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GLOBAL SOLUTION AND BLOW-UP OF LOGARITHMIC KLEIN-GORDON EQUATION

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Abstract. The initial-boundary value problem for a class of semilinear Klein-Gordon equation with logarithmic nonlinearity in bounded domain is studied. The existence of global solution for this problem is proved by using potential well method, and obtain the exponential decay of global solution through introducing an appropriate Lyapunov function. Meanwhile, the blow-up of solution in the unstable set is also obtained.

1. Introduction

In this paper, we are concerned with the initial-boundary problem for the following logarithmic Klein-Gordon equation

$$u_{tt} - \Delta u + u + u_t = u \log |u|^2, \quad (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.1}$$

with the initial-boundary value conditions

$$u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0, \; (x,t) \in \partial\Omega \times \mathbb{R}^+, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ so that the divergence theorem can be applied.

The problem (1.1)-(1.3) is applied in many branches of physics such as nuclear physics, optics and geophysics [6, 22, 24]. When there is no linear dissipative term u_t in equation (1.1), P. Gorka [18] deals with the following problem

$$u_{tt} - \Delta u + u = \varepsilon u \log |u|^2, \quad (x,t) \in \mathcal{O} \times \mathbb{R}^+, \tag{1.4}$$

$$u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), \quad x \in \mathcal{O}, \tag{1.5}$$

$$u(x,t) = 0, \; (x,t) \in \partial\mathcal{O} \times \mathbb{R}^+, \tag{1.6}$$

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where $\mathcal{O} = [a, b]$, the parameter ε measures the force of the nonlinear interactions. The problem (1.4)-(1.6) is a relativistic version of logarithmic quantum mechanics introduced by I. Bialynicki-Birula and J. Mycielski [4,5]. In [18], the author proved the global existence of weak solution for the problem (1.4)-(1.6) by applying the Galerkin method, logarithmic Sobolev inequality and compactness theorem. In [3], K. Bartkowski and P. Korka showed the existence of classical solution and investigated weak solution for the corresponding Cauchy problem of the equation (1.4) for $\mathcal{O} = R$.

T. Hiramatsu et al. [21] introduced the following equation

$$u_{tt} - \Delta u + u + u_t + |u|^2 u = u \log |u|^2, \quad (x,t) \in \Omega \times \mathbb{R}^+ \tag{1.7}$$

for studying the dynamics of Q-ball in theoretical physics. The logarithmic nonlinearity appears in inflation cosmology and supersymmetric field theories, quantum mechanics and nuclear physics [2, 11, 23]. By using the same method and technique as the one in [18], X. S. Han [20] studied the weak solution of the equation (1.7) with initial-boundary value conditions (1.2) and (1.3) in three dimensional case.

For the initial-boundary value problem of a damped wave equation

$$u_{tt} - \Delta u + \mu u_t = a|u|^{p-2}u, \ x \in \Omega, \ t > 0, \tag{1.8}$$

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega, \tag{1.9}$$

$$u(x,t) = 0, \ x \in \partial\Omega, \ t \geq 0. \tag{1.10}$$

F. Gazzola and M. Squassina [12] study the existence of local and global solution of the problem (1.8)-(1.10), and not only finite time blow-up for solution starting in the unstable set is proved, but also high energy initial data for which the solution blows up are constructed. Later, S. Gerbi and B. Said-Houari [13–15] consider the problem (1.8)-(1.10) or the equation (1.8) with dynamic boundary conditions. They obtained the existence of local and global solution and gave the asymptotic stability and the blow-up result of solution.

Introducing a logarithmic nonlinear term $u \log |u|^2$ makes the problem (1.1)-(1.3) different from the one (1.8)-(1.10). For this reason less results are, at the present time, known for the logarithmic Klein-Gordon equation and many problems remain unsolved.

Let us finally mention that a semilinear heat equation with logarithmic nonlinearity was studied by H. Chen and S. Y. Tian [9] and H. Chen, P. Luo and G. W. Liu [10]. Moreover, there have been some works on the logarithmic Schrödinger equation [7, 8, 16, 17].

Motivated by the above researches, in this paper, we prove the global existence for the problem (1.1)-(1.3) by applying the potential well theory introduced by D. H. Sattinger [28] and L. Payne and D. H. Sattinger [27]. Furthermore, we obtain the exponential decay and the blow-up result of solution for this problem.

