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EXACT FORMULA FOR JACOBI-EISENSTEIN SERIES OF SQUARE FREE DISCRIMINANT LATTICE INDEX

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Abstract. In this paper we give an exact formula for the Fourier coefficients of the Jacobi-Eisenstein series of square free discriminant lattice index. For a special case the discriminant of lattice is prime we show that the Jacobi-Eisenstein series corresponds to a well known Eisenstein series of modular forms.

1. Introduction

Jacobi forms of lattice index, whose theory can be viewed as an extension of the theory of classical Jacobi forms, play important roles in many mathematical fields, such as the theory of orthogonal modular forms, the theory of vertex operator algebras. A classical example of Jacobi form is Jacobi-Eisenstein series. In [1], Ajouz defined Jacobi-Eisenstein series of lattice index and studied its basic properties. In [6], Mocanu gave the first formula of Fourier expansion for Jacobi-Eisenstein series of lattice index and, for the trivial case, she showed that the Fourier coefficients in fact are special values of Dirichlet L-functions up to finite Euler factors. In [7], Woitalla considered this for the lattice $\bigoplus_{1\le j\le N} A_1$ where A_1 is the scalar lattice $(\mathbb{Z},(x,y)\to 2xy)$. In this short paper we give an exact formula for Fourier coefficients of Jacobi-Eisenstein series of square free discriminant lattice index. This type lattice has occurred in several problems.

This paper is organized as follows: In Section 2 we review some basic facts for Jacobi forms of lattice index briefly. An exact formula for Jacobi-Eisenstein series of square free discriminant index is given and proved in Section 3. Finally, for the case the discriminant of lattice is prime we refer that the Jacobi-Eisenstein series corresponds to a well known Eisenstein series of elliptic modular forms.

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2. Jacobi forms of lattice index

Throughout this paper, for a complex number t, write $e(t) := e^{2\pi i t}$, $q = e(\tau)$ for τ of the complex upper half plane $\mathbb{H}$. For a prime p, a rational number m, denote $\nu_p(m)$ the p-adic valuation of m. We put $\Gamma = SL_2(\mathbb{Z})$ and, for a positive integer $N > 1$, set the congruence subgroup $\Gamma_0(N) := \left\{\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma : N \mid c\right\}$.

We call a pair $\underline{L} = (L, \beta)$ a positive definite even lattice if L is a $\mathbb{Z}$-module of finite rank $r_{\underline{L}}$, equipped with a positive definite integral quadratic form β. Denote by $L^\sharp$ the dual lattice

$$L^\sharp := \{y \in L \otimes_{\mathbb{Z}} \mathbb{Q} : \beta(y, x) \in \mathbb{Z} \quad \text{for all } x \in L\}.$$

The discriminant form of $\underline{L}$ is defined as

$$D_{\underline{L}} := \left(L^\sharp/L, x + L \to \beta(x) + \mathbb{Z}\right).$$

Note that $\operatorname{card}\left(L^\sharp/L\right) = \det(\underline{L}) := \det(F)$ where F is the Gram matrix corresponding to a $\mathbb{Z}$-basis of L. For $r \in L^\sharp$, let ω_r be its order in $L^\sharp/L$. Define the discriminant of $\underline{L}$ as

$$\Delta_{\underline{L}} := \begin{cases} (-1)^{\lfloor \frac{r_{\underline{L}}}{2} \rfloor} \det(\underline{L}), & r_{\underline{L}} \text{ even}, \\ (-1)^{\lfloor \frac{r_{\underline{L}}}{2} \rfloor} 2\det(\underline{L}), & r_{\underline{L}} \text{ odd}. \end{cases}$$

It is well known that $\Delta_{\underline{L}} \equiv 0, 1 \bmod 4$. For $a \in \mathbb{Z}$, $D \in \mathbb{Q}$ such that $\Delta_{\underline{L}} D \in \mathbb{Z}$, denote $\chi_{\underline{L}}(D, a) := \left(\frac{\Delta_{\underline{L}} D}{a}\right)$ and in particular put $\chi_{\underline{L}}(a) := \chi_{\underline{L}}(1, a)$.

