , any downset of  $D$  is a downset of  $\mathcal{C}^\circ$ , so  $\mathcal{D}_{\text{nt}}(D)$  is a subset of  $\mathcal{D}_{\text{nt}}(\mathcal{C}^\circ)$ . Since both are ordered by inclusion  $\mathcal{D}_{\text{nt}}(D) \leq \mathcal{D}_{\text{nt}}(\mathcal{C}^\circ)$ .

Now it is enough to show that  $\mathbf{T} : L \rightarrow \mathcal{D}_{\text{nt}}(D)$  a bijection. We do this by induction on the size of  $\mathcal{J}$ . Let  $[\alpha_i, \beta^j] \in \mathcal{J}$ ,  $\mathcal{J}' = \mathcal{J} \setminus [\alpha_i, \beta^j]$ ,  $L' = \mathcal{P} \setminus \bigcup \mathcal{J}'$ , and  $D' = D_{\mathcal{P}}(\mathcal{J}')$ . By induction we have that  $\mathbf{T} : L' \rightarrow \mathcal{D}_{\text{nt}}(D')$  is a bijection.

We must show for  $\mathbf{T}(x) \in \mathcal{D}_{\text{nt}}(D')$ , that  $\mathbf{T}(x) \notin \mathcal{D}_{\text{nt}}(D)$  if and only if  $x \in [\alpha_i, \beta^j]$ . Now  $\mathbf{T}(x)$  being a downset in  $D'$ , but not in  $D$  which we get from  $D'$  by adding the arc  $((\beta + 1)_j, \alpha_i)$ , means exactly that  $\alpha_i \in \mathbf{T}(x)$  and  $(\beta + 1)_j \notin \mathbf{T}(x)$ . This is to say  $\alpha \leq x_i$  but  $(\beta + 1) > x_j$ , which means that  $x \in [\alpha_i, \beta^j]$ , as needed.

To see that  $T^{-1} = S \mapsto \bigvee_{\mathcal{P}} S$  just observe that  $T$  is defined by its definition on irreducible elements, and for irreducible  $x$ , the downset  $\langle x \rangle \cap \mathcal{C}^\circ$  is the principle downset  $\langle x \rangle_{\mathcal{C}^\circ}$  in  $\mathcal{C}^\circ$ , so

$$\bigvee_{\mathcal{P}} \langle x \rangle \cap \mathcal{C}^\circ = \bigvee_{\mathcal{P}} \langle x \rangle_{\mathcal{C}^\circ} = \bigvee_{\mathcal{C}^\circ} \langle x \rangle_{\mathcal{C}^\circ} = x. \quad \square$$

## 4.2. One-to-one Correspondence

By Theorem 4.1 we have that any sublattice of  $\mathcal{P}$  can be expressed as the lattice  $\mathcal{D}_{\text{nt}}(D)$  of non-trivial downsets of some spanned extension  $D$  of  $\mathcal{C}^\circ$ . On the other hand, it is clear that for a spanned extension  $D$  of  $\mathcal{C}^\circ$  the family

$$\mathcal{J}_D = \{[\alpha_i, \beta^j] \mid ((\beta + 1)_j, \alpha_i) \in D \setminus \mathcal{C}^\circ\}$$

is such that  $\mathcal{D}_{\text{nt}}(D)$  is the sublattice  $\mathcal{P} \setminus \bigcup \mathcal{J}_D$  of  $\mathcal{P}$ . While several digraphs  $D$  may yield the same sublattice of  $\mathcal{P}$ , as may families of intervals have the same union, it is clear that there is a unique closed family of intervals defining a given sublattice. So by Proposition 3.7 we get the following.

**Corollary 4.2.** *Let  $\mathcal{P} = \prod \mathcal{C}$  be a product of chains. The map  $L \mapsto D_{\mathcal{P}}(L)$  gives a one-to-one correspondence between the sublattices of  $\mathcal{P}$  and the spanned preorder extensions of  $\mathcal{C}^\circ$ .  $\square$*

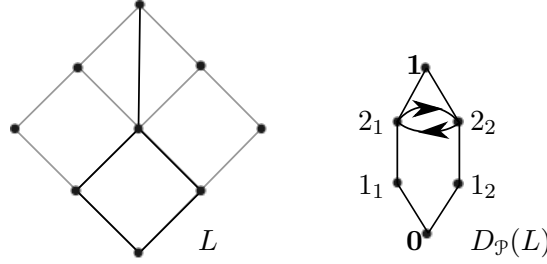
This solves one of our main goals, ‘reversing the point of view’ of correspondence given in Corollary 2.5.

## 4.3. Classification of Sublattices

Recall that a sublattice  $L$  of  $\mathcal{P}$  is a  $\{0, 1\}$ -sublattice if it contains the extremal elements  $\mathbf{0}$  and  $\mathbf{1}$  of  $\mathcal{P}$ . It is *subdirect* if for each  $i \in [d]$ , the projection  $\pi_i : L \rightarrow C_i$  is surjective; this is necessarily a  $\{0, 1\}$ -sublattice. A  $\{0, 1\}$ -sublattice is *tight* if its covers are covers of  $\mathcal{P}$ .

It was shown in [6] that every tight sublattice of a product of chains is subdirect. The converse was also claimed, but the proof was flawed: indeed, the lattice  $L = Z_3 \times Z_3 \setminus ([2_1, 1^2] \cup [2_2, 1^1])$  shown in Figure 3 is a subdirect sublattice of  $Z_3 \times Z_3$ , but not tight. One notices in this example that  $D_{\mathcal{P}}(L)$  has a cycle. We will see that this is indicative of non-tight sublattices.

