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ON THE BERGMAN KERNEL FOR SOME HARTOGS DOMAINS

Jong-Do Park

Abstract. In this paper, we compute the Bergman kernel for

$$\Omega_{p,q,r} = \{(z, z', w) \in \mathbb{C}^2 \times \Delta : |z|^{2p} < (1 - |z'|^{2q})(1 - |w|^2)^r\},$$

where $p, q, r > 0$ in terms of multivariable hypergeometric series. As a consequence, we obtain the behavior of

$$K_{\Omega_{p,q,r}}(z, 0, 0; z, 0, 0)$$

when $(z, 0, 0)$ approaches to the boundary of $\Omega_{p,q,r}$.

1. Introduction

For any bounded domain $D \subset \mathbb{C}^N$, let $L^2_a(D)$ be the space of all holomorphic square-integrable functions on D. For any $z \in D$, $\Phi_z : L^2_a(D) \to \mathbb{C}$ defined by $\Phi_z(f) = f(z)$ is a bounded linear functional on $L^2_a(D)$. By Riesz representation theorem, there exists the unique element $K_z(\cdot) \in L^2_a(D)$ such that $\Phi_z(f) = \langle f(\cdot), K_z(\cdot)\rangle$, namely

$$f(z) = \int_D f(w)\overline{K_z(w)}dV(w)$$

for all $f \in L^2_a(D)$.

Definition. The Bergman kernel function $K_D(z; w)$ for D is given by $K_D(z; w) = \overline{K_z(w)}$, where $z, w \in D$.

The Bergman kernel is defined for arbitrary bounded domains, but it is hard to compute the explicit form of the Bergman kernel except for special cases. For the unit ball, bounded symmetric domain [9, 15], bounded homogeneous domains [7], the explicit form of the Bergman kernel has been computed. See more [2–4, 8, 12, 13, 17, 18] for other special domains.

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Recently, many mathematicians have investigated the properties of the Hartogs domain (over Ω)

$$\widetilde{\Omega} := \{(z,\zeta) \in \Omega \times \mathbb{C} : |\zeta|^2 < p(z)\}, \tag{1}$$

where $p(z)$ is a positive-valued function on Ω. See [1, 11, 14, 16].

In [1], Ahn and the author computed the Bergman kernel for Hartogs domain, where Ω is a product domain $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_q$ of irreducible bounded *symmetric* domains and $p(z) = p(z_1)p(z_2)\cdots p(z_q)$ with generic norm $p(z_i)$ for each Ω_i. This result is generalized to the case [11] when each Ω_i is a bounded *homogeneous* domain.

So far, the Hartogs domains have been considered only when $p(z)$ is a generic norm in (1). For example [1, 11, 14], if Ω is a polydisk, then

$$p(z) = (1-|z_1|^2)^{\mu_1}\cdots(1-|z_n|^2)^{\mu_n}.$$

In this paper, we study the Bergman kernel for some Hartogs domain whose $p(z)$ *is not a generic norm.* More precisely, we express the Bergman kernel for

$$\Omega_{p,q,r}=\{(z,z',w)\in\mathbb{C}^3 : |z|^{2p} < (1-|z'|^{2q})(1-|w|^2)^r\}$$

in terms of sums of Appell hypergeometric functions $F_2(a;b_1,b_2,c_1,c_2;x,y)$ which are two-dimensional Gauss hypergeometric series. As a consequence, we obtain the boundary behavior of the Bergman kernel for $\Omega_{p,q,r}$.

Remark 1.1. If $q = 1$, then the domain $\Omega_{p,1,r}$ is a Hartogs domain whose defining function is a generic norm with respect to the polydisk. The Bergman kernel for $\Omega_{p,1,r}$ has been computed in [1]. This paper is the first research on the Bergman kernel for Hartogs domain when $p(z)$ is not a generic norm.

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2. Hypergeometric functions

In this section we review the hypergeometric function and the Bergman kernel for complex ellipsoid. For $a, b, c \in \mathbb{R}$ and $|z| < 1$, we define the Gauss hypergeometric series

$$\begin{aligned}{}_2F_1(a,b;c;z) &= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n\\ &= 1+\frac{a\cdot b}{c\cdot 1}z+\frac{a(a+1)\cdot b(b+1)}{c(c+1)\cdot 1\cdot 2}z^2+\cdots,\end{aligned}$$

where a rising factorial $(a)_n$ is given by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)\cdots(a+n-1), & n\geq 1,\\ 1, & n=0.\end{cases}$$

The Bergman kernel for complex ellipsoid is expressed in terms of Appell's multivariable hypergeometric functions which are infinite series. Recall that an Appell's hypergeometric function is defined by

$$\begin{aligned} &F_A^{(n)}(a; b_1, \ldots, b_n; c_1, \ldots, c_n; z_1, \ldots, z_n) \\ &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{m_1!\cdots m_n!(c_1)_{m_1}\cdots(c_n)_{m_n}} z^m, \end{aligned}$$

where $z^m := z_1^{m_1} \cdots z_n^{m_n}$.

