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SMOOTH POINTS OF $\mathcal{L}_s({}^n l_\infty^2)$

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Abstract. For $n \geq 2$, we characterize the smooth points of the unit ball of $\mathcal{L}_s({}^n l_\infty^2)$.

1. Introduction

The main result about smooth points is known as *the Mazur density theorem.* Recall that the Mazur density theorem [5, p. 171] says that the set of all the smooth points of a solid closed convex subset of a separable Banach space is a residual subset of its boundary. We denote by B_E the closed unit ball of a real Banach space E and also by E^* the dual space of E. $x \in B_E$ is called a *smooth point* of B_E if there is a unique $f \in E^*$ so that $f(x) = 1 = \|f\|$. We denote by smB_E the set of smooth points of B_E. For $n \in \mathbb{N}$, we denote by $\mathcal{L}({}^n E)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1, 1\leq k \leq n} |T(x_1, \ldots, x_n)|$. A n-linear form T is symmetric if $T(x_1, \ldots, x_n) = T(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for every permutation σ on $\{1, 2, \ldots, n\}$. We denote by $\mathcal{L}_s({}^n E)$ the Banach space of all continuous symmetric n-linear forms on E. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a unique $T \in \mathcal{L}_s({}^n E)$ such that $P(x) = T(x, \ldots, x)$ for every $x \in E$. In this case it is convenient to write $T = \check{P}$. We denote by $\mathcal{P}({}^n E)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [3].

Choi and Kim ([1, 2]) initiated and characterized the smooth points of the unit balls of $\mathcal{P}({}^2 l_1^2)$ and $\mathcal{P}({}^2 l_2^2)$. Grecu [4] characterized the smooth 2-homogeneous polynomials on Hilbert spaces. Kim ([6, 8]) characterized the smooth points of the unit balls of $\mathcal{L}_s({}^2 l_\infty^2)$ and $\mathcal{L}_s({}^3 l_\infty^2)$. Kim [7] classified the smooth points of the unit ball of $\mathcal{P}({}^2 d_*(1, w)^2)$, where $d_*(1, w)^2 = \mathbb{R}^2$ with the octagonal norm of weight w. Recently, Kim [9] classified the smooth points

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of the unit ball of $\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})$, where $\mathbb{R}^2_{h(\frac{1}{2})} = \mathbb{R}^2$ with the hexagonal norm of weight $\frac{1}{2}$.

Let

$$u_j := [(1,-1),(1,-1),\ldots,(1,-1),(1,1),\ldots,(1,1)] \text{ for } 0 \le j \le n,$$

where -1 appears j-times in u_j. Let

$$\mathcal{U}_n := \{u_j : 0 \le j \le n\}.$$

In this paper, for $n \ge 2$, we characterize the smooth points of the unit ball of $\mathcal{L}_s(^n l^2_\infty)$ as follows: Let $n \ge 2$ and $T \in \mathcal{L}_s(^n l^2_\infty)$ with $\|T\| = 1$. Then, $T \in smB_{\mathcal{L}_s(^n l^2_\infty)}$ if and only if $|Norm(T) \cap \mathcal{U}_n| = 1$, where

$$Norm(T) := \{((a_1,b_1),\ldots,(a_n,b_n)) \in B_{l^2_\infty} \times \cdots \times B_{l^2_\infty} : |T((a_1,b_1),\ldots,(a_n,b_n))| = \|T\|\}.$$

2. Results

Let

$$F_0((x_1,y_1),\ldots,(x_n,y_n)) := x_1 \cdots x_n,\ F_n((x_1,y_1),\ldots,(x_n,y_n)) := y_1 \cdots y_n$$

and for $1 \le j \le n-1$,

$$F_j((x_1,y_1),\ldots,(x_n,y_n)) := \sum_{\{k_1,\ldots,k_j,s_{j+1},\ldots,s_n\}=\{1,\ldots,n\}} y_{k_1}\cdots y_{k_j} x_{s_{j+1}} \cdots x_{s_n}.$$

Note that $\{F_0, F_1, \ldots, F_n\}$ is a basis for $\mathcal{L}_s(^n l^2_\infty)$. If $T \in \mathcal{L}_s(^n l^2_\infty)$, then

$$T((x_1,y_1),\ldots,(x_n,y_n)) = \sum_{0 \le j \le n} a_j F_j((x_1,y_1),\ldots,(x_n,y_n))$$

for some $a_j \in \mathbb{R}$. For simplicity, we write $T = (a_0, a_1, \ldots, a_n)^t$. Note that $F_0(u_j) = 1$ for every $0 \le j \le n$.

Theorem 2.1. *Let $n \ge 2$ and $T \in \mathcal{L}_s(^n l^2_\infty)$. Then*

$$\|T\| = \sup\{|T(u_j)| : 0 \le j \le n\}.$$

Proof. It follows from the Krein-Milman theorem, symmetry and n-linearity of T. □

Theorem 2.2. *Let $n \ge 2$ and $T \in \mathcal{L}_s(^n l^2_\infty)$ with $\|T\| = 1$. Then, $T \in smB_{\mathcal{L}_s(^n l^2_\infty)}$ if and only if $|Norm(T) \cap \mathcal{U}_n| = 1$.*

Proof. ($\Rightarrow$) Otherwise. There exist $u_{k_1}, u_{k_2} \in Norm(T) \cap \mathcal{U}_n$ for some $0 \le k_1 \ne k_2 \le n$. Then,

$$sign(T(u_{k_1}))\delta_{u_{k_1}} \ne sign(T(u_{k_2}))\delta_{u_{k_2}}$$

and

$$sign(T(u_{k_i}))\delta_{u_{k_i}}(T) = |T(u_{k_i})| = 1 = \|sign(T(u_{k_i}))\delta_{u_{k_i}}\| \ (i = 1, 2),$$

where $\delta_{u_{k_i}} \in \mathcal{L}_s({}^n l_\infty^2)^*$ is the evaluation functional by u_{k_i}. Hence,

$$T \notin smB_{\mathcal{L}_s({}^n l_\infty^2)}.$$

This is a contradiction.