The *height* of a finite lattice is the maximum number of elements in chain of the lattice. It is well known that for a distributive lattice, every cover is in a maximum


 Figure 3: Non-tight subdirect sublattice of  $Z_3 \times Z_3$ 

chain. It follows that a sublattice  $L$  of  $\mathcal{P}$  is tight if and only if  $L$  and  $\mathcal{P}$  have the same height. An arc  $(x, y)$  in a digraph  $D$  is an *in-arc* of  $y$  and an *out-arc* of  $x$ .

**Lemma 4.3.** *Let  $L$  be a sublattice of  $\mathcal{P}$  and  $D = D_{\mathcal{P}}(L)$ .*

- (i)  *$L$  is a  $\{0, 1\}$ -sublattice if and only if  $\mathbf{0}$  is a source (has no in-arcs) in  $D$  and  $\mathbf{1}$  is a sink (has no out-arcs).*
- (ii)  *$L$  is subdirect if and only if  $D$  has no down edges: those of the form  $\alpha_i \rightarrow (\alpha - 1)_i$ .*
- (iii)  *$L$  is tight if and only if  $D$  is a poset.*

*Proof.* We have by Theorem 4.1 that  $\mathbf{T} : L \rightarrow \mathcal{D}_{\text{nt}}(D)$  is an isomorphism.

(i) Clearly  $\mathbf{T}(\mathbf{0}) = \{\mathbf{0}\}$  is a downset of  $D$  if and only if  $\mathbf{0}$  is a source in  $D$ . Just as clearly,  $\mathbf{T}(\mathbf{1}) = \mathcal{C}^\vee$  is a downset if and only if  $\mathbf{1}$  is a sink. The result follows.

(ii) We have  $\alpha_i \rightarrow (\alpha - 1)_i$  in  $D$  if and only if there is no downset  $\mathbf{T}(x)$  of  $D$  containing  $(\alpha - 1)_i$  but not  $\alpha_i$ . This is if and only if there is no element  $x \in L$  such that  $x_i = \alpha - 1$ .

(iii) First assume that  $L$  is tight, and assume, towards contradiction that  $D$  contains a cycle  $(\alpha_1)_{i_1} \rightarrow (\alpha_2)_{i_2} \cdots \rightarrow (\alpha_\ell)_{i_\ell} \rightarrow (\alpha_1)_{i_1}$ , for some  $\ell \geq 2$ . We show that no vertex  $x$  in  $L$  has  $x_{i_1} = \alpha_1$ , which contradicts the fact that  $L$  is tight. Indeed, if  $x$  did have  $x_{i_1} = \alpha_1$ , then  $x_{i_1} \geq \alpha_1$  so by Lemma 3.6  $x_{i_2} > \alpha_2$  so  $x_{i_3} > \alpha_3$  etc., and we get that  $x_{i_1} > \alpha_1$ , a contradiction.

On the other hand, assume that  $D$  has no cycles. Then its vertices can be ordered so that all arcs go up. Visiting vertices from the bottom of this ordering, one at a time, we get an ascending walk in  $L \cong \mathcal{D}_{\text{nt}}(D)$  from  $\mathbf{0}$  to  $\mathbf{1}$  of size  $|D|$ , showing that  $L$  has height equal to the height of  $\mathcal{P}$ . Thus  $L$  is a tight sublattice.  $\square$

**Remark 4.4.** Though our goal was a correspondence including **all** sublattices of a product  $\mathcal{P}$  of chains, there is some satisfaction in restricting now to  $\{0, 1\}$ -sublattices so that the sublattices are corresponded to spanned extensions of  $J(\mathcal{P})$  rather than of  $\mathcal{C}^*$ .

For any preorder  $D$  with named elements  $\mathbf{0}$  and  $\mathbf{1}$ , let  $D^*$  be the subgraph induced by  $D \setminus \{\mathbf{0}, \mathbf{1}\}$ . For a  $\{0, 1\}$ -sublattice  $L$  of  $\mathcal{P}$  it is a simple matter to extend (3.1) and show that the map  $\mathcal{D}_{\text{nt}}(D_{\mathcal{P}}(L)) \rightarrow \mathcal{D}(D_{\mathcal{P}}^*(L)) : S \mapsto S \setminus \{\mathbf{1}\}$  is an isomorphism. Composing this with  $\mathbf{T} : L \rightarrow \mathcal{D}_{\text{nt}}(D_{\mathcal{P}}(L))$ , Theorem ?? tells us that

$$\mathbf{S} : L \rightarrow \mathcal{D}(D_{\mathcal{P}}(L)^*) : x \mapsto \langle x \rangle \cap \mathcal{C}$$

is a lattice isomorphism. When  $L$  is a tight sublattice of  $\mathcal{P}$ , we have by Lemma 4.3 that  $D_{\mathcal{P}}(L)$ , and so  $D_{\mathcal{P}}(L)^*$ , is a poset. It follows that  $D_{\mathcal{P}}(L)^* = J(L)$  in this case. So this restriction of Theorem 4.1 is even symbolically an extension of Theorem 2.1.