Let $\underline{L} = (L, \beta)$ be a positive definite even lattice and k an integer. The space $J_{k,\underline{L}}(\Gamma)$ of Jacobi forms of weight k and index $\underline{L}$ consists of all holomorphic functions $\phi(\tau, z)$ on $\mathbb{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C})$ which satisfy

(1) For any $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$ and $h = (x, y) \in L \times L$, the following identities hold:

$$\phi(\tau, z) = \left(\phi|_{k,\underline{L}} A\right)(\tau, z) := (c\tau + d)^{-k} e\left(\frac{-c\beta(z)}{c\tau + d}\right) \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right),$$

$$\phi(\tau, z) = \left(\phi|_{k,\underline{L}} h\right)(\tau, z) := e\left(\tau\beta(x) + \beta(x, z)\right) \phi\left(\tau, z + x\tau + y\right);$$

(2) The Fourier expansion of ϕ is of the form

$$\phi(\tau, z) = \sum_{\substack{n \in \mathbb{Z}, r \in L^\sharp \\ \beta(r) - n \le 0}} c_\phi(n, r)\, q^n \zeta_\beta^r.$$

Here and in the following, for $z \in L \otimes_{\mathbb{Z}} \mathbb{C}$ and $r \in L^\sharp$ we write $\zeta_\beta^r := e(\beta(r, z))$.

By [1, Chapter 2, Proposition 2.4.3], the quality of $c_\phi(n, r)$ depends on the value of $\beta(r) - n$ and $r \bmod L$. In the following we write $D := \beta(r) - n$ and $C_\phi(D, r) = c_\phi(n, r)$ as usual.

Define the Jacobi theta series $\vartheta_{\underline{L}}(\tau, z)$ as

$$\vartheta_{\underline{L}}(\tau, z) := \sum_{x \in L} q^{\beta(x)} \zeta_\beta^x.$$

For an even integer $k > \frac{r_L}{2}+2$, the Jacobi-Eisenstein series $E_{k,\underline{L}}(\tau,z)$ is defined as

$$E_{k,\underline{L}}(\tau,z) := \sum_{A\in\Gamma_\infty\backslash\Gamma} \left(\vartheta_{\underline{L}}(\tau,z)\right)\big|_{k,\underline{L}}A,$$

where $\Gamma_\infty = \{\pm\left[\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right] : n\in\mathbb{Z}\}$. We have that $E_{k,\underline{L}}(\tau,z)\in J_{k,\underline{L}}(\Gamma)$.

3. Fourier expansion of Jacobi-Eisenstein series of square free discriminant index

From now on we assume that $\Delta_{\underline{L}}$, the discriminant of $\underline{L}$ is square free. This implies that $r_{\underline{L}}$ is even and $D_{\underline{L}} \approx \bigoplus_{\substack{p|\Delta_{\underline{L}}\\ p \text{ prime}}} \left(\mathbb{Z}/p\mathbb{Z}, \frac{a_p x^2}{p}\right)$. With notations as before the main result of this paper is stated as follows:

Theorem 3.1. *Let $\underline{L}=(L,\beta)$ be a positive definite even lattice of square free discriminant. Then for even integer k with $k>\frac{r_L}{2}+2$, the Jacobi-Eisenstein series $E_{k,\underline{L}}(\tau,z)$ of weight k and index $\underline{L}$ has the following Fourier expansion*

$$E_{k,\underline{L}}(\tau,z) = \vartheta_{\underline{L}}(\tau,z) + \sum_{\substack{n\in\mathbb{Z}, r\in L^\sharp\\ \beta(r)<n}} c(n,r)q^n\zeta_\beta^r,$$

where the qualities of $c(n,r)$ are given by

$$c(n,r) = \frac{2(-1)^{\lceil \frac{r_L}{4}\rceil}}{L(1-k+\frac{r_L}{2},\chi_{\underline{L}})} \sum_{d\left||\Delta_{\underline{L}}D|\right.} d^{k-\frac{r_L}{2}-1}\chi_{\underline{L}}(|\Delta_{\underline{L}}|D/d)$$
$$\times \prod_{\substack{p|\Delta_{\underline{L}}\\ p\nmid\omega_r}} \left(1+p^{\nu_p(|\Delta_{\underline{L}}D|)(1-k+\frac{r_L}{2})}\chi_{\underline{L}}(a_p\Delta_{\underline{L}}D/p^{\nu_p(\Delta_{\underline{L}}^2D)},p)\chi_{\underline{L}}(-1/p,p^{\nu_p(\Delta_{\underline{L}}D)})\right).$$