Remark 2.1. In this paper we will write $F_A^{(1)} = {}_2F_1$ and $F_A^{(2)} = F_2$.

Proposition 2.2 ([5,6])**.** *The Bergman kernel for the complex ellipsoid*

$$\{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1|^{2p_1} + \cdots + |z_n|^{2p_n} < 1\}$$

is given by

$$K(z; w) = \frac{\prod_{j=1}^n p_j}{\pi^n} \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \frac{\Gamma\left(1 + \sum_{j=1}^n \frac{k_j+1}{p_j}\right)}{\prod_{j=1}^n \Gamma\left(\frac{k_j+1}{p_j}\right)} (z\overline{w})^k,$$

where $(z\overline{w})^k := (z_1\overline{w}_1)^{k_1} \cdots (z_n\overline{w}_n)^{k_n}$*. In particular, if* $p_1, \ldots, p_n \in \mathbb{N}$*, then the Bergman kernel is expressed in terms of Appell's hypergeometric functions as follows*:

$$\begin{aligned} K(z; w) = \frac{\prod_{j=1}^n p_j}{\pi^n} \sum_{r_1=0}^{p_1-1} \cdots \sum_{r_n=0}^{p_n-1} \frac{\Gamma\left(1 + \sum_{j=1}^n \frac{r_j+1}{p_j}\right)}{\prod_{j=1}^n \Gamma\left(\frac{r_j+1}{p_j}\right)} (z\overline{w})^r \\ \times F_A^{(n)}\left(1 + \sum_{j=1}^n \frac{r_j+1}{p_j}; \mathbf{1}; \frac{\mathbf{r}+\mathbf{1}}{\mathbf{p}}; (z\overline{w})^p\right), \end{aligned}$$

where $\mathbf{1} = (1, 1, \ldots, 1)$ *and* $\frac{\mathbf{r}+\mathbf{1}}{\mathbf{p}} = (\frac{r_1+1}{p_1}, \ldots, \frac{r_n+1}{p_n})$.

Using Proposition 2.2, the Bergman kernels for some complex ellipsoids have been computed. See [12, 13].

3. The Bergman kernel for $\Omega_{p,q,r}$

In this section we review the method [10] of computing the Bergman kernel for some Hartogs domains. Throughout this paper, let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in $\mathbb{C}$.

Let Ω be a bounded Reinhardt domain in $\mathbb{C}^{n+m}$. Let α_j be a positive constant for all $1 \leq j \leq n$. Define

$$U^\alpha := \{(z, z', w) \in \mathbb{C}^n \times \mathbb{C}^m \times \mathbb{C} : (f_\alpha(z, w), z') \in \Omega,\ |w| < 1\},$$

where

$$f_\alpha(z,w) := \left(\frac{z_1}{(1-|w|^2)^{\frac{\alpha_1}{2}}}, \dots, \frac{z_n}{(1-|w|^2)^{\frac{\alpha_n}{2}}} \right).$$

For fixed $w \in \Delta$, we define the slice domain

$$U_w^\alpha := \{(z,z') \in \mathbb{C}^n \times \mathbb{C}^m : (z,z',w) \in U^\alpha\}.$$

Then for any $w \in \Delta$, the domain U_w^α is biholomorphic to Ω. Define

$$h(z,w,\eta) := \left(z_1 \left(\frac{1-|\eta|^2}{1-w\overline{\eta}} \right)^{\alpha_1}, \dots, z_n \left(\frac{1-|\eta|^2}{1-w\overline{\eta}} \right)^{\alpha_n} \right)$$

and

$$D_{U^\alpha} := \frac{(1-|\eta|^2)^{\alpha\cdot\mathbf{1}}}{\pi(1-w\overline{\eta})^{2+\alpha\cdot\mathbf{1}}} \left(I + \sum_{j=1}^n \alpha_j \left(1 + z_j \frac{\partial}{\partial z_j} \right) \right).$$

In [10], Huo computed the Bergman kernel for U^α.

