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A NOTE ON COHOMOLOGICAL DIMENSION OVER COHEN-MACAULAY RINGS

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Abstract. Let $(R, \mathfrak{m})$ be a Noetherian local Cohen-Macaulay ring and I be a proper ideal of R. Assume that $\beta_R(I, R)$ denotes the constant value of $\operatorname{depth}_R(R/I^n)$ for $n \gg 0$. In this paper we introduce the new notion $\gamma_R(I, R)$ and then we prove the following inequalities:

$$\beta_R(I, R) \leq \gamma_R(I, R) \leq \dim R - \operatorname{cd}(I, R) \leq \dim R/I.$$

Also, some applications of these inequalities will be included.

1. Introduction

Let R denote a commutative Noetherian ring (with identity), I, J be two ideals of R, and M be a finitely generated R-module. Ratliff in [15] conjectured that the set $\operatorname{Ass}_R(M/J^nM)$ stabilizes for $n \gg 0$, when R is a Noetherian domain. Subsequently, Brodmann in [4] showed that if R is a Noetherian ring, then the sets $\operatorname{Ass}_R(M/J^nM)$ and $\operatorname{Ass}_R(J^nM/J^{n+1}M)$ are ultimately constant for large n. Also, based on this result, in [5] he showed that if R is a Noetherian ring, then the integers $\operatorname{grade}(I, M/J^nM)$ and $\operatorname{grade}(I, J^nM/J^{n+1}M)$ take constant values for large n. In particular, the integers $\operatorname{depth}_R(M/J^nM)$ and $\operatorname{depth}_R(J^nM/J^{n+1}M)$ take constant values for large n, when $(R, \mathfrak{m})$ is a Noetherian local ring. In the sequel let $\beta_R(I, M)$ denote the constant value of $\operatorname{depth}_R(M/I^nM)$ for $n \gg 0$, when R is local. In [1] it has been proved that if M is a non-zero finitely generated R-module and I is an ideal of R with $\beta_R(I, M) = \dim M$, then M annihilates by some power of I and so M is a Cohen-Macaulay R-module.

For an R-module M, the ith local cohomology module of M with support in $V(I)$ is defined as:

$$H^i_I(M) = \varinjlim_{n \geq 1} \operatorname{Ext}^i_R(R/I^n, M).$$

We refer the reader to [6] or [10] for more details about local cohomology.

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For an R-module M, the notion $\operatorname{cd}(I, M)$, *the cohomological dimension of M with respect to I*, is defined as:

$$\operatorname{cd}(I, M) = \sup\{i \in \mathbb{N}_0 : H_I^i(M) \neq 0\}$$

with the usual convention that the supremum of the empty set of integers is interpreted as $-\infty$. This notion have been studied by several authors (see [3, 7–9, 11, 12]).

In this paper first we shall introduce the new notion $\gamma_R(I, R)$ and then we will establish some new inequalities between $\operatorname{cd}(I, R)$, $\beta_R(I, R)$ and $\gamma_R(I, R)$. Also, some applications of these inequalities will be included.

Throughout this paper, $(R, \mathfrak{m})$ will always be a commutative Noetherian local ring with non-zero identity. For each R-module M, we denote by $\mathrm{E}_R(M)$ the injective envelope (or injective hull) of M. Also, for any ideal $\mathfrak{b}$ of R, *the radical of* $\mathfrak{b}$, denoted by $\operatorname{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b}$ for some $n \in \mathbb{N}\}$. For any other unexplained notation and terminology we refer the reader to [6] and [13].

2. The results

The main purpose of this section is to prove Theorem 2.2. But, first we need to introduce the new notion $\gamma_R(I, M)$.

Definition. Let $(R, \mathfrak{m})$ be a local ring and I be an ideal of R. For every finitely generated R-module M, set $G(I, M) := \bigoplus_{n=1}^{\infty} M/I^n M$. Then we define $\gamma_R(I, M)$ as:

$$\gamma_R(I, M) := \inf\{i \in \mathbb{N}_0 : I \nsubseteq \operatorname{Rad}(\operatorname{Ann}_R H_{\mathfrak{m}}^i(G(I, M)))\},$$

with the convention that the infimum of the empty set of integers is interpreted as ∞.

From the definition it follows that there exists a positive integer k such that

$$I^k \subseteq \operatorname{Ann}_R \bigoplus_{j=0}^{\gamma_R(I,M)-1} H_{\mathfrak{m}}^j(G(I, M)).$$

Furthermore, since the local cohomology functor $H_{\mathfrak{m}}^i(-)$ commutes with the direct sums it follows that $I^k H_{\mathfrak{m}}^j(M/I^n M) = 0$ for each $0 \leq j \leq \gamma_R(I, M) - 1$ and each $n \geq 1$.