We adopt the usual notation and convention. Let $H^1(\Omega)$ denote the Sobolev space with the usual scalar products and norm. $H_0^1(\Omega)$ denotes the closure in $H^1(\Omega)$ of $C_0^\infty(\Omega)$. For simplicity of notations, hereafter we denote by $\|\cdot\|_r$ the Lebesgue space $L^r(\Omega)$ norm and $\|\cdot\|$ denotes $L^2(\Omega)$ norm, we write the equivalent norm $\|\nabla\cdot\|$ instead of $H_0^1(\Omega)$ norm $\|\cdot\|_{H_0^1(\Omega)}$. In addition, C_i $(i = 1, 2, \ldots)$ denote various positive constants which depend on the known constants and may be different at each appearance.

This paper is organized as follows: In the next section, we are going to give some preliminaries. In Section 3, we will study the existence and exponential decay of global solution of the problem (1.1)-(1.3). Section 4 is devoted to the proof of blow-up result of solution in finite time.

2. Preliminaries

At first, we define the following functionals

$$J(u) = \frac{1}{2}(\|\nabla u\|^2 - \int_\Omega u^2 \log |u|^2 dx) + \|u\|^2, \tag{2.1}$$

$$K(u) = \|\nabla u\|^2 - \int_\Omega u^2 \log |u|^2 dx + \|u\|^2, \tag{2.2}$$

for $u(t) \in H_0^1(\Omega)$, and denote the total energy related to the equation (1.1) by

$$E(t) = \frac{1}{2}(\|u_t\|^2 + \|\nabla u\|^2 - \int_\Omega u^2 \log |u|^2 dx) + \|u\|^2 = \frac{1}{2}\|u_t\|^2 + J(u), \tag{2.3}$$

for $u \in H_0^1(\Omega),\ t \geq 0$ and

$$E(0) = \frac{1}{2}(\|u_1\|^2 + \|\nabla u_0\|^2 - \int_\Omega u_0^2 \log |u_0|^2 dx) + \|u_0\|^2 \tag{2.4}$$

is the initial total energy.

As in [27], The mountain pass value of $J(u)$ (also known as potential well depth) is defined as

$$d = \inf\{\sup_{\lambda \geq 0} J(\lambda u) : u \in H_0^1(\Omega)/\{0\}\}. \tag{2.5}$$

Now, we define the so called Nehari manifold (see [26, 29]) as follows

$$\mathcal{N} = \{u \in H_0^1(\Omega)/\{0\};\ K(u) = 0\}.$$

$\mathcal{N}$ separates the two unbounded sets

$$\mathcal{N}^+ = \{u \in H_0^1(\Omega)/\{0\};\ K(u) > 0\} \cup \{0\}$$

and

$$\mathcal{N}^- = \{u \in H_0^1(\Omega)/\{0\};\ K(u) < 0\}.$$

Then, the stable set $\mathcal{W}$ and the unstable set $\mathcal{U}$ can be defined respectively by

$$\mathcal{W} = \{u \in H_0^1(\Omega) : J(u) \leq d\} \cap \mathcal{N}^+$$

and

$$\mathcal{U} = \{u \in H_0^1(\Omega) : J(u) \le d\} \cap \mathcal{N}^-.$$

It is readily seen that the potential well depth d defined in (2.5) may also be characterized as

$$d = \inf_{u \in \mathcal{N}} J(u). \tag{2.6}$$

For the applications through this paper, we introduce the definition of the weak solution for the problem (1.1)-(1.3) and list up some known lemmas.

Definition 2.1. A function u is said to be a weak solution of (1.1)-(1.3) on $[0,T]$ if

$$u \in C([0,T], H_0^1(\Omega)), \quad u_t \in C([0,T], L^2(\Omega)), \quad u_{tt} \in C([0,T], H^{-1}(\Omega)),$$

and u satisfies

$$\int_\Omega u_{tt}\varphi dx + \int_\Omega \nabla u \nabla \varphi dx + \int_\Omega u_t \varphi dx + \int_\Omega u\varphi dx = \int_\Omega u \log |u|^2 \varphi dx$$

for each test function $\varphi \in H_0^1(\Omega)$ and for almost all $t \in [0,T]$.