Proposition 3.1 ([10])**.** *For $(z,z',w),(\zeta,\zeta',\eta) \in U^\alpha$, we have*

$$K_{U^\alpha}(z,z',w;\zeta,\zeta',\eta) = D_{U^\alpha} K_{U_\eta^\alpha}(h(z,w,\eta),z';\zeta,\zeta').$$

Now we explain how we can compute the Bergman kernel for

$$\Omega_{p,q,r} := \{(z,z',w) \in \mathbb{C}^3 : |z|^{2p} < (1-|z'|^{2q})(1-|w|^2)^r\}$$

following Huo's notations. Let

$$\Omega := \{(z,z') \in \mathbb{C}^2 : |z|^{2p} + |z'|^{2q} < 1\}$$

and $\alpha := \frac{r}{p}$ for $r > 0$. Then $f_\alpha(z,w) = \frac{z}{(1-|w|^2)^{\frac{\alpha}{2}}}$. It follows that

$$U^\alpha = \left\{ (z,z',w) \in \mathbb{C}^2 \times \Delta : \left| \frac{z}{(1-|w|^2)^{\frac{\alpha}{2}}} \right|^{2p} + |z'|^{2q} < 1 \right\},$$

or equivalently,

$$U^\alpha = \left\{ (z,z',w) \in \mathbb{C}^2 \times \Delta : |z|^{2p} < (1-|z'|^{2q})(1-|w|^2)^r \right\},$$

which will be written as $\Omega_{p,q,r}$ in this paper.

It is clear that for any fixed $w \in \Delta$,

$$U_w^\alpha := \left\{ (z,z') \in \mathbb{C}^2 : \left| \frac{z}{(1-|w|^2)^{\frac{\alpha}{2}}} \right|^{2p} + |z'|^{2q} < 1 \right\} \tag{2}$$

is biholomorphic to $\Omega = \{(z,z') \in \mathbb{C}^2 : |z|^{2p} + |z'|^{2q} < 1\}$ with respect to the linear map.

The following is the special case of Proposition 2.2.

Proposition 3.2. *The Bergman kernel for*

$$D_{p,q} := \left\{(z, z') \in \mathbb{C}^2 : |z|^{2p} + |z'|^{2q} < 1\right\}$$

is

$$K_{D_{p,q}}(z, z'; \zeta, \zeta') = \frac{pq}{\pi^2} \sum_{\beta=0}^{\infty} \sum_{\gamma=0}^{\infty} c_{\beta,\gamma} (z\bar{\zeta})^{\beta} (z'\overline{\zeta'})^{\gamma},$$

where $c_{\beta,\gamma} = \frac{\Gamma(\frac{\beta+1}{p}+\frac{\gamma+1}{q}+1)}{\Gamma(\frac{\beta+1}{p})\Gamma(\frac{\gamma+1}{q})}$.

Note that for each $\eta \in \Delta$, a linear mapping

$$\phi_\eta(z, z') := \left(\frac{z}{(1-|\eta|^2)^{\frac{\alpha}{2}}}, z'\right)$$

is a biholomorphic mapping from U_η^α onto $D_{p,q}$, where U_η^α is given as (2).

Lemma 3.3. *Let $\phi : \Omega_1 \to \Omega_2$ be a biholomorphic mapping from Ω_1 onto Ω_2. Then we have*

$$K_{\Omega_1}(z; \zeta) = \det J\phi(z) K_{\Omega_2}(\phi(z); \phi(\zeta)) \overline{\det J\phi(\zeta)},$$

where $z, \zeta \in \Omega_1$.

By Proposition 3.2 and Lemma 3.3, for any $\eta \in \Delta$, it follows that

$$\begin{aligned} K_{U_\eta^\alpha}(z, z'; \zeta, \zeta') &= \frac{1}{(1-|\eta|^2)^{\alpha}} K_{D_{p,q}}\left(\frac{z}{(1-|\eta|^2)^{\frac{\alpha}{2}}}, z'; \frac{\zeta}{(1-|\eta|^2)^{\frac{\alpha}{2}}}, \zeta'\right) \\ &= \frac{pq}{\pi^2(1-|\eta|^2)^{\alpha}} \sum_{\beta=0}^{\infty} \sum_{\gamma=0}^{\infty} c_{\beta,\gamma} \left(\frac{z\bar{\zeta}}{(1-|\eta|^2)^{\alpha}}\right)^{\beta} (z'\overline{\zeta'})^{\gamma}. \qquad (3) \end{aligned}$$

We need the following lemma in order to express (3) in terms of Appell hypergeometric functions. We write $F_2 := F_A^{(2)}$.