Proof. The application $\Gamma_\infty A\mapsto \pm(0,1)A$ gives a bijection

$$\Gamma_\infty\backslash\Gamma \to \left\{(c,d)\in\mathbb{Z}^2 : \gcd(c,d)=1\right\}/\{\pm1\}$$

therefore by the definition of Jacobi-Eisenstein series,

$$E_{k,\underline{L}}(\tau,z) = \vartheta_{\underline{L}}(\tau,z) + \sum_{c\geq1}\sum_{\substack{d\in\mathbb{Z}\\ \gcd(c,d)=1}}\sum_{x\in L}\frac{e\left(\frac{a\tau+b}{c\tau+d}\beta(x)+\frac{\beta(x,z)}{c\tau+d}-\frac{c\beta(z)}{c\tau+d}\right)}{(c\tau+d)^k},$$

where a,b are chosen such that $\left[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right]$ is in Γ. Using the identity

$$\frac{a\tau+b}{c\tau+d}\beta(x)+\frac{\beta(x,z)}{c\tau+d}-\frac{c\beta(z)}{c\tau+d} = -\frac{c}{c\tau+d}\beta\left(z-\frac{x}{c}\right)+\frac{a}{c}\beta(x)$$

we obtain

$$\sum_{x\in L}\frac{e\left(\frac{a\tau+b}{c\tau+d}\beta(x)+\frac{\beta(x,z)}{c\tau+d}-\frac{c\beta(z)}{c\tau+d}\right)}{(c\tau+d)^k}$$

$$= \sum_{c\geq 1} \frac{1}{c^k} \sum_{\substack{d \bmod c\\ \gcd(c,d)=1}} \sum_{x\in L/cL} e(a\beta(x)) F\left(\tau + \frac{d}{c}, z - \frac{x}{c}\right),$$

where

$$F(\tau, z) := \sum_{h\in\mathbb{Z}} \sum_{y\in L} \frac{e\left(-\frac{\beta(z+y)}{\tau+h}\right)}{(\tau+h)^k}.$$

Applying Possion summation we obtain

$$F(\tau, z) = \sum_{n\in\mathbb{Z}, r\in L^\sharp} \omega(n,r)\, q^n \zeta_\beta^r$$

with

$$\begin{aligned}\omega(n,r) &= \int_{\mathbb{R}\times(L\otimes_{\mathbb{Z}}\mathbb{R})} \tau^{-k} e\left(-\frac{\beta(z)}{\tau}\right) e\left(-(n\tau + \beta(r,z))\right) d\tau dz\\ &= \int_{\Im\tau = v} \tau^{-k} e(D\tau) d\tau \int_{L\otimes_{\mathbb{Z}}\mathbb{R}} e\left(-\frac{\beta(z+r\tau)}{\tau}\right) dz.\end{aligned}$$

The inner integral equals to $\frac{\tau^{r_{\underline{L}}/2}}{i^{r_{\underline{L}}/2}\sqrt{\det(\underline{L})}}$. By [5, Page 19],

$$\omega(n,r) = \frac{(2\pi)^{k-\frac{r_{\underline{L}}}{2}} i^k |D|^{k-\frac{r_{\underline{L}}}{2}-1}}{\sqrt{\det(\underline{L})}\Gamma\left(k - \frac{r_{\underline{L}}}{2}\right)}$$

if $D < 0$ and 0 otherwise. Summing up we find

$$E_{k,\underline{L}}(\tau, z) = \vartheta_{\underline{L}}(\tau, z) + \sum_{\substack{n\in\mathbb{Z}, r\in L^\sharp\\ \beta(r)<n}} \frac{(2\pi)^{k-\frac{r_{\underline{L}}}{2}} i^k |D|^{k-\frac{r_{\underline{L}}}{2}-1}}{\sqrt{\det(\underline{L})}\Gamma\left(k - \frac{r_{\underline{L}}}{2}\right)} L(\gamma_{n,r}, k) q^n \zeta_\beta^r, \tag{1}$$