Lemma 2.1. *Let r be a number with $2 \le r < +\infty$ if $n \le 2$ and $2 \le r \le \frac{2n}{n-2}$ if $n > 2$. Then there is a constant C depending on Ω and r such that*

$$\|u\|_r \le C\|\nabla u\|, \ \forall u \in H_0^1(\Omega).$$

Lemma 2.2 ([9, 10, 19])**.** *If $u \in H_0^1(\Omega)$, then for each $a > 0$, one has the inequality*

$$\int_\Omega |u|^2 \log |u| dx \le \|u\|^2 \log \|u\| + \frac{a^2}{2\pi}\|\nabla u\|^2 - \frac{n}{2}(1 + \log a)\|u\|^2.$$

Lemma 2.3. *Let $u(t)$ be a solution of the problem* (1.1)-(1.3)*. Then $E(t)$ is a non-increasing function for $t > 0$ and*

$$E'(t) = -\|u_t\|^2 \le 0. \tag{2.7}$$

We conclude this section by stating a local existence result of the problem (1.1)-(1.3), which can be established by a similar way as done in combination of the arguments in [1, 12, 25].

Theorem 2.1 (Local existence)**.** *Suppose that $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. Then there exists $T > 0$ such that the problem* (1.1)-(1.3) *has a unique local solution $u(t)$ which satisfies*

$$u \in C([0,T);\ H_0^1(\Omega)), \quad u_t \in C([0,T);\ L^2(\Omega)).$$

Moreover, at least one of the following statements holds true: (1) $\|u_t\|^2 + \|\nabla u\|^2 + \|u\|^2 \to \infty$ *as* $t \to T^-$; (2) $T = +\infty$.

3. Global existence and exponential decay

We can now proceed in the study of the existence of global solution for the problem (1.1)-(1.3). For this purpose, we need the following lemmas.

Lemma 3.1. *If $u \in H_0^1(\Omega)$ and $\|u\| \neq 0$, then we have*

$$K(\lambda u) = \lambda \frac{d}{d\lambda} J(\lambda u) \begin{cases} > 0, & 0 < \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda^* < \lambda < +\infty, \end{cases}$$

where

$$\lambda^* = \exp\left(\frac{\|\nabla u\|^2 + \|u\|^2 - 2\int_\Omega u^2 \log |u| dx}{2\|u\|^2} \right).$$

Proof. Since

$$J(\lambda u) = \frac{\lambda^2}{2}\|\nabla u\|^2 + \lambda^2 \|u\|^2 - (\lambda^2 \log \lambda)\|u\|^2 - \lambda^2 \int_\Omega u^2 \log |u| dx,$$

we get

$$\frac{d}{d\lambda} J(\lambda u) = \lambda(\|\nabla u\|^2 + \|u\|^2 - 2\int_\Omega u^2 \log |u| dx) - 2(\lambda \log \lambda)\|u\|^2. \tag{3.1}$$

Let $\frac{d}{d\lambda} J(\lambda u) = 0$, then we have

$$\lambda^* = \exp\left(\frac{\|\nabla u\|^2 + \|u\|^2 - 2\int_\Omega u^2 \log |u| dx}{2\|u\|^2} \right).$$

It follows from (2.2) that

$$K(\lambda u) = \lambda^2(\|\nabla u\|^2 + \|u\|^2 - 2\int_\Omega u^2 \log |u| dx) - 2(\lambda^2 \log \lambda)\|u\|^2. \tag{3.2}$$

By (3.1) and (3.2), the conclusion in Lemma 3.1 is valid. $\square$

Lemma 3.2. *Suppose that $u \in H_0^1(\Omega)$. Then*

$$d = \frac{1}{2}\pi^{\frac{n}{2}} e^n. \tag{3.3}$$

Proof. We have from the logarithmic Sobolev inequality in Lemma 2.2 that

$$\begin{aligned} K(u) &= \|\nabla u\|^2 - \int_\Omega u^2 \log |u|^2 dx + \|u\|^2 \\ &\geq \left(1 - \frac{a^2}{\pi}\right)(\|\nabla u\|^2 + \|u\|^2) + [n(1 + \log a) - 2\log \|u\|] \cdot \|u\|^2 \end{aligned} \tag{3.4}$$

for any $a > 0$. Taking $a = \sqrt{\pi}$, we obtain from (3.4) that

$$K(u) \geq [n(1 + \log a) - 2\log \|u\|] \cdot \|u\|^2. \tag{3.5}$$

We have from Lemma 3.1 that

$$\sup_{\lambda\geq 0} J(\lambda u) = J(\lambda^* u) = \frac{1}{2}K(\lambda^* u) + \frac{1}{2}\|\lambda^* u\|^2. \tag{3.6}$$