Lemma 3.4. *Let $c_{\beta,\gamma}$ be as in Proposition 3.2. Then*

$$\begin{aligned} &\sum_{\beta=0}^{\infty} \sum_{\gamma=0}^{\infty} c_{\beta,\gamma} x^{\beta} y^{\gamma} \\ &= \sum_{s=0}^{p-1} \sum_{u=0}^{q-1} c_{s,u} x^s y^u F_2\left(\frac{s+1}{p} + \frac{u+1}{q} + 1; 1, 1; \frac{s+1}{p}, \frac{u+1}{q}; x^p, y^q\right). \end{aligned}$$

Proof. If we write $\beta = pr+s$ and $\gamma = qt+u$ with $0 \le s \le p-1$ and $0 \le u \le q-1$, then

$$\begin{aligned} &\sum_{\beta=0}^{\infty} \sum_{\gamma=0}^{\infty} \frac{\Gamma(\frac{\beta+1}{p} + \frac{\gamma+1}{q} + 1)}{\Gamma(\frac{\beta+1}{p})\Gamma(\frac{\gamma+1}{q})} x^{\beta} y^{\gamma} \\ &= \sum_{s=0}^{p-1} \sum_{u=0}^{q-1} \sum_{r,t=0}^{\infty} \frac{\Gamma(r + t + \frac{s+1}{p} + \frac{u+1}{q} + 1)}{\Gamma(r + \frac{s+1}{p})\Gamma(t + \frac{u+1}{q})} x^{pr+s} y^{qt+u}, \end{aligned}$$

which completes the proof. □

For simplicity, we write

$$A := \frac{z\overline{\zeta}}{(1-w\overline{\eta})^{\alpha}}, \quad B = z'\overline{\zeta'}.$$

Note that

$$h(z,w,\eta) = z\left(\frac{1-|\eta|^2}{1-w\overline{\eta}}\right)^{\alpha}.$$

Then by (2) and Lemma 3.4, we have

$$\begin{aligned}
&K_{U_{\eta}^{\alpha}}(h(z,w,\eta),z';\zeta,\zeta') \\
&= \frac{pq}{\pi^2(1-|\eta|^2)^{\alpha}}\sum_{\beta=0}^{\infty}\sum_{\gamma=0}^{\infty} c_{\beta,\gamma}\left(\frac{z\overline{\zeta}}{(1-w\overline{\eta})^{\alpha}}\right)^{\beta}(z'\overline{\zeta'})^{\gamma} \\
&= \frac{pq}{\pi^2(1-|\eta|^2)^{\alpha}}\sum_{s=0}^{p-1}\sum_{u=0}^{q-1} c_{s,u}(A^sB^u) \\
&\qquad \times F_2\left(\frac{s+1}{p}+\frac{u+1}{q}+1;1,1;\frac{s+1}{p},\frac{u+1}{q};A^p,B^q\right).
\end{aligned}$$

Recall that by Proposition 3.1,

$$K_{U^{\alpha}}(z,z',w;\zeta,\zeta',\eta) = D_{U^{\alpha}}K_{U_{\eta}^{\alpha}}(h(z,w,\eta),z';\zeta,\zeta'), \tag{4}$$

where

$$D_{U^{\alpha}} = \frac{(1-|\eta|^2)^{\alpha}}{\pi(1-w\overline{\eta})^{2+\alpha}}\left((1+\alpha)I + \alpha z\frac{\partial}{\partial z}\right).$$

Thus we need the following formula for a partial derivative of Appell's hypergeometric functions.

Lemma 3.5. *For any $(x,y)\in\mathbb{C}^2$ satisfying $|x|+|y|<1$, we have*

$$\frac{\partial}{\partial x}F_2(a;b_1,b_2;c_1,c_2;x,y) = \frac{ab_1}{c_1}F_2(a+1;b_1+1,b_2;c_1+1,c_2;x,y).$$

By (4), we have

$$\begin{aligned}
&K_{U^{\alpha}}(z,z',w;\zeta,\zeta',\eta) \\
&= \frac{(1-|\eta|^2)^{\alpha}}{\pi(1-w\overline{\eta})^{2+\alpha}}\left\{(1+\alpha)I+\alpha z\frac{\partial}{\partial z}\right\}K_{U_{\eta}^{\alpha}}(h(z,w,\eta),z';\zeta,\zeta') \\
&= \frac{pq}{\pi^3(1-w\overline{\eta})^{2+\alpha}}\sum_{s=0}^{p-1}\sum_{u=0}^{q-1}c_{s,u}B^u\left\{(1+\alpha)I+\alpha z\frac{\partial}{\partial z}\right\} \\
&\qquad \times\left[A^sF_2\left(\frac{s+1}{p}+\frac{u+1}{q}+1;1,1;\frac{s+1}{p},\frac{u+1}{q};A^p,B^q\right)\right].
\end{aligned}$$

For the simplicity, we write

$$F_2 := F_2\left(\frac{s+1}{p}+\frac{u+1}{q}+1;1,1;\frac{s+1}{p},\frac{u+1}{q};A^p,B^q\right).$$