Observing that a spanned extension  $D_{\mathcal{P}}(L)$  of  $\mathcal{C}^\diamond$  yields a spanned extension  $D_{\mathcal{P}}^*(L)$  of  $(\mathcal{C}^\diamond)^* = J(\mathcal{P})$ , Corollary 4.2 says that the map  $L \mapsto D_{\mathcal{P}}^*(L)$  gives a one-to-one correspondence between the  $\{0, 1\}$ -sublattices of  $\mathcal{P}$  and the spanned preorder extensions of  $J(\mathcal{P})$ . Tight sublattices of  $\mathcal{P}$  correspond to spanned poset extensions of  $J(\mathcal{P})$ .

## 5. Back to Embeddings

We would now like to extend Corollary 2.5 by fixing a finite distributive lattice  $L$ , and cataloging all embeddings into products of chains. An embedding  $e : L \rightarrow \mathcal{P}$ , defines a sublattice  $e(L)$  of  $\mathcal{P}$  so gives, by Corollary 4.2, a preorder extension  $D = D_{\mathcal{P}}(e(L))$  of  $\mathcal{C}^\diamond$  such that  $\mathcal{D}_{\text{nt}}(D) \cong L$ . But as neither  $D$  or  $\mathcal{C}^\diamond$  are canonical for  $L$ , this is a little unwieldy for our purposes. Luckily all such  $D$  have a canonical quotient that turns out to be  $J(L)$ . Viewing an extension  $D$  of  $\mathcal{C}^\diamond$  rather as a bijective homomorphism  $\phi : \mathcal{C}^\diamond \rightarrow D$ , we get a surjective homomorphism of  $\mathcal{C}^\diamond \rightarrow J(L)$  by passing to the quotient. As such, every embedding  $e$  of  $L$  into a product of chains corresponds to a surjective homomorphism of some structure to  $J(L)$ . Giving the proper definitions now, we show in this section that this correspondence is one-to-one.

A *pointed union of chains* is any poset that can be represented as  $\mathcal{C}^\diamond = J^\diamond(\mathcal{P})$  for some product  $\mathcal{P} = \prod \mathcal{C}$  of chains. (Recalling this construction, we get a pointed union of chains  $\mathcal{C}^\diamond$  from the parallel sum of the chains of a family  $\mathcal{C}$  of chains by appending a new zero  $\mathbf{0}$  and unit  $\mathbf{1}$ . A homomorphism (relation preserving map)  $\phi : \mathcal{C}^\diamond \rightarrow D$  of a pointed union  $\mathcal{C}^\diamond$  of chains to a preorder  $D$  is a *pointed chain cover* if it is surjective, and a *pointed chain decomposition* if it is bijective.

### 5.1. The Canonical Quotient of $D_{\mathcal{P}}(L)$

The *non-preference* relation ' $\sim$ ' defined on a preorder  $D$  by letting  $x \sim y$  if  $x$  and  $y$  are in a directed cycle, is an equivalence relation. The quotient  $\tilde{D} := D/\sim$  is well known to be a poset. Classes of  $\tilde{D}$  satisfy  $[a] \leq [b]$  if and only if  $a \leq b$ .

The *quotient homomorphism*  $q : D \rightarrow \tilde{D} : a \mapsto [a]$  induces set maps between the power-sets  $2^D$  and  $2^{\tilde{D}}$  of  $D$  and  $\tilde{D}$ . For  $T \subseteq D$ , we let  $[T] := \{[x] \mid x \in T\} \subseteq \tilde{D}$ , and for  $S \subseteq \tilde{D}$  we let  $\bigcup S = \bigcup_{[x] \in S} [x]$ .

**Lemma 5.1.** *Let  $D$  be a preorder, then  $\mathcal{D}_{\text{nt}}(D) \rightarrow \mathcal{D}_{\text{nt}}(\tilde{D}) : T \mapsto [T]$  is an isomorphism with inverse  $S \mapsto \bigcup S$ . Thus  $\tilde{D} \cong J^\circ(L)$ , where  $L = \mathcal{D}_{\text{nt}}(D)$ .*

*Proof.* As  $[x]$  contains only elements in the downset of  $x$ , for a downset  $T$ ,  $[T]$  is a partition of  $T$  and so is a downset of  $\tilde{D}$ . Thus  $T \mapsto [T]$  is a well-defined map from  $\mathcal{D}_{\text{nt}}(D)$  to  $\mathcal{D}_{\text{nt}}(\tilde{D})$ , and  $S \mapsto \bigcup S$  is its inverse. As  $T \rightarrow [T]$  is clearly a bijection, it is also the inverse of  $S \mapsto \bigcup S$ . That it preserves order is also clear as the order on both posets is inclusion, and  $[T]$  partitions  $T$  with respect to an underlying fixed partition of  $D$ .

As  $\mathcal{D}_{\text{nt}}(\tilde{D}) = \mathcal{D}_{\text{nt}}(D) = L = \mathcal{D}_{\text{nt}}(J^\circ(L))$ , the second fact follows by (3.2).  $\square$

Now for any pointed chain decomposition  $\phi : \mathcal{C}^\circ \rightarrow D$  of a preorder  $D$  we get a surjective homomorphism  $\tilde{\phi} = q \circ \phi : \mathcal{C}^\circ \rightarrow \tilde{D}$  by composing with the quotient map. And while the quotient map  $q : D \rightarrow \tilde{D} : a \mapsto [a]$  is not invertible, the homomorphism  $\tilde{\phi}$  tells us what elements were identified in passing from  $D$  to  $\tilde{D}$ . As identified elements induce a complete digraph in the transitive digraph  $D$ , we can uniquely resolve the pair  $(\tilde{\phi}, \tilde{D})$  back to the pair  $(\phi, D)$  they came from.