We obtain from (3.5) and Lemma 3.1 that

$$0 = K(\lambda^* u) \geq [n(1+\log a) - 2\log\|\lambda^* u\|]\cdot\|\lambda^* u\|^2,$$

then

$$\|\lambda^* u\|^2 \geq a^n e^n. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\sup_{\lambda\geq 0} J(\lambda u) \geq \frac{1}{2}a^n e^n. \tag{3.8}$$

By (2.5) and (3.8), we have that $d = \frac{1}{2}\pi^{\frac{n}{2}} e^n$. □

Lemma 3.3. *Assume that $E(0) < d$. If $u_0 \in \mathcal{N}^+$ and $u_1 \in L^2(\Omega)$, then $u(t) \in \mathcal{N}^+$ for each $t \in [0, T)$.*

Proof. From (2.3) and Lemma 2.3, we have

$$E(t) = \frac{1}{2}\|u_t\|^2 + J(u) \leq \frac{1}{2}\|u_1\|^2 + J(u_0) = E(0) < d,$$

for $\forall t \in [0, T)$, which implies that

$$J(u) < d. \tag{3.9}$$

Suppose that there exists a number $t^* \in [0, T)$ such that $u(t) \in \mathcal{N}^+$ on $[0, t^*)$ and $u(t^*) \notin \mathcal{N}^+$. Then, in virtue of the continuity of $u(t)$, we see $u(t^*) \in \partial\mathcal{N}^+$. From the definition of $\mathcal{N}^+$ and the continuity of $K(u)$ with respect to t, we have

$$K(u(t^*)) = 0. \tag{3.10}$$

Assume that (3.10) holds, then we get from (3.5) that

$$\|u(t^*)\|^2 \geq 2d. \tag{3.11}$$

By (2.1), (2.2), (3.10) and (3.11), we have

$$J(u(t^*)) = \frac{1}{2}\|u(t^*)\|^2 + \frac{1}{2}K(u(t^*)) \geq d, \tag{3.12}$$

which is contradictive with (3.9). Hence, the case (3.10) is impossible. Thus, we conclude that $u(t) \in \mathcal{N}^+$ on $[0, T)$. □

Theorem 3.1. *If $u_0 \in \mathcal{W}$, $u_1 \in L^2(\Omega)$ and $E(0) < d$, then the local solution furnished in Theorem 2.1 is a global solution and T may be taken arbitrarily large.*

Proof. It suffices to show that $\|u_t\|^2 + \|\nabla u\|^2 + \|u\|^2$ is bounded independently of t. Under the hypotheses in Theorem 3.1, we get from Lemma 3.3 that $u \in \mathcal{W}$ on $[0, T)$. So, the following formula holds on $[0, T)$ by Lemma 2.2.

$$\begin{aligned} J(u) &= \frac{1}{2}\|\nabla u\|^2 - \frac{1}{2}\int_\Omega u^2 \log|u|^2 dx + \|u\|^2 \\ &\geq \frac{1}{2}\left(1 - \frac{a^2}{\pi}\right)\|\nabla u\|^2 + \left[1 + \frac{n}{2}(1 + \log a) - \log\|u\|\right]\|u\|^2. \end{aligned} \tag{3.13}$$

By (2.1), (2.2) and $u \in \mathcal{W}$, we obtain that

$$J(u) = \frac{1}{2}\|u(t)\|^2 + \frac{1}{2}K(u) \geq \frac{1}{2}\|u\|^2, \tag{3.14}$$

which implies that

$$\|u\|^2 \leq 2J(u) \leq 2d. \tag{3.15}$$

It follows from (3.13) and (3.15) that

$$J(u) \geq \frac{1}{2}\left(1 - \frac{a^2}{\pi}\right)\|\nabla u\|^2 + \left[1 + \frac{n}{2}(1 + \log a) - \frac{1}{2}\log(2d)\right]\|u\|^2. \tag{3.16}$$