By Lemma 3.5, we have

$$\frac{\partial}{\partial z}F_2 = \frac{\frac{s+1}{p}+\frac{u+1}{q}+1}{\frac{s+1}{p}}F_2^*\cdot\left(\frac{\partial}{\partial z}A^p\right) = \frac{\frac{s+1}{p}+\frac{u+1}{q}+1}{\frac{s+1}{p}}F_2^*\cdot pA^{p-1}\frac{\partial A}{\partial z},$$

where

$$F_2^* := F_2\left(\frac{s+1}{p}+\frac{u+1}{q}+2;2,1;\frac{s+1}{p}+1,\frac{u+1}{q};A^p,B^q\right).$$

Since $z\frac{\partial A}{\partial z}=A$, we have

$$\begin{aligned}
&\left\{(1+\alpha)I+\alpha z\frac{\partial}{\partial z}\right\}(A^sF_2)\\
&=(1+\alpha)A^sF_2+\alpha zsA^{s-1}\frac{\partial A}{\partial z}F_2+\alpha zA^s\frac{\partial}{\partial z}F_2\\
&=(1+\alpha+\alpha s)A^sF_2+\frac{\frac{s+1}{p}+\frac{u+1}{q}+1}{\frac{s+1}{p}}\cdot\alpha pA^{s+p}F_2^*.
\end{aligned}$$

Since $\alpha=\frac{r}{p}$, we obtain the following.

Theorem 3.6. *The Bergman kernel $K_{\Omega_{p,q,r}}(z,z',w;\zeta,\zeta',\eta)$ is given by*

$$\begin{aligned}
&K_{\Omega_{p,q,r}}(z,z',w;\zeta,\zeta',\eta)\\
&=\frac{pq}{\pi^3(1-w\overline{\eta})^{2+\frac{r}{p}}}\sum_{s=0}^{p-1}\sum_{u=0}^{q-1}\left[\left(1+\frac{r}{p}+\frac{rs}{p}\right)c_{s,u}A^sB^uF_2+rc_{s,u}^*A^{s+p}B^uF_2^*\right],
\end{aligned}$$

where $A=\frac{z\overline{\zeta}}{(1-w\overline{\eta})^{\frac{r}{p}}}$ and $B=z'\overline{\zeta'}$. The hypergeometric functions F_2 and F_2^* are given by

$$\begin{aligned}
F_2 &= F_2\left(\frac{s+1}{p}+\frac{u+1}{q}+1;1,1;\frac{s+1}{p},\frac{u+1}{q};A^p,B^q\right),\\
F_2^* &= F_2\left(\frac{s+1}{p}+\frac{u+1}{q}+2;2,1;\frac{s+1}{p}+1,\frac{u+1}{q};A^p,B^q\right).
\end{aligned}$$

The constants $c_{s,u}$ and $c_{s,u}^*$ are given by

$$c_{s,u}=\frac{\Gamma(\frac{s+1}{p}+\frac{u+1}{q}+1)}{\Gamma(\frac{s+1}{p})\Gamma(\frac{u+1}{q})},\quad c_{s,u}^*=\frac{\Gamma(\frac{s+1}{p}+\frac{u+1}{q}+2)}{\Gamma(\frac{s+1}{p}+1)\Gamma(\frac{u+1}{q})}.$$

4. Behavior of the Bergman kernel $K_{\Omega_{p,q,r}}(z,0,0;z,0,0)$

As a consequence of Theorem 3.6, we obtain the behavior of

$$K_{\Omega_{p,q,r}}(z,0,0;z,0,0)$$

when $(z,0,0)$ approaches to the boundary of $\Omega_{p,q,r}$.

At first we need the following Euler's transformation formula.

Lemma 4.1.

$$_2F_1(a,b;c;x) = (1-x)^{c-a-b}{}_2F_1(c-a,c-b;c;x).$$

The following explicit value of $F(a,b;c;x)$ at $x=1$ is well-known.

Lemma 4.2. *If* $\Re(c) > \Re(a+b)$, *then*

$$\lim_{x\to 1-} {}_2F_1(a,b;c;x) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

The above lemma comes from the Euler representation

$$B(b,c-b)_2F_1(a,b;c;x) = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dx,$$

where $B(\cdot,\cdot)$ is a beta function.