Thus we have the following.

**Fact 5.2.** *The map  $(\phi, D) \mapsto (\tilde{\phi}, \tilde{D})$  is a one-to-one correspondence between pointed chain decompositions of preorders, and pointed chain covers of posets.*

## 5.2. The Full Correspondence Extending Corollary 2.5

Fix a lattice  $L$  and let  $J^\circ = J^\circ(L)$ . An embedding  $e : L \rightarrow \mathcal{P}$  of  $L$  yields by Corollary 4.2 a spanned preorder extension  $D = D_{\mathcal{P}}(e(L))$  of  $\mathcal{C}^\circ$ , which we view as a pointed chain decomposition  $\phi'_e : \mathcal{C}^\circ \rightarrow D$ . Extending this by the quotient  $q : D \rightarrow \tilde{D}$  gives a homomorphism  $\phi_e : \mathcal{C}^\circ \rightarrow \tilde{D}$  of a pointed union of chains to  $\tilde{D} = J^\circ$ .

Using the isomorphism  $T \circ e : L \rightarrow \mathcal{D}_{\text{nt}}(D) : x \mapsto \langle e(x) \rangle \cap \mathcal{C}^\circ$  that we get from Theorem 4.1, we will see that we can write  $\phi_e$  explicitly as

$$(5.1) \quad \phi_e^{-1}(x) = (\langle e[x] \rangle_{\mathcal{P}} \setminus \langle e[x] \rangle_{\mathcal{P}}) \cap \mathcal{C}^\circ.$$

On the other hand, for any surjective homomorphism  $\phi : \mathcal{C}^\circ \rightarrow J^\circ$  from a pointed union of chains, we get, by Fact 5.2, a unique pointed chain decomposition  $\phi' : \mathcal{C}^\circ \rightarrow D$  of a preorder  $D$  such that  $\phi = q \circ \phi'$  and  $\tilde{D} = J^\circ$ . By Lemma 5.1 we have  $L \cong \mathcal{D}_{\text{nt}}(D)$  and Theorem 4.1 gives us an embedding

$$T^{-1} : \mathcal{D}_{\text{nt}}(D) \rightarrow \mathcal{P} : S \mapsto \bigvee_{\mathcal{P}} (\phi')^{-1}(S)$$

of  $L$  into  $\mathcal{P}$ , where we write  $(\phi')^{-1}(S)$  in place of  $S$  as  $S$  is only considered a subset of  $\mathcal{P}$  via the bijection  $\phi'$ . As  $q$  induces an isomorphism  $\mathcal{D}_{\text{nt}}(D) \rightarrow \mathcal{D}_{\text{nt}}(\tilde{D}) \cong \mathcal{D}_{\text{nt}}(J^\diamond)$ , this is an embedding

$$(5.2) \quad e_\phi := (T \circ q)^{-1} : \mathcal{D}_{\text{nt}}(J^\diamond) \rightarrow \mathcal{P} : S \mapsto \bigvee_{\mathcal{P}} \phi^{-1}(S)$$

of  $L$  into a product of chains.

We are ready for our generalisation of Corollary 2.5. Recall that a pointed chain cover of  $J^\diamond(L)$  is a surjective map from a pointed union of chains to  $J^\diamond(L)$ .

**Theorem 5.3.** *For any finite distributive lattice  $L$ , the maps  $e \rightarrow \phi_e$  and  $\phi \rightarrow e_\phi$  defined in equations (5.1) and (5.2) give a one-to-one correspondence between pointed chain covers of  $J^\diamond(L)$ , and embeddings of  $\mathcal{D}_{\text{nt}}(J^\diamond(L)) \cong L$  into products of chains.*

*Proof.* It is enough to show that the maps  $e \rightarrow \phi_e$  and  $\phi \rightarrow e_\phi$  are inverses. For a downset  $S$  of  $J^\diamond$  we have that  $\phi_e^{-1}(S) = \bigcup_{x \in S} \phi_e^{-1}(x)$ . But this is just  $\langle e \langle x \rangle \rangle_{\mathcal{P}} \cap \mathcal{C}^\diamond$ , because anything in  $\langle e \langle x \rangle \rangle_{\mathcal{P}}$  is in  $\langle e \langle x' \rangle \rangle_{\mathcal{P}}$  for some other  $x' \in S$ . It follows that  $e = e_{\phi_e}$ ; indeed, recalling for the last equality that  $p \rightarrow \langle p \rangle \cap \mathcal{C}^\diamond$  and  $S \rightarrow \bigvee S$  are inverse :

$$e_{\phi_e} \langle x \rangle = \bigvee \phi_e^{-1} \langle x \rangle = \bigvee \langle e \langle x \rangle \rangle_{\mathcal{P}} \cap \mathcal{C}^\diamond = e \langle x \rangle.$$

To see that  $\phi_{e_\phi} = \phi$ , let  $x$  be in  $J^\diamond$ . Using in the third line that  $\langle p \rangle_{\mathcal{P}} \setminus \langle p \rangle_{\mathcal{P}} = p$  for any  $p$ , we get

$$\begin{aligned} \phi_{e_\phi}^{-1}(x) &= (\langle e_\phi \langle x \rangle \rangle_{\mathcal{P}} \setminus \langle e_\phi \langle x \rangle \rangle_{\mathcal{P}}) \cap \mathcal{C}^\diamond \\ &= (\langle \bigvee \phi^{-1} \langle x \rangle \rangle_{\mathcal{P}} \setminus \langle \bigvee \phi^{-1} \langle x \rangle \rangle_{\mathcal{P}}) \cap \mathcal{C}^\diamond \\ &= (\bigvee \phi^{-1} \langle x \rangle) \cap \mathcal{C}^\diamond \\ &= \phi^{-1}(x). \quad \square \end{aligned}$$

**Remark 5.4.** When removing  $\mathbf{0}$  and  $\mathbf{1}$ , it is not hard to see that this yields a one-to-one correspondence between full embeddings of  $L$  into product of chains, and *chain covers* of  $J(L)$ – surjective homomorphisms from a disjoint unions of chains to  $J(L)$ . Subdirect embeddings yield homomorphisms that are injective on each chain; and tight embeddings correspond with bijections– so-called chain decompositions.