($\Leftarrow$) Let $T = \sum_{0\le j\le n} a_j F_j = (a_0, a_1, \ldots, a_n)^t$ for some $a_j \in \mathbb{R}$. Suppose that

$$Norm(T) \cap \mathcal{U}_n = \{u_{k_0}\}$$

for some $0 \le k_0 \le n$. Then,

$$|T(u_{k_0})| = 1 > |T(u_k)| \ (0 \le k \ne k_0 \le n).$$

Let $f \in \mathcal{L}_s({}^n l_\infty^2)^*$ be such that $f(T) = 1 = \|f\|$. For simplicity we denote $f = (f(F_0), f(F_1), \ldots, f(F_n))$. Let

$$\alpha_j := f(F_j) \text{ for } 0 \le j \le n.$$

Claim: $\alpha_l = F_l(u_{k_0})\alpha_0$ for $1 \le l \le n$.

Case 1: $F_l(u_{k_0}) \ne 0$.

Let $m \in \mathbb{N}$ be such that

$$|T(u_k)| + \frac{1}{m}|1 - \frac{F_l(u_k)}{F_l(u_{k_0})}| < 1 \ (0 \le k \ne k_0 \le n).$$

Let

$$T_1 := T + \frac{1}{m}(F_0 - \frac{1}{F_l(u_{k_0})}F_l)$$

and

$$T_2 := T - \frac{1}{m}(F_0 - \frac{1}{F_l(u_{k_0})}F_l).$$

By Theorem 2.1, it follows that, for $i = 1, 2$,

$$\begin{aligned}\|T_i\| &= \max\{|T(u_j) \pm \frac{1}{m}(F_0(u_j) - \frac{F_l(u_j)}{F_l(u_{k_0})})| : 0 \le j \le n\} \\ &= \max\{|T(u_{k_0})|, |T(u_k)| + \frac{1}{m}|1 - \frac{F_l(u_k)}{F_l(u_{k_0})}| : 0 \le k \ne k_0 \le n\} \\ &\quad \text{(because of } F_0(u_{k_0}) = F_0(u_k) = 1) \\ &= 1.\end{aligned}$$

Since

$$1 \ge f(T_i) = f(T) \pm \frac{1}{m} f(F_0 - \frac{1}{F_l(u_{k_0})}F_l) = 1 \pm \frac{1}{m}(\alpha_0 - \frac{1}{F_l(u_{k_0})}\alpha_l) \ (i = 1, 2),$$

$$\alpha_0 = \frac{1}{F_l(u_{k_0})}\alpha_l,$$

so

$$\alpha_l = F_l(u_{k_0})\alpha_0.$$

Case 2: $F_l(u_{k_0}) = 0$.

Let $m \in \mathbb{N}$ be such that

$$|T(u_k)| + \frac{1}{m}|F_l(u_k)| < 1 \ (0 \le k \ne k_0 \le n).$$

Let

$$R_1 := T + \frac{1}{m}F_l, \ R_2 := T - \frac{1}{m}F_l.$$

By Theorem 2.1, it follows that, for $i = 1, 2$,

$$\begin{aligned}\|R_i\| &= \max\{|T(u_j)| + \frac{1}{m}|F_l(u_j)| : 0 \le j \le n\} \\ &= \max\{|T(u_{k_0})|, |T(u_k)| + \frac{1}{m}|F_l(u_k)| : 0 \le k \ne k_0 \le n\} \\ &= 1.\end{aligned}$$

Since

$$1 \ge f(R_i) = f(T) \pm \frac{1}{m}f(F_l) = 1 \pm \frac{1}{m}\alpha_l \ (i = 1, 2),$$

which shows that $\alpha_l = 0 = F_l(u_{k_0})\alpha_0$. We have shown the claim. It follows that

$$\begin{aligned}1 &= f(T) \\ &= \sum_{0\le j\le n} a_j\alpha_j \\ &= a_0\alpha_0 + (\sum_{1\le j\le n} a_jF_j(u_{k_0}))\alpha_0 \\ &= (a_0F_0(u_{k_0}) + \sum_{1\le j\le n} a_jF_j(u_{k_0}))\alpha_0 \ \text{(because of } F_0(u_{k_0}) = 1) \\ &= T(u_{k_0})\alpha_0,\end{aligned}$$

which shows that $\alpha_0 = sign(T(u_{k_0}))$. Therefore,

$$\alpha_l = sign(T(u_{k_0}))F_l(u_{k_0})$$

for $1 \le l \le n$. Since f is unique, $T \in smB_{\mathcal{L}_s({}^n l^2_\infty)}$. □

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