By Theorem 3.6, we have

$$\begin{aligned} &K_{\Omega_{p,q,r}}(z,0,0;z,0,0) \\ &= \frac{pq}{\pi^3}\sum_{s=0}^{p-1}\Bigg[\left(1+\frac{r}{p}+\frac{rs}{p}\right)c_{s,0}|z|^{2s}{}_2F_1\left(\frac{s+1}{p}+\frac{1}{q}+1,1;\frac{s+1}{p};|z|^{2p}\right) \\ &\qquad +(\alpha p)c^*_{s,0}|z|^{2s+2p}{}_2F_1\left(\frac{s+1}{p}+\frac{1}{q}+2,2;\frac{s+1}{p}+1;|z|^{2p}\right)\Bigg], \end{aligned}$$

where

$$c_{s,0} = \frac{\Gamma(\frac{s+1}{p}+\frac{1}{q}+1)}{\Gamma(\frac{s+1}{p})\Gamma(\frac{1}{q})}, \quad c^*_{s,0} = \frac{\Gamma(\frac{s+1}{p}+\frac{1}{q}+2)}{\Gamma(\frac{s+1}{p}+1)\Gamma(\frac{1}{q})}.$$

Now we will find the constant $R>0$ such that

$$\lim_{z\to z^0}(1-|z|^{2p})^R K_{\Omega_{p,q,r}}(z,0,0;z,0,0)$$

converges to a nonzero number. By Lemma 4.1,

$$\begin{aligned} &{}_2F_1\left(\frac{s+1}{p}+\frac{1}{q}+1,1;\frac{s+1}{p};|z|^{2p}\right) \\ &= (1-|z|^{2p})^{-\frac{1}{q}-2}{}_2F_1\left(-\frac{1}{q}-1,\frac{s+1}{p}-1;\frac{s+1}{p};|z|^{2p}\right) \end{aligned}$$

and

$${}_2F_1\left(\frac{s+1}{p}+\frac{1}{q}+2,2;\frac{s+1}{p}+1;|z|^{2p}\right)$$

$$= (1-|z|^{2p})^{-\frac{1}{q}-3} {}_2F_1\left(-\frac{1}{q}-1, \frac{s+1}{p}-1; \frac{s+1}{p}+1; |z|^{2p}\right).$$

By Lemma 4.2, we have

$$\lim_{|z|\to 1-} {}_2F_1\left(-\frac{1}{q}-1, \frac{s+1}{p}-1; \frac{s+1}{p}; |z|^{2p}\right) = \frac{\Gamma(\frac{s+1}{p})\Gamma(2+\frac{1}{q})}{\Gamma(\frac{s+1}{p}+\frac{1}{q}+1)}$$

and

$$\lim_{|z|\to 1-} {}_2F_1\left(-\frac{1}{q}-1, \frac{s+1}{p}-1; \frac{s+1}{p}+1; |z|^{2p}\right) = \frac{\Gamma(\frac{s+1}{p}+1)\Gamma(3+\frac{1}{q})}{\Gamma(\frac{s+1}{p}+\frac{1}{q}+2)}.$$

Combining the above identities, we obtain

$$\begin{aligned}
&\lim_{|z|\to 1-}(1-|z|^{2p})^{\frac{1}{q}+3} K_{\Omega_{p,q,r}}(z,0,0;z,0,0) \\
&= \frac{pq}{\pi^3}\sum_{s=0}^{p-1} rc^*_{s,0} \lim_{|z|\to 1-} {}_2F_1\left(-\frac{1}{q}-1, \frac{s+1}{p}-1; \frac{s+1}{p}+1; |z|^{2p}\right) \\
&= \frac{p^2qr}{\pi^3}\frac{\Gamma(3+\frac{1}{q})}{\Gamma(\frac{1}{q})}.
\end{aligned}$$

Theorem 4.3. *Let $(z^0,0,0)$ be a boundary point of $\Omega_{p,q,r}$. For $(z,0,0) \in \Omega_{p,q,r}$, we have*

$$\lim_{(z,0,0)\to(z^0,0,0)}(1-|z|^{2p})^{\frac{1}{q}+3} K_{\Omega_{p,q,r}}(z,0,0;z,0,0) = \frac{p^2qr}{\pi^3}\frac{\Gamma(3+\frac{1}{q})}{\Gamma(\frac{1}{q})}.$$

5. Higher dimensional cases

Finally, we consider the Bergman kernel for higher-dimensional Hartogs domain defined by

$$\Omega_{\mathbf{p},\mathbf{q},r} := \left\{(z,z',w) \in \mathbb{C}^n \times \mathbb{C}^m \times \Delta : \sum_{j=1}^n |z|^{2p_j} < (1-|w|^2)^r \left(1-\sum_{j=1}^m |z'|^{2q_j}\right)\right\},$$

where $\mathbf{p} = (p_1,\ldots,p_n)$ and $\mathbf{q} = (q_1,\ldots,q_m)$. Define

$$\Omega = \left\{(z,z') \in \mathbb{C}^n \times \mathbb{C}^m : \sum_{j=1}^n |z_j|^{2p_j} + \sum_{j=1}^m |z'_j|^{2q_j} < 1\right\}$$

for $p_j, q_j \in \mathbb{N}$ and $\alpha = (\frac{r}{p_1},\ldots,\frac{r}{p_n})$. Let

$$U^\alpha = \{(z,z',w) \in \mathbb{C}^n \times \mathbb{C}^m \times \mathbb{C} : (f_\alpha(z,w),z') \in \Omega, |w|<1\},$$

where

$$f_\alpha(z,w) = \left(\frac{z_1}{(1-|w|^2)^{\frac{r}{2p_1}}},\ldots,\frac{z_n}{(1-|w|^2)^{\frac{r}{2p_n}}}\right).$$