where $L(\gamma_{n,r}, k) := \sum_{c\geq 1} \frac{\gamma_{n,r}(c)}{c^k}$ with

$$\gamma_{n,r}(c) := \sum_{\substack{d\in\mathbb{Z}\\ \gcd(c,d)=1}} \sum_{x\in L/cL} e\left(d(\beta(x) + \beta(r,x) + n)\right)$$

(For the expression of $\gamma_{n,r}(c)$ we replaced x by dx). Note that $\gamma_{n,r}(c)$ is multiplicative in c. Thus

$$L(\gamma_{n,r}, k) = \prod_p L_p(\gamma_{n,r}, k) \quad \text{with} \quad L_p(\gamma_{n,r}, k) := \sum_{\nu\geq 0} \frac{\gamma_{n,r}(p^\nu)}{p^{k\nu}}.$$

By [3, Theorem 7], [3, Theorem 11],

$$\prod_{p\nmid \Delta_{\underline{L}}} L_p(\gamma_{n,r}, k) = \frac{1}{L(\chi_{\underline{L}}, k - \frac{r_{\underline{L}}}{2})} \sum_{d \big| |\Delta_{\underline{L}} D|} d^{1-k+\frac{r_{\underline{L}}}{2}} \chi_{\underline{L}}(d).$$

Applying the functional equation for L-function we have

$$\tag{2}
\begin{aligned}
&\frac{(2\pi)^{k-\frac{r_{\underline{L}}}{2}} i^k |D|^{k-\frac{r_{\underline{L}}}{2}-1}}{\sqrt{\det(\underline{L})}\Gamma\left(k-\frac{r_{\underline{L}}}{2}\right)} \prod_{p\nmid\Delta_{\underline{L}}} L_p(\gamma_{n,r},k)\\
&= \frac{2(-1)^{\lceil \frac{r_{\underline{L}}}{4}\rceil}}{L(\chi_{\underline{L}}, 1-k+\frac{r_{\underline{L}}}{2})} \sum_{d\,\big|\,|\Delta_{\underline{L}}D|} d^{k-\frac{r_{\underline{L}}}{2}-1}\chi_{\underline{L}}(|\Delta_{\underline{L}}|D/d).
\end{aligned}
$$

Now we calculate $L_p(\gamma_{n,r},k)$ for $p \mid \Delta_{\underline{L}}$.

If prime p does not divide ω_r, then

$$\tag{3}
\gamma_{n,r}(p^\nu) = \sum_{\substack{d \bmod p^\nu\\ \gcd(d,p^\nu)=1}} e_{p^\nu}(-d\omega_r^2 D) \sum_{x\in L/p^\nu L} e_{p^\nu}(d(\beta(x))).
$$

We have that the quadratic form β is $\mathbb{Z}_p$-isomorphic to the form $x_1^2+\cdots+a'_p x_{r_{\underline{L}}-1}^2 + pa_p x_{r_{\underline{L}}}^2$ where $a'_p \in \mathbb{Z}$ such that $a_p a'_p = 2^{r_{\underline{L}}}\det(\underline{L})/p$ (see [4, Chapter 11, Theorem 2]). Thus we calculate the inner sum