By Lemma 3.2 and $0 < a \leq \sqrt{\pi}$, we get that

$$1 - \frac{a^2}{\pi} \geq 0, \quad 1 + \frac{n}{2}(1 + \log a) - \frac{1}{2}\log(2d) > 0.$$

Thus, we have from (3.16) that

$$J(u) \geq C_1\left(\|\nabla u\|^2 + \|u\|^2\right), \tag{3.17}$$

where

$$C_1 = \min\left\{\frac{1}{2} - \frac{a^2}{2\pi}, \quad 1 + \frac{n}{2}(1 + \log a) - \frac{1}{2}\log(2d)\right\}.$$

We have from (3.17) that

$$\frac{1}{2}\|u_t\|^2 + C_1(\|\nabla u\|^2 + \|u\|^2) \leq \frac{1}{2}\|u_t\|^2 + J(u) = E(t) \leq E(0) < d, \tag{3.18}$$

which implies that

$$\|u_t\|^2 + \|\nabla u\|^2 + \|u\|^2 \leq \frac{d}{C_2} < +\infty,$$

where $C_2 = \min\{C_1,\ 1\}$. The above inequality and the continuation principle lead to the global existence of the solution u for the problem (1.1)-(1.3). $\square$

Remark. Theorem 3.1 is still valid for the equation (1.1) without the dissipative term u_t.

The following theorem shows that the global solution of the problem (1.1)-(1.3) is exponential decay.

Theorem 3.2. *Assume that* $E(0) < \frac{1}{2}\pi^{\frac{n}{2}}e^n\alpha \leq d$, *where* α *is a positive number which satisfies* $0 < \alpha \leq 1$. *If* $u_0 \in \mathcal{W}$, $u_1 \in L^2(\Omega)$, *then there exist two positive constants* M *and* k *independent of* t *such that the global solution has the following exponential decay property*

$$0 < E(t) \leq Me^{-kt}, \forall t \geq 0.$$

Proof. By Lemma 3.3, we see that $u(t) \in \mathcal{N}^+$ for all $t \geq 0$. Thus, we have $0 < E(t) < d$ for all $t \geq 0$. In order to prove the exponential decay of global solution. We define

$$F(t) = E(t) + \varepsilon \int_\Omega uu_t dx, \tag{3.19}$$

where $\varepsilon > 0$ will be determined later.

It is easy to prove that there exist two positive constants ξ_1 and ξ_2 depending on ε such that

$$\xi_1 E(t) \leq F(t) \leq \xi_2 E(t), \tag{3.20}$$

for $\forall t \geq 0$. In fact, we get from (3.18) and (3.19) that

$$F(t) \leq E(t) + \frac{\varepsilon}{2}(\|u_t\|^2 + \|u\|^2) \leq (1 + \varepsilon + \frac{\varepsilon}{2C_1})E(t) = \xi_2 E(t). \tag{3.21}$$

On the other hand, by (3.18) and (3.19), we obtain the following inequality

$$\begin{aligned} F(t) &\geq E(t) - \frac{\varepsilon}{2}\|u_t\|^2 - \frac{\varepsilon}{2}\|u\|^2 \\ &\geq \frac{1}{2}(1-\varepsilon)\|u_t\|^2 + J(u) - \frac{\varepsilon}{2C_1}E(t). \end{aligned} \tag{3.22}$$

By choosing ε small enough such that $0 < \varepsilon \leq \min\{1, \frac{2C_1}{2C_1+1}\}$, it follows from (3.22) that

$$F(t) \geq (1 - \varepsilon - \frac{\varepsilon}{2C_1})E(t) = \xi_1 E(t). \tag{3.23}$$

From (3.21) and (3.23), the inequality (3.20) is valid.