Note that $(f_\alpha(z,w),z') \in \Omega$ is equivalent to

$$\frac{\sum_{j=1}^n |z_j|^{2p_j}}{(1-|w|^2)^\gamma} + \sum_{j=1}^m |z'_j|^{2q_j} < 1,$$

so that

$$\sum_{j=1}^n |z_j|^{2p_j} < (1-|w|^2)^r \left(1 - \sum_{j=1}^m |z'_j|^{2q_j}\right).$$

Thus U^α becomes $\Omega_{\mathbf{p},\mathbf{q},r}$. At first we need to compute the Bergman kernel for

$$U^\alpha_\eta = \left\{(z,z') \in \mathbb{C}^n \times \mathbb{C}^m : \sum_{j=1}^n \left|\frac{z_j}{(1-|\eta|^2)^{\frac{\alpha_j}{2}}}\right|^{2p_j} + \sum_{j=1}^m |z'_j|^{2q_j} < 1\right\}$$

for fixed $\eta \in \Delta$. Define Ω_1 and Ω_2 by

$$\Omega_1 = \{(z,z') \in \mathbb{C}^n \times \mathbb{C}^m : \sum_{j=1}^n \left|\frac{z_j}{R_j}\right|^{2p_j} + \sum_{j=1}^m |z'_j|^{2q_j} < 1\},$$

$$\Omega_2 = \{(z,z') \in \mathbb{C}^n \times \mathbb{C}^m : \sum_{j=1}^n |z_j|^{2p_j} + \sum_{j=1}^m |z'_j|^{2q_j} < 1\}.$$

Consider $\phi(z,z') = (\frac{z_1}{R_1},\ldots,\frac{z_n}{R_n},z')$. Then ϕ is biholomorphic from Ω_1 onto Ω_2. By Lemma 3.3,

$$\begin{aligned} K_{\Omega_1}(z,z';\zeta,\zeta') &= \det J\phi(z,z') K_{\Omega_2}(\phi(z,z'),\phi(\zeta,\zeta'))\overline{\det J\phi(\zeta,\zeta')} \\ &= \frac{1}{(R_1\cdots R_n)^2} K_{\Omega_2}(\phi(z,z'),\phi(\zeta,\zeta')). \end{aligned}$$

Let $R_j = (1-|\eta|^2)^{\frac{\alpha_j}{2}}$. Note that

$$\begin{aligned} K_{\Omega_2}(z,z';\zeta,\zeta') = \frac{\prod p_j \prod q_j}{\pi^{n+m}} \sum_{\substack{\beta_1,\ldots,\beta_n \\ \gamma_1,\ldots,\gamma_m}} &\frac{\Gamma(\sum \frac{\beta_j+1}{p_j} + \sum \frac{\gamma_j+1}{q_j} + 1)}{\prod \Gamma(\frac{\beta_j+1}{p_j}) \prod \Gamma(\frac{\gamma_j+1}{q_j})} \\ &\times (z_1\overline{\zeta}_1)^{\beta_1}\cdots(z_n\overline{\zeta}_n)^{\beta_n}(z'_1\overline{\zeta}'_1)^{\gamma_1}\cdots(z'_n\overline{\zeta}'_n)^{\gamma_m}. \end{aligned}$$

It follows that

$$\begin{aligned} K_{U^\alpha_\eta}(z,z';\zeta,\zeta') = \frac{\prod p_j \prod q_j}{\pi^{n+m}} \frac{1}{(R_1\cdots R_n)^2} \sum_{\substack{\beta_1,\ldots,\beta_n \\ \gamma_1,\ldots,\gamma_m}} &\frac{\Gamma(\sum \frac{\beta_j+1}{p_j} + \sum \frac{\gamma_j+1}{q_j} + 1)}{\prod \Gamma(\frac{\beta_j+1}{p_j}) \prod \Gamma(\frac{\gamma_j+1}{q_j})} \\ &\times (\frac{z_1\overline{\zeta}_1}{R_1^2})^{\beta_1}\cdots(\frac{z_n\overline{\zeta}_n}{R_n^2})^{\beta_n}(z'_1\overline{\zeta}'_1)^{\gamma_1}\cdots(z'_n\overline{\zeta}'_n)^{\gamma_m}. \end{aligned}$$