$$
\begin{aligned}
&\sum_{x\in L/p^\nu L} e_{p^\nu}\left(d\beta(x)\right)\\
&= \prod_{j=0}^{r_{\underline{L}}-2}\left(\sum_{x_j \bmod p^\nu} e_{p^\nu}(dx_j^2)\right) \sum_{x_{r_{\underline{L}}-1} \bmod p^\nu} e_{p^\nu}(da'_p x_{r_{\underline{L}}-1}^2) \sum_{x_{r_{\underline{L}}} \bmod p^\nu} e_{p^\nu}(da_p p x_{r_{\underline{L}}}^2)\\
&= p^{\frac{\nu(r_{\underline{L}}-1)}{2}} \left(\frac{d}{p^\nu}\right)^{r_{\underline{L}}-1} \left(\frac{a'_p}{p^\nu}\right) \sqrt{\left(\frac{-1}{p^\nu}\right)}^{r_{\underline{L}}-1} \times p^{\frac{\nu+1}{2}} \left(\frac{a_p d}{p^{\nu-1}}\right) \sqrt{\left(\frac{-1}{p^{\nu-1}}\right)}\\
&= p^{\frac{\nu r_{\underline{L}}+1}{2}} \left(\frac{a'_p a_p}{p^\nu}\right)\left(\frac{a_p d}{p}\right) \sqrt{\left(\frac{-1}{p^\nu}\right)}^{r_{\underline{L}}-1} \sqrt{\left(\frac{-1}{p^{\nu-1}}\right)}\\
&= p^{\frac{\nu r_{\underline{L}}+1}{2}} \left(\frac{\det(\underline{L})/p}{p^\nu}\right)\left(\frac{a_p d}{p}\right) \sqrt{\left(\frac{-1}{p^\nu}\right)}^{r_{\underline{L}}-1} \sqrt{\left(\frac{-1}{p^{\nu-1}}\right)}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\sqrt{\left(\frac{-1}{p^\nu}\right)}^{r_{\underline{L}}-1} \sqrt{\left(\frac{-1}{p^{\nu-1}}\right)}\\
&= \sqrt{\left(\frac{-1}{p}\right)}^{\nu(r_{\underline{L}}-1)} \left(\frac{-1}{p}\right)^{\lfloor\frac{\nu}{2}\rfloor(r_{\underline{L}}-1)} \sqrt{\left(\frac{-1}{p}\right)}^{\nu-1} \left(\frac{-1}{p}\right)^{\lfloor\frac{\nu-1}{2}\rfloor}\\
&= \sqrt{\left(\frac{-1}{p}\right)}^{\nu r_{\underline{L}}} \sqrt{\left(\frac{-1}{p}\right)}^{-1} \left(\frac{-1}{p}\right)^{\lfloor\frac{\nu}{2}\rfloor+\lfloor\frac{\nu-1}{2}\rfloor}
\end{aligned}
$$

$$= \left(\frac{(-1)^{\frac{r_{\underline{L}}}{2}}}{p^\nu}\right)\sqrt{\left(\frac{-1}{p}\right)}\left(\frac{-1}{p}\right)^{\lfloor\frac{\nu}{2}\rfloor+\lfloor\frac{\nu-1}{2}\rfloor+1}$$
$$= \sqrt{\left(\frac{-1}{p}\right)}\left(\frac{(-1)^{\frac{r_{\underline{L}}}{2}+1}}{p^\nu}\right).$$

Therefore

$$\sum_{x\in L/p^\nu L} e_{p^\nu}\left(d\beta(x)\right) = p^{\frac{\nu r_{\underline{L}}+1}{2}}\sqrt{\left(\frac{-1}{p}\right)}\left(\frac{-\Delta_{\underline{L}}/p}{p^\nu}\right)\left(\frac{a_p d}{p}\right).$$

Inserting this into (3) and applying [1, Lemma 2.1.14] we find

$$\gamma_{n,r}(p^\nu) = \begin{cases} p^{\nu(\frac{r_{\underline{L}}}{2}+1)}\left(\frac{-\Delta_{\underline{L}}/p}{p^\nu}\right)\left(\frac{a_p\omega_r^2 D/p^{\nu_p(\omega_r^2 D)}}{p}\right), & p^\nu \| pD,\\ 0, & \text{otherwise.}\end{cases} \tag{4}$$