## 6. Lattices of Independent Sets

Adjusting our construction, we extend and complement results of Koh from [5].

### 6.1. Background

The correspondence taking a downset of a poset  $P$  to its subset of maximal elements gives one-to-one correspondence between the downsets  $\mathcal{D}(P)$  of  $P$  and the

antichains  $\mathcal{A}(P)$  of  $P$ . It is not hard to see that by defining the following ordering on  $\mathcal{A}(P)$ , the correspondence can be extended to a lattice isomorphism: for antichains  $I, I' \in \mathcal{A}(P)$ , set  $I \leq I'$  if for all  $a \in I$  there is some  $a' \in I'$  such that  $a \leq a'$ . The *width* of a poset is the size of its maximum antichain. In [3], Dilworth, using his famous chain decomposition theorem to decompose a poset  $P$  of width  $d$  into  $d$  chains, showed the following.

**Theorem 6.1.**([3]) *The poset  $\mathcal{A}_m(P) \leq \mathcal{A}(P)$  of all maximum sized antichains of a poset  $P$  is a distributive lattice.*

In [5], Koh showed a converse: every finite distributive lattice  $L$  is  $\mathcal{A}_m(P)$  for some poset  $P$ . In doing so he showed something stronger. Koh defines a construction  $(J(L), \mathcal{C}) \mapsto P_{\mathcal{C}}$  which yields a poset  $P_{\mathcal{C}}$  for every chain decomposition  $\mathcal{C}$  of the poset  $J(L)$ . He then showed the following.

**Theorem 6.2.**([5]) *For every finite distributive lattice  $L$ , and every chain decomposition  $\mathcal{C}$  of  $J(L)$ ,  $L \cong \mathcal{A}_m(P_{\mathcal{C}})$ .*

There are posets  $P$  for which  $L \cong \mathcal{A}_m(P)$  that are not accounted for by Koh's construction. A small alteration of the reverse point of view of this paper yields a simple generalisation of Koh's construction. With this we show that every poset  $P$  for which  $L \cong \mathcal{A}_m(P)$  arises from a chain cover of  $J(L)$ . In fact, the generalisation gives a digraph in place of  $P$  for every embedding of  $L$  into a product of chains, and this digraph is a poset if and only if the embedding is subdirect.

The alteration of the reverse point of view yields transitive digraph extensions of a union of tournaments rather than preorder extensions of a pointed union of sets. We start with definitions required to define the antichain lattice of such digraphs.

## 6.2. Definitions and Setup

A directed path  $x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_d$  in a digraph is *non-trivial* if  $d \geq 2$ . Note that a loopless vertex induces a trivial path, while a vertex  $x$  with a loop induces the non-trivial directed path  $x \rightarrow x$ .

**Definition 6.3.** A subset  $I$  of a digraph  $D$  is *independent* if there is no non-trivial  $xy$ -path in  $D$  for  $x$  and  $y$  in  $I$ .

This is a much stronger notion of independence than is usually defined for digraphs, it implies independence in the transitive closure of the graph. In fact, no vertex that is in a cycle, including a looped vertex, can be in an independent set.

**Definition 6.4.**( $\mathbb{A}_d(A)$ ) Let  $\mathbb{A}_d(A)$  be the family of independent sets of size  $d$  of a digraph  $A$ , and order it as follows. For  $I, I' \in \mathbb{A}_d(A)$ , let  $I \leq I'$  if for each  $a' \in I'$  there is an  $aa'$ -path in  $A$  with  $a \in I$ .

**Remark 6.5.** For a poset  $P$  let  $A$  be transitive acycle digraph that we get from  $P$  by removing loops. It is easy to see that  $\mathbb{A}_d(A) = \mathcal{A}_m(P)$  when  $d$  is the width of  $P$ . As it will simplify exposition, we will translate Koh's results into our notation, and remind the reader of this by saying he *essentially* proved such and such.

Our basic setup now is as follows. Given a product  $\mathcal{P} = \prod \mathcal{C}$  of a family  $\mathcal{C} = \{C_1, C_2, \dots, C_d\}$  of  $d$  chains,  $\mathcal{T}$  will be the (disjoint) *union of tournaments* we get from the parallel sum of the chains in  $\mathcal{C}$  by removing the loops from all vertices. As we will now be referring to elements of a chain  $C_i$  individually, we affix a superscript, for example, denoting the element 3 or  $x_j$ , of  $C_i$  by  $3^{(i)}$  or  $x_j^{(i)}$  respectively. For a vertex  $x = (x_1, \dots, x_d)$  of  $\mathcal{P}$ , let  $I_x$  be the set

$$I_x := \{x_1^{(1)}, \dots, x_d^{(d)}\}.$$

By Remark 6.5 the  $d$  element independent sets of  $\mathcal{T}$  are exactly  $I_x$  for  $x \in \mathcal{P}$ .