We now differentiate (3.19), by using the equation (1.1) and Lemma 2.3, to obtain

$$F'(t) = (\varepsilon - 1)\|u_t\|^2 - \varepsilon\|\nabla u\|^2 - \varepsilon\|u\|^2 - \varepsilon \int_\Omega uu_t dx + \varepsilon \int_\Omega u^2 \log|u|^2 dx. \tag{3.24}$$

For any $\gamma > 0$, We have from Young's inequality that

$$\left| \int_\Omega uu_t dx \right| \leq \frac{1}{4\gamma}\|u_t\|^2 + \gamma\|u\|^2. \tag{3.25}$$

Therefore, inserting (3.25) into (3.24), we get

$$F'(t) \leq (\varepsilon + \frac{\varepsilon}{4\gamma} - 1)\|u_t\|^2 - \varepsilon\|\nabla u\|^2 + \varepsilon(\gamma - 1)\|u\|^2 + \varepsilon \int_\Omega u^2 \log|u|^2 dx. \tag{3.26}$$

By using (2.3) and (3.26), for any positive constant Λ, we obtain that

$$\begin{aligned} F'(t) \leq & -\Lambda\varepsilon E(t) + \left[\varepsilon(1+\frac{\Lambda}{2}+\frac{1}{4\gamma})-1\right]\|u_t\|^2 \\ & + \varepsilon\left(\frac{\Lambda}{2}-1\right)\|\nabla u\|^2 + \varepsilon(\Lambda+\gamma-1)\|u\|^2 \\ & + \varepsilon\left(1-\frac{\Lambda}{2}\right)\int_\Omega u^2\log|u|^2 dx. \end{aligned} \tag{3.27}$$

Now, choosing $0<\Lambda\leq 1$, and by Lemma 2.2 and (3.15), we have

$$\begin{aligned} F'(t) \leq & -\Lambda\varepsilon E(t) + \left[\varepsilon(1+\frac{\Lambda}{2}+\frac{1}{4\gamma})-1\right]\|u_t\|^2 \\ & - \varepsilon\left(1-\frac{\Lambda}{2}\right)\left(1-\frac{a^2}{\pi}\right)\|\nabla u\|^2 \\ & + \varepsilon\left\{\Lambda+\gamma-1+\left(1-\frac{\Lambda}{2}\right)[\log(2J(t))-n(1+\log a)]\right\}\|u\|^2. \end{aligned} \tag{3.28}$$

By $0<\Lambda\leq 1$ and $J(t)<E(0)<\frac{1}{2}\pi^{\frac{n}{2}}e^n\alpha\leq d$, we select the constant a to meet $\sqrt{\pi}\alpha^{\frac{1}{n}}\leq a\leq\sqrt{\pi}$, and take $\gamma>0$ small sufficiently such that

$$\begin{aligned} \gamma & < 1-\Lambda+\left(\frac{\Lambda}{2}-1\right)[\log(2J(t))-n(1+\log a)] \\ & < 1-\Lambda+\left(\frac{\Lambda}{2}-1\right)[\log(\pi^{\frac{n}{2}}e^n\alpha)-n(1+\log a)] \\ & = 1-\Lambda+\left(\frac{\Lambda}{2}-1\right)\log\frac{\pi^{\frac{n}{2}}\alpha}{a^n}. \end{aligned}$$

Then, we obtain

$$F'(t)\leq -\Lambda\varepsilon E(t)+\left[\varepsilon(1+\frac{\Lambda}{2}+\frac{1}{4\gamma})-1\right]\|u_t\|^2. \tag{3.29}$$

Now, choosing ε so small enough that

$$\varepsilon(1+\frac{\Lambda}{2}+\frac{1}{4\gamma})-1<0,$$

then the inequality (3.29) implies that

$$F'(t)\leq -\Lambda\varepsilon E(t),\ \forall t\geq 0. \tag{3.30}$$

We conclude from (3.20) and (3.30) that

$$F'(t)\leq -kF(t),\ \forall t\geq 0, \tag{3.31}$$

where $k=\Lambda\varepsilon/\xi_2>0$.

Integrating the differential inequality (3.31) from 0 to t gives the following exponential decay estimate for function $F(t)$

$$F(t)\leq F(0)e^{-kt},\ \forall t\geq 0. \tag{3.32}$$

Consequently, we obtain from (3.20) once again that

$$E(t) \leq Me^{-kt},\ \forall t \geq 0,$$

where $M = F(0)/\xi_1$. This completes the proof of Theorem 3.2. $\square$

4. Blow-up result

In this section, we are concerned with the blow-up property of solution for the initial boundary value problem (1.1)-(1.3) and give the estimate of the lifespan of solution. For this purpose, we give the following lemma.