If $p \mid \omega_r$, then the quadratic polynomial $\beta(x)+\beta(r,x)$ is $\mathbb{Z}_p$-isomorphic to the form

$$x_1^2+\cdots+a_p' x_{r_{\underline{L}}-1}^2 + pa_p x_{r_{\underline{L}}}^2 + 2(r_1x_1+\cdots+a_p' r_{r_{\underline{L}}-1}x_{r_{\underline{L}}-1} + a_p r_{r_{\underline{L}}} x_{r_{\underline{L}}}),$$

where $r_1,\ldots,r_{r_{\underline{L}}}\in\mathbb{Z}$ with $p\nmid r_{r_{\underline{L}}}$. For $\nu\ge 1$ and each u mod p^ν, the equation $px_{r_{\underline{L}}}^2 + 2a_p r_{r_{\underline{L}}} x_{r_{\underline{L}}} \equiv u \bmod p^\nu$ has exact one solution in $\mathbb{Z}/p^\nu\mathbb{Z}$, which implies that

$$\sum_{x_{r_{\underline{L}}} \bmod p^\nu} e_{p^\nu}\left(d(px_{r_{\underline{L}}}^2 + 2a_p r_{r_{\underline{L}}} x_{r_{\underline{L}}})\right) = \sum_{u \bmod p^\nu} e_{p^\nu}(du) = 0.$$

Therefore $\gamma_{n,r}(p^\nu)=0$ for $\nu\ge 1$.

By above discussions, for $p\mid\Delta_{\underline{L}}$, if $p\mid\omega_r$, then $L_p(\gamma_{n,r},k)=1$ and if $p\nmid\omega_r$, then rewrite the first item of (4) in terms of the character associated with lattice we obtain

$$L_p(\gamma_{n,r},k) = 1+p^{\nu_p(|\Delta_{\underline{L}}D|)(1-k+\frac{r_{\underline{L}}}{2})}\chi_{\underline{L}}(a_p\Delta_{\underline{L}}D/p^{\nu_p(\Delta_{\underline{L}}^2 D)},p)\chi_{\underline{L}}(-1/p,p^{\nu_p(\Delta_{\underline{L}}D)}).$$

Inserting (2) and the formula for $L_p(\gamma_{n,r},k)$ with $p\mid\Delta_{\underline{L}}$ into (1) we complete the proof of Theorem 3.1. □

4. Corresponding to Eisenstein series of modular forms

In this section we suppose that the discriminant of $\underline{L}$ is prime p thus $D_{\underline{L}} \approx \left(\mathbb{Z}/p\mathbb{Z}, x\to\frac{a_px^2}{p}\right)$. In [1, Chapter 5] Ajouz constructed a surjective map from $J_{k,\underline{L}}(\Gamma)$ to $M_{k-\frac{r_{\underline{L}}}{2}}(p,\chi_{\underline{L}})$, the space of modular forms of weight $k-\frac{r_{\underline{L}}}{2}$ on $\Gamma_0(p)$ with character $\chi_{\underline{L}}$. Explicitly the application

$$\Omega : J_{k,\underline{L}}(\Gamma)\to M^{\chi_{\underline{L}}(a_p)}_{k-\frac{r_{\underline{L}}}{2}}(p,\chi_{\underline{L}})$$

given by

$$\sum_{\substack{n\in\mathbb{Z}, r\in L^\sharp \\ \beta(r)\le n}} c_\phi(n,r)q^n\zeta_\beta^r \to \sum_{n\ge 0} \sum_{\substack{x\in L^\sharp/L \\ -n/p\equiv\beta(x) \bmod \mathbb{Z}}} C_\phi(-n/p,x)q^n$$

is an isomorphism, where

$$M^{\chi_{\underline{L}}(a_p)}_{k-\frac{r_L}{2}}(p,\chi_{\underline{L}}) = \left\{ f(\tau)\in M_{k-\frac{r_L}{2}}(p,\chi_{\underline{L}}) : f(\tau) = \sum_{\chi_{\underline{L}}(-n)\ne -\chi_{\underline{L}}(a_p)} a_f(n)q^n \right\}.$$

Let the two Eisenstein series of $M_{k-\frac{r_L}{2}}(p,\chi_{\underline{L}})$ be

$$E^{1,\chi_{\underline{L}}}_{k-\frac{r_L}{2}}(\tau) = 1 + \frac{2}{L(1-k+\frac{r_L}{2},\chi_{\underline{L}})} \sum_{n>0}\sum_{d|n} d^{k-\frac{r_L}{2}-1}\chi_{\underline{L}}(d)q^n,$$

$$E^{\chi_{\underline{L}},1}_{k-\frac{r_L}{2}}(\tau) = \sum_{n>0}\sum_{d|n} d^{k-\frac{r_L}{2}-1}\chi_{\underline{L}}(n/d)q^n.$$