A digraph  $A$  is a *spanned extension* of  $\mathcal{T}$  if  $\mathcal{T}$  is a spanning subgraph of  $A$ . For such digraphs all  $d$ -element independent sets are of the form  $I_x$ , and so the ordering on  $\mathbb{A}_d(A)$  defined in Definition 6.4 has a simpler definition, essentially observed in [5] in the case that  $A$  is a poset.

**Lemma 6.6.** *Let  $A$  be a spanned extension of a union  $\mathcal{T}$  of  $d$  tournaments. For  $I_x, I_y \in \mathbb{A}_d(A)$ , we have  $I_x \leq I_y$  if and only if  $x_i \leq y_i$  for all  $i$ .*

*Proof.* The ‘if’ direction is immediate from Definition 6.4 as all arcs of  $\mathcal{T}$  are in  $A$ . For the ‘only if’ direction, assume that  $I_x \leq I_y$ , and let  $i \in [d]$ . We must show that  $x_i \leq y_i$ . Assume that it is not, so  $y_i < x_i$ , and so there is a directed path in  $A$  from  $y_i$  to  $x_i$ . As  $I_x \leq I_y$  there is a path in  $A$  from  $x_i$  to  $y_j$  for some  $j \in [d]$ . If  $i = j$  then  $x_i$  and  $y_i$  are in a directed cycle, contradicting the fact that  $x_i$  is in an independent set. If  $i \neq j$  then there is a path from  $y_i$  to  $x_i$  and one from  $x_i$  to  $y_j$ , contradicting the fact that  $y_i$  and  $y_j$  are in an independent set.  $\square$

From here, given a digraph  $D$  of appropriately defined width  $d$ , it is not hard, adapting Dilworth’s chain decomposition theorem to digraphs to find as a spanning subgraph a union  $\mathcal{T}$  of  $d$  tournaments. Then one can prove that the set  $\mathbb{A}_d(D)$  is a lattice under the above ordering. We will use Dilworth’s chain decomposition in this way in Corollary 6.11, but for now we follow our reverse point of view, and relate such digraphs  $D$  to sublattices of a given product of chains.

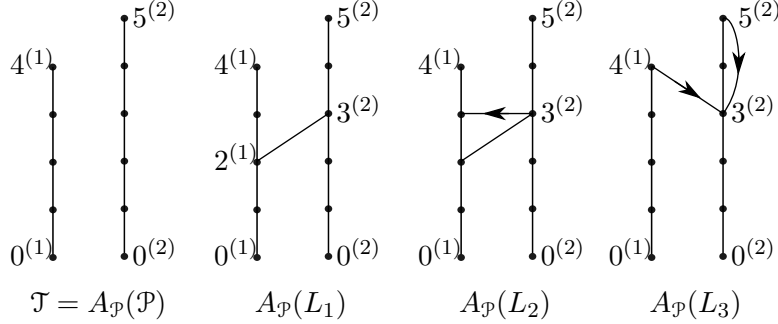
### 6.3. The Alternate Rival Construction $A_{\mathcal{P}}(L)$

Compare the following to Definition 3.2.

**Definition 6.7.** ( $A_{\mathcal{P}}(\mathcal{J})$  and  $A_{\mathcal{P}}(L)$ ) For a family  $\mathcal{C} = \{C_1, \dots, C_d\}$  of chains and any family of irreducible intervals  $\mathcal{J}$  of  $\mathcal{P} = \prod \mathcal{C}$ , let  $A_{\mathcal{P}}(\mathcal{J})$  be the spanned extension of  $\mathcal{T}$  that we get from  $\mathcal{T}$  by letting  $\beta^{(j)} \rightarrow \alpha^{(i)}$  for each  $[\alpha_i, \beta_j] \in \mathcal{J}$ . For a sublattice  $L$  of  $\mathcal{P}$ , let  $A_{\mathcal{P}}(L) = A_{\mathcal{P}}(\mathcal{J}_L)$ .

See Figure 4 for examples of  $A_{\mathcal{P}}(L)$  corresponding to the sublattices  $L$  of  $\mathcal{P}$  in Figure 2. The digraph  $A_{\mathcal{P}}(L)$  is still the transitive closure of the depicted digraph, where all unoriented arcs are oriented up, but it is no longer reflexive. Note that in the fourth figure in Figure 4 corresponding to  $L_3 = \mathcal{P} \setminus [3_2, 4^1]$ , we have both of the arcs  $(4^{(1)}, 3^{(2)})$  and  $(5^{(2)}, 3^{(2)})$ , as the intervals  $[3_2, 4^1]$  and  $[3_2, 5^2]$  are the




 Figure 4: Digraphs  $A_{\mathcal{P}}$  for sublattices of  $\mathcal{P} = Z_5 \times Z_6$  from Fig. 2

same. The graphs in Figure 4 are all irreflexive except  $A_{\mathcal{P}}(L_3)$  which, because it is transitive, has loops on the vertices  $3^{(2)}$ ,  $4^{(2)}$ , and  $5^{(3)}$ .