Lemma 4.1. *Let $u(t)$ be a solution of (1.1)-(1.3) which is given by Theorem 2.1. If $u_0 \in \mathcal{U}$ and $E(0) < d$, then $u(t) \in \mathcal{U}$ and $E(t) < d$, for all $t \geq 0$.*

Proof. It follows from the conditions in Lemma 4.1 and Lemma 2.3 that

$$E(t) \leq E(0) < d,\ \forall t \in [0, T).$$

Therefore, we have from (2.3) that

$$J(u) \leq E(t) < d,\ \forall t \in [0, T). \tag{4.1}$$

Next, let us assume by contradiction that there exists $t^* \in [0, T)$ such that $u(t^*) \notin \mathcal{U}$, then by continuity, we have $K(u(t^*)) = 0$. This implies that $u(t^*) \in \mathcal{N}$. We get from (2.6) that $J(u(t^*)) \geq d$, which is contradiction with (4.1). Consequently, the conclusion in Lemma 4.1 holds. $\square$

Theorem 4.1. *Assume that $u_0 \in \mathcal{U},\ u_1 \in L^2(\Omega)$ satisfies $\int_\Omega u_0(x)u_1(x)dx \neq 0$ and $0 < E(0) < \min\{d,\ \frac{3}{4}\pi^{\frac{n}{2}}e^n\}$. Then the solution $u(t)$ in Theorem 2.1 of the problem (1.1)-(1.3) blows up in finite $T_* < +\infty$, this means that*

$$\lim_{t \to T_*^-} \|u(t)\|^2 = +\infty.$$

Proof. By $u_0 \in \mathcal{U}$, $E(0) < d$ and Lemma 4.1, we obtain $u \in \mathcal{U}$ for all $t \in [0, T]$. Thus, we have

$$K(u) = \|\nabla u\|^2 + \|u\|^2 - \int_\Omega u^2 \log |u|^2 dx < 0 \tag{4.2}$$

for all $t \in [0, T]$. We have from (4.2) and Lemma 2.2 that

$$(1 - \frac{a^2}{\pi})\|\nabla u\|^2 + \|u\|^2 + [n(1 + \log a) - \log \|u\|^2] \cdot \|u\|^2 < 0. \tag{4.3}$$

We conclude from $a = \sqrt{\pi}$ and (4.3) that

$$n(1 + \log a) - \log \|u\|^2 < 0,$$

which implies that

$$\|u(t)\|^2 > 2d,\ \forall t \in [0, T]. \tag{4.4}$$

Assume by contradiction that the solution $u(t)$ is global. Then for any $T > 0$, we define $\Theta(t) : [0, T] \to [0, +\infty)$ by

$$\Theta(t) = \|u(t)\|^2 + \int_0^t \|u(s)\|^2 ds + (T - t)\|u_0\|^2. \tag{4.5}$$

Noting that $\Theta(t) > 0$ for all $t \in [0, T]$. By the continuity of the function $\Theta(t)$, there exists $\mu > 0$ (independent of the choice of T) such that

$$\Theta(t) \geq \mu > 0, \ \forall t \in [0, T]. \tag{4.6}$$

By differentiating on both sides of (4.5), we get

$$\begin{aligned} \Theta'(t) &= 2\int_\Omega uu_t dx + \|u(t)\|^2 - \|u_0\|^2 \\ &= 2\int_\Omega uu_t dx + 2\int_0^t \int_\Omega u(s)u_t(s) dx ds. \end{aligned} \tag{4.7}$$

Taking the derivative of the function $\Theta'(t)$ in (4.7), we obtain

$$\Theta''(t) = 2\|u_t\|^2 + 2\int_\Omega uu_{tt} dx + 2\int_\Omega uu_t dx. \tag{4.8}$$

We get from (1.1) and (4.8) that

$$\Theta''(t) = 2(\|u_t(t)\|^2 + \int_\Omega u(t)^2 \log |u(t)|^2 dx - \|\nabla u(t)\|^2 - \|u(t)\|^2). \tag{4.9}$$