Note that

$$h(z,w,\eta) = (z_1 a^{\alpha_1},\ldots,z_n a^{\alpha_n}),$$

where $a = \frac{1-|\eta|^2}{1-w\overline{\eta}}$. Then

$$K_{U_\eta^\alpha}(h(z,w,\eta),z';\zeta,\zeta') = \frac{\prod p_j \prod q_j}{\pi^{n+m}(1-|\eta|^2)^{\sum \alpha_j}} \sum_{\beta,\gamma} c_{\beta,\gamma} A_1^{\beta_1} \cdots A_n^{\beta_n} B_1^{\gamma_1} \cdots B_m^{\gamma_m},$$

where

$$A_j = \frac{z_j \overline{\zeta}_j}{(1-w\overline{\eta})^{\alpha_j}}, \quad B_j = z_j' \overline{\zeta}_j'.$$

It follows that

$$\begin{aligned}&K_{U_\eta^\alpha}(h(z,w,\eta),z';\zeta,\zeta')\\ &= \frac{\prod p_j \prod q_j}{\pi^{n+m}(1-|\eta|^2)^{\sum \alpha_j}} \sum_{s_1=0}^{p_1-1} \cdots \sum_{s_n=0}^{p_n-1} \sum_{u_1=0}^{q_1-1} \cdots \sum_{u_m=0}^{q_m-1} c_{s,u} \prod A_j^{s_j} \prod B_j^{u_j}\\ &\quad F_A^{(n)}\left(a;1,\ldots,1;c_1,\ldots,c_n,c_1',\ldots,c_m';A_1^{p_1},\ldots,A_n^{p_n},B_1^{q_1},\ldots,B_m^{q_m}\right),\end{aligned}$$

where

$$\begin{aligned}a &= \sum_{j=1}^n \frac{s_j+1}{p_j} + \sum_{j=1}^m \frac{u_j+1}{q_j} + 1,\\ c_j &= \frac{s_j+1}{p_j} \ (1 \le j \le n),\\ c_j' &= \frac{u_j+1}{q_j} \ (1 \le j \le m).\end{aligned}$$

Since $\alpha_j = \frac{r}{p_j}$, Proposition 3.1 implies

$$\begin{aligned}&K_{\Omega_{\mathbf{p},\mathbf{q},r}}(z,z',w;\zeta,\zeta',\eta)\\ &= \frac{\prod p_j \prod q_j}{\pi^{n+m+1}(1-w\overline{\eta})^{2+r\sum\frac{1}{p_j}}} \sum_{s_1=0}^{p_1-1} \cdots \sum_{s_n=0}^{p_n-1} \sum_{u_1=0}^{q_1-1} \cdots \sum_{u_m=0}^{q_m-1} \prod_{j=1}^n A_j^{s_j} \prod_{j=1}^m B_j^{u_j} c_{s,u}\\ &\quad \left\{(1+r\sum_{j=1}^n \frac{1}{p_j}(1+s_j))F_A^{(n)} + r\sum_{j=1}^n \frac{\sum_{j=1}^n \frac{s_j+1}{p_j} + \sum_{j=1}^m \frac{u_j+1}{q_j} + 1}{\frac{s_j+1}{p_j}} A_j^{p_j} F_{A,j}^{(n)*}\right\},\end{aligned}$$

where

$$\begin{aligned}F_A^{(n)} &= F_A^{(n)}\left(a;1,\ldots,1;c_1,\ldots,c_n,c_1',\ldots,c_m';A_1^{p_1},\ldots,A_n^{p_n},B_1^{q_1},\ldots,B_m^{q_m}\right),\\ F_{A,j}^{(n)*} &= F_A^{(n)}\left(a+1;\mathbf{1}+\mathbf{e}_j^{n+m};\mathbf{c}+\mathbf{e}_j^n;\mathbf{c}';A_1^{p_1},\ldots,A_n^{p_n},B_1^{q_1},\ldots,B_m^{q_m}\right),\end{aligned}$$

where $\mathbf{1} = (1,1,\ldots,1) \in \mathbb{R}^{n+m}$, $\mathbf{c} = (c_1,c_2,\ldots,c_n) \in \mathbb{R}^n$, $\mathbf{c}' = (c_1',c_2',\ldots,c_m')$ $\in \mathbb{R}^m$. Here $\mathbf{e}_j^{n+m} \in \mathbb{R}^{n+m}$ and $\mathbf{e}_j^n \in \mathbb{R}^n$ has its jth component equal to 1 and all other components equal to 0.

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Jong-Do Park
Department of Mathematics and Research Institute for Basic Sciences
Kyung Hee University
Seoul 02447, Korea
Email address: mathjdpark@khu.ac.kr