It is easy to see that  $\mathbb{A}_2(\mathcal{T}) \cong \mathcal{P} = Z_5 \times Z_6$  where  $\mathcal{T}$  is the first picture in Figure 4. This is an example of the base case of Theorem 6.8. To prove Theorem 6.8 we simply observe what we must do to  $\mathcal{T}$  to maintain this isomorphism when we remove an irreducible interval from  $\mathcal{P}$ . Remove the interval  $[3_1, 3^2]$  as we do to get the second lattice  $L_2 = L_1 \setminus [3_1, 3^2]$  of Figure 2. The second figure of Figure 4 shows what we must add to  $\mathcal{T}$ . To ensure that, for example,  $I_{(2,3)} = \{2^{(1)}, 3^{(2)}\}$  is no longer independent, we add the arc  $(2^{(1)}, 3^{(2)})$  to  $\mathcal{T}$ . This also ensures that such sets as  $I_{(2,4)}$  and  $I_{(1,5)}$  are no longer independent.

**Theorem 6.8.** *Let  $L$  be a sublattice of  $\mathcal{P} = \prod \mathcal{C}$ , where  $\mathcal{C} = \{C_1, \dots, C_d\}$  is a family of disjoint chains, and let  $A = A_{\mathcal{P}}(L)$ . The map*

$$I : L \rightarrow \mathbb{A}_d(A) : x \mapsto I_x := \{x_1^{(1)}, \dots, x_d^{(d)}\}$$

*is a lattice isomorphism.*

*Proof.* The theorem holds by Lemma 6.6 in the simplest case where  $L = \mathcal{P}$  and so  $A = \mathcal{T}$ . We proceed by induction on the size of  $A$ , by adding arcs. As adding arcs to  $\mathcal{T}$  or any spanned extension cannot create new independent sets, we have that  $\mathbb{A}_d(A)$  is a subset of  $\mathcal{P} = \mathbb{A}_d(\mathcal{T})$ ; and so by Lemma 6.6 it is a subposet.

Thus it is enough to show, inductively, that  $I : L \rightarrow \mathbb{A}_d(A)$  is a bijection. We must show that an independent set  $I_x \in \mathbb{A}_d(A')$  is not in  $\mathbb{A}_d(A)$  if and only if  $x$  is in  $[\alpha_i, \beta^j]$ . That is, we must show that  $I_x \in \mathbb{A}_d(A') \setminus \mathbb{A}_d(A)$  if and only if  $x_i \geq \alpha$  and  $\beta \geq x_j$ . For  $I_x \in \mathbb{A}_d(A')$ , clearly  $I_x \notin \mathbb{A}_d(A)$  if and only if  $(\beta^{(j)}, \alpha^{(i)})$  completes a path between  $x_j$  and  $x_i$ , so if and only if  $x_i \geq \alpha$  and  $\beta \geq x_j$ , as needed.  $\square$

Observe that this is essentially a generalisation of Koh's Theorem 6.2. Indeed, as a tight embedding of  $L$  into a product of chains  $\mathcal{P}$  corresponds to a chain decomposition of  $J(L)$  by Corollary 4.2 and Remark 4.4, Theorem 6.8 gives, for any

chain decomposition of  $J(L)$ , a digraph  $A$  such that  $\mathbb{A}_d(A) \cong L$ . As Dilworth's decomposition theorem assures there is such a decomposition when  $d$  is the width of  $J(L)$ , this essentially gives Theorem 6.2.

We continue, and complement Theorem 6.2 by finding every poset  $P$  such that  $\mathcal{A}_m(P) = L$ . As per Remark 6.5, this is done by finding every acyclic transitive graph  $A$  such that  $\mathbb{A}_d(A) = L$ . The main elements required are anti-chain analogues of Corollary 4.2 and Lemma 4.3. Instead of 'repeating' the proofs of these earlier sections, we observe that the anti-chain analogues follow from the results of Section 4 by comparing the constructions of  $D_{\mathcal{P}}(A)$  and  $A_{\mathcal{P}}(L)$ .

#### 6.4. Properties of $A_{\mathcal{P}}(L)$ via Koh's Construction

Comparing definitions, or comparing Figure 2 to Figure 4, it is easy to see how to get  $A_{\mathcal{P}}(L)$  directly from  $D_{\mathcal{P}}(L)$ . Indeed for a spanned preorder extension  $D$  of  $\mathcal{C}^\circ$ , let  $A = K_{\mathcal{P}}(D)$  be the digraph on the vertices of  $\mathcal{T}$  with arcset

$$\{(\beta^{(j)}, \alpha^{(i)}) \in \mathcal{T}^2 \mid ((\beta + 1)_j, \alpha_i) \in D\}.$$

One can easily check that  $K_{\mathcal{P}}(\mathcal{C}^\circ) = \mathcal{T}$ , and more generally that  $K_{\mathcal{P}}(D_{\mathcal{P}}(\mathcal{J})) = A_{\mathcal{P}}(\mathcal{J})$  for any closed family  $\mathcal{J}$  of irreducible intervals. (We require that  $\mathcal{J}$  is closed for this as an out-arc of  $\mathbf{1}$  in  $D$  yields several arcs in  $K_{\mathcal{P}}(D)$ , as is seen in the fourth figure in Figure 4).

It is a simple matter to show the transitivity of  $K_{\mathcal{P}}(D)$  from that of  $D$ , and so get from Proposition 3.7 that  $A_{\mathcal{P}}(\mathcal{J})$  is transitive if  $\mathcal{J}$  is closed. Observing that the construction  $D \mapsto K_{\mathcal{P}}(D)$  invertible when considering only transitive extensions  $A$  of  $\mathcal{T}$ , (i.e. tournament decompositions of  $A$ ) we see that its inverse is essentially a generalisation of Koh's construction from [5]. The only difference is that Koh's construction would always have loops on  $\mathcal{T}$ . As he considered only posets, this difference would be cosmetic for him, but it is significant for us. The construction can be used to easily translate statements about  $D_{\mathcal{P}}(L)$  to statements about  $A_{\mathcal{P}}(L)$ . We give two examples, omitting the simple proofs.