We have from (4.5), (4.7) and (4.9) that

$$\begin{aligned} &\Theta(t)\Theta''(t) - \frac{3}{2}\Theta'(t)^2 \\ &= 2\Theta(t)[\|u_t(t)\|^2 + \int_\Omega u(t)^2 \log |u(t)|^2 dx - \|\nabla u(t)\|^2 - \|u(t)\|^2] \\ &\quad - 6[\Theta(t) - (T - t)\|u_0\|^2] \times \left[\|u_t(t)\|^2 + \int_0^t \|u_t(s)\|^2 ds\right] + 6\chi(t), \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} \chi(t) = &\left[\|u(t)\|^2 + \int_0^t \|u(s)\|^2 ds\right] \times \left[\|u_t(t)\|^2 + \int_0^t \|u_t(s)\|^2 ds\right] \\ &- \left[\int_\Omega uu_t dx + \int_0^t \int_\Omega u(s)u_t(s) dx ds\right]^2. \end{aligned} \tag{4.11}$$

By using Schwarz inequality, we obtain

$$\left(\int_\Omega uu_t dx\right)^2 \leq \|u(t)\|^2 \|u_t(t)\|^2, \tag{4.12}$$

$$\left(\int_0^t \int_\Omega uu_t dx ds\right)^2 \leq \int_0^t \|u(s)\|^2 ds \int_0^t \|u_t(s)\|^2 ds, \tag{4.13}$$

and

$$2\int_0^t\int_\Omega u(s)u_t(s)dxds\int_\Omega uu_tdx$$
$$\leq \|u_t(t)\|^2\int_0^t\|u(s)\|^2ds+\|u(t)\|^2\int_0^t\|u_t(s)\|^2ds. \tag{4.14}$$

These inequalities (4.11)-(4.14) entail $\chi(t)\geq 0$ for all $t\in[0,T]$. Therefore, we reach the following differential inequality from (4.10) that

$$\Theta\Theta''(t)-\frac{3}{2}\Theta'(t)^2\geq\Theta(t)\Gamma(t),\forall t\in[0,T], \tag{4.15}$$

where

$$\Gamma(t)=2[\|u_t(t)\|^2+\int_\Omega u(t)^2\log|u(t)|^2dx-\|\nabla u(t)\|^2-\|u(t)\|^2]$$
$$-6\left[\|u_t(t)\|^2+\int_0^t\|u_t(s)\|^2ds\right]. \tag{4.16}$$

We have from (2.3) and Lemma 2.2 that

$$\Gamma(t)\geq\ -8E(t)+6\|u(t)\|^2+2\left(1-\frac{a^2}{\pi}\right)\|\nabla u(t)\|^2$$
$$+[\log\|u(t)\|^2-n(1+\log a)]\|u(t)\|^2-6\int_0^t\|u_t(s)\|^2ds. \tag{4.17}$$

By (3.3), (4.4) and $a=\sqrt{\pi}$, we have from (4.17) that

$$\Gamma(t)\geq -8E(t)+6\|u(t)\|^2-6\int_0^t\|u_t(s)\|^2ds. \tag{4.18}$$

By (2.7), we get

$$\Gamma(t)\geq -8E(0)+6\|u(t)\|^2+2\int_0^t\|u_t(s)\|^2ds. \tag{4.19}$$

Hence, we conclude from (4.4) and $E(0)<d$ that

$$\Gamma(t)>-8E(0)+12d=8[d-E(0)]+4d>0. \tag{4.20}$$

Therefore, there exists $\eta>0$ which is independent of T such that

$$\Gamma(t)\geq\eta>0,\ \forall t\geq 0. \tag{4.21}$$

It follows from (4.6), (4.15) and (4.21) that

$$\Theta\Theta''(t)-\frac{3}{2}\Theta'(t)^2\geq\mu\eta>0,\ \forall t\in[0,T]. \tag{4.22}$$

By the differential inequality (4.22), we have

$$\Theta(t)\geq\frac{\Theta(0)}{\left(1-\frac{\Theta'(0)}{2\Theta(0)}t\right)^2}. \tag{4.23}$$

Hence, there exists T_* such that

$$0 < T_* < \frac{2\Theta(0)}{\Theta'(0)} \leq T, \tag{4.24}$$

and we have

$$\lim_{t \to T_*^-} \Theta(t) = +\infty. \tag{4.25}$$

From the definition (4.5) of $\Theta(t)$, (4.25) means that

$$\lim_{t \to T_*^-} \|u(t)\|^2 = +\infty.$$

Thus we can not suppose that the solution of (1.1)-(1.3) is global.

This finishes the proof of Theorem 4.1. □

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