By [2, (14)], the unique Eisenstein series of $M^{\chi_{\underline{L}}(a_p)}_{k-\frac{r_L}{2}}(p,\chi_{\underline{L}})$ is

$$E^{\chi_{\underline{L}}(a_p)}_{k-\frac{r_L}{2}}(\tau) := E^{1,\chi_{\underline{L}}}_{k-\frac{r_L}{2}}(\tau) + \frac{2\chi_{\underline{L}}(-a_p)}{L(1-k+\frac{r_L}{2},\chi_{\underline{L}})} E^{\chi_{\underline{L}},1}_{k-\frac{r_L}{2}}(\tau).$$

We now show that $\Omega(E_{k,\underline{L}}(\tau,z)) = E^{\chi_{\underline{L}}(a_p)}_{k-\frac{r_L}{2}}(\tau)$. It is deduced by the table in [1, Page 92] that $(-1)^{\lceil \frac{r_L}{4}\rceil}\chi_{\underline{L}}(-1) = \chi_{\underline{L}}(a_p)$. By this we rewrite the Fourier expansion of $E_{k,\underline{L}}(\tau,z)$ as

$$E_{k,\underline{L}}(\tau,z) = \sum_{x\in L} q^{\beta(x)}\zeta_\beta^x + \sum_{\substack{n\in\mathbb{Z}, r\in L^\sharp \\ \beta(r)<n}} c(n,r)q^n\zeta_\beta^r$$

with

$$c(n,r) = \frac{2}{L(1-k+\frac{r_L}{2},\chi_{\underline{L}})} \times \begin{cases} \chi_{\underline{L}}(-a_p)\sum_{d|-pD} d^{k-\frac{r_L}{2}-1}\chi_{\underline{L}}(-pD/d) + \sum_{d|-pD} d^{k-\frac{r_L}{2}-1}\chi_{\underline{L}}(d), & r\in L, \\ \chi_{\underline{L}}(-a_p)\sum_{d|-pD} d^{k-\frac{r_L}{2}-1}\chi_{\underline{L}}(-pD/d), & r\notin L. \end{cases}$$

The condition $D\in\mathbb{Z}$ is equivalent to $r\in L$, and for $r\notin L$, the equation $\beta(x) = -D$ has exact $1+\chi_{\underline{L}}(-a_pD)$ solutions in $L^\sharp/L$. Therefore

$$\Omega\left(E_{k,\underline{L}}(\tau,z)\right) = 1 + \frac{2}{L(1-k+\frac{r_L}{2},\chi_{\underline{L}})}$$

$$\times \sum_{n>0,p|n} \left(\sum_{d|n} d^{k-\frac{r_{\underline{L}}}{2}-1} \chi_{\underline{L}}(d) + \chi_{\underline{L}}(-a_p) \sum_{d|n} d^{k-\frac{r_{\underline{L}}}{2}-1} \chi_{\underline{L}}(n/d) \right) q^n$$
$$+ \frac{2\chi_{\underline{L}}(-a_p)}{L(1-k+\frac{r_{\underline{L}}}{2}, \chi_{\underline{L}})} \sum_{n>0,p\nmid n} \left(1+\chi_{\underline{L}}(-a_p n)\right) d^{k-\frac{r_{\underline{L}}}{2}-1} \chi_{\underline{L}}(n/d) q^n$$
$$= E^{1,\chi_{\underline{L}}}_{k-\frac{r_{\underline{L}}}{2}}(\tau) + \frac{2\chi_{\underline{L}}(-a_p)}{L(1-k+\frac{r_{\underline{L}}}{2}, \chi_{\underline{L}})} E^{\chi_{\underline{L}},1}_{k-\frac{r_{\underline{L}}}{2}}(\tau)$$
$$= E^{\chi_{\underline{L}}(a_p)}_{k-\frac{r_{\underline{L}}}{2}}(\tau).$$

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