Using Corollary 4.2, we get the following analogue.

**Corollary 6.9.** *Let  $\mathcal{P} = \prod \mathcal{C}$  be product of chains. The map  $L \mapsto A_{\mathcal{P}}(L)$  gives a one-to-one correspondence between the sublattices of  $\mathcal{P}$  and the spanned transitive digraph extensions of  $\mathcal{T}$ .*

The classification lemma, Lemma 4.3, yields the following analogue.

**Lemma 6.10.** *Let  $L$  be a sublattice of  $\mathcal{P}$  and  $A = A_{\mathcal{P}}(L)$ .*

- (i)  *$L$  is a  $\{0, 1\}$ -sublattice if and only if the minimum and maximum element of each tournament  $T$  in  $\mathcal{T}$  are loopless in  $A$ .*
- (ii)  *$L$  is a subdirect sublattice if and only if  $A$  is loopless, which holds if and only if  $A$  contains no directed cycles.*
- (iii)  *$L$  is a tight sublattice if and only if  $A$  contains no directed cycle or alternating cycle:  $v_1 \rightarrow u_1 \leftarrow v_2 \rightarrow u_2 \leftarrow \cdots \rightarrow u_\ell \leftarrow v_1$ .*

For tight sublattices  $L$  we have that  $D_{\mathcal{P}}(L) = J^{\circ}(L)$  and so  $A = K_{\mathcal{P}}(J^{\circ})$ . An important part of Koh's proof from [5] was that his version of  $K_{\mathcal{P}}(J^{\circ})$  contains no crown, which is essentially what item (iii) says.

### 6.5. Complement to Koh

For a lattice  $L$ , and an embedding  $e$  of  $L$  into a product  $\mathcal{P}$  of chains, Theorem 5.3 gives a pointed chain cover  $\tilde{\phi} : \mathcal{C}^{\circ} \rightarrow J^{\circ}(L)$ . This gives, by Fact 5.2, a pointed chain decomposition  $\phi : \mathcal{C}^{\circ} \rightarrow D_e$  of some preorder  $D_e$  with  $D_{\mathcal{P}}(D_e) = L$ . Viewing  $D_e$  as a preorder extension of  $\mathcal{C}^{\circ}$ , we can apply construction  $K_{\mathcal{P}}(J^{\circ}(L))$  to get a transitive digraph extension  $A_e = K_{\mathcal{P}}(D_e)$  of  $K_{\mathcal{P}}(\mathcal{C}^{\circ}) = \mathcal{T}$ . By Theorem 6.8 we have  $L \cong \mathbb{A}_d(A_e)$ . In particular, if the embedding  $e$  was subdirect, then by Lemma 6.10  $A_e$  is essentially a poset— we simply have to add loops to get Koh's poset.

As a complement to Koh's theorem, we want to show that for any poset  $P$  such that  $\mathcal{A}_m(P) = L$ , there is some  $e$  such that  $A_e$  is  $P$  with loops removed. More generally, we have the following.

**Corollary 6.11.** *Let  $L$  be a finite distributive lattice. For any transitive digraph  $A$  such that  $L \cong \mathbb{A}_d(A)$  where  $d$  is the width of  $A$ , there is an embedding  $e$  of  $L$  into a product of chains  $\mathcal{P}$  such that  $A = A_e$ .*

*Proof.* Let  $A$  be a transitive digraph with  $L \cong \mathbb{A}_d(A)$ . The main task is to find a tournament decomposition  $\phi : \mathcal{T} \rightarrow A$  of  $A$  into  $d$  tournaments.

Where  $A^r$  is the preorder we get from  $A$  by making it reflexive, and  $P = q(A^r)$  is the poset we get by quotienting by the non-preference relation, we have by Dilworth's Decomposition Theorem that  $P$  has a decomposition  $\phi' : \mathcal{C} \rightarrow P$  into  $d$  chains. Removing loops from  $\mathcal{C}$ , we alter  $\phi'$  to a tournament cover  $\tilde{\phi}$  of  $P$  so that it lifts to a tournament decomposition  $\phi$  of  $A$  by Fact 5.2, which clearly holds with loops removed. This is easy: for any vertex  $x \in C \in \mathcal{C}$  such that  $\phi'(x) = [y]$ , add  $|[y]| - 1$  new elements to  $C$  just above  $x$  and let  $\tilde{\phi}$  map  $x$  and all of these vertices to  $[y]$ . By Fact 5.2  $(\tilde{\phi}, P)$  lifts to some  $(\phi, A')$  where to get  $A'$  we replace an element  $y$  of  $P$  with a transitive cycle of size  $q^{-1}(y)$ . So  $A' = A$ , and  $\phi$  is a tournament decomposition of  $A$ . Thus  $A = A_{e_{\phi}}$  where  $e_{\phi}$  is the embedding of  $L$  we get by Theorem 5.3.  $\square$

**Corollary 6.12.** *For every finite poset  $P$  such that  $\mathcal{A}_m(P) \cong L$  there is a subdirect embedding of  $L$  into a product of chains  $\mathcal{P}$  such that  $P = A_{\mathcal{P}}^r(L)$ .*

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