Now we consider the following question.

Question 1: Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring and I be an ideal of R. Whether $\gamma_R(I, R) = \dim R - \operatorname{cd}(I, R)$?

In this paper we shall prove that the inequality $\gamma_R(I, R) \leq \dim R - \operatorname{cd}(I, R)$ always holds. So, Question 1 reduces to the following question:

Question 2: Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring and I be an ideal of R. Whether $\gamma_R(I, R) \geq \dim R - \operatorname{cd}(I, R)$?

The following easy lemma is needed in the proof of Theorem 2.2.

Lemma 2.1. *Let $(R, \mathfrak{m})$ be a local ring, I be an ideal of R and M be a finitely generated R-module. If $\operatorname{cd}(I, M) = t \geq 1$, then $H_I^t(M) = IH_I^t(M)$.*

In particular,

$$I \not\subseteq \operatorname{Rad}(\operatorname{Ann}_R H_I^t(M)).$$

Proof. In view of [2, Lemma 2.8], $H_I^t(M) = IH_I^t(M)$. Since, $H_I^t(M) \neq 0$ it follows that $I^k H_I^t(M) = H_I^t(M) \neq 0$ for each $k \geq 1$ and hence

$$I \not\subseteq \operatorname{Rad}(\operatorname{Ann}_R H_I^t(M)).$$

□

Now we are ready to state and prove the main result of this paper.

Theorem 2.2. *Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring and I be an ideal of R. Then,*

$$\beta_R(I, R) \leq \gamma_R(I, R) \leq \dim R - \operatorname{cd}(I, R) \leq \dim R/I.$$

In particular, if $\gamma_R(I, R) = \dim R/I$, then

$$\gamma_R(I, R) = \dim R - \operatorname{cd}(I, R) \text{ and } \operatorname{cd}(I, R) = \operatorname{height} I.$$

Proof. Assume that $\beta_R(I, R) = t$. If $t = 0$, then it is clear that

$$\beta_R(I, R) \leq \gamma_R(I, R).$$

Now assume that $t \geq 1$. Then, by the definition there is a positive integer k such that for each $n \geq k$, $\operatorname{depth}_R R/I^n = t$. Thus, for each $0 \leq j \leq t-1$ and each $n \geq k$, $H_{\mathfrak{m}}^j(R/I^n) = 0$. On the other hand, for each $0 \leq j \leq t-1$ and each $n < k$ it is clear that $I^k H_{\mathfrak{m}}^j(R/I^n) = 0$. Therefore, for each $0 \leq j \leq t-1$, $I^k H_{\mathfrak{m}}^j(G(I, R)) = 0$ and hence $I \subseteq \operatorname{Rad}(\operatorname{Ann}_R H_{\mathfrak{m}}^j(G(I, R)))$. Now, it follows from the definition that $\gamma_R(I, R) \geq t = \beta_R(I, R)$.

Let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R. Then $(\widehat{R}, \mathfrak{m}\widehat{R})$ is a Noetherian local Cohen-Macaulay ring such that $\gamma_{\widehat{R}}(I\widehat{R}, \widehat{R}) = \gamma_R(I, R)$, $\dim \widehat{R} = \dim R$, $\dim \widehat{R}/I\widehat{R} = \dim R/I$ and $\operatorname{cd}(I\widehat{R}, \widehat{R}) = \operatorname{cd}(I, R)$. Therefore, in order to prove the inequality $\gamma_R(I, R) \leq \dim R - \operatorname{cd}(I, R)$, without loss of generality we may assume that R is a Noetherian complete local Cohen-Macaulay ring.

Let $\gamma_R(I, R) = \ell$ and assume that ω_R denotes the canonical module of R. Then by the definition there exists a positive integer k such that for each $0 \leq j \leq \ell - 1$ and each $n \geq 1$, $I^k H_{\mathfrak{m}}^j(R/I^n) = 0$. Therefore, in view of *Local Duality Theorem* for each $0 \leq j \leq \ell - 1$ and each $n \geq 1$, $I^k \operatorname{Ext}_R^{\dim R - j}(R/I^n, \omega_R) \simeq I^k D(H_{\mathfrak{m}}^j(R/I^n)) = 0$, where $D(-)$ denotes the Matlis dual functor $\operatorname{Hom}_R(-, E_R(R/\mathfrak{m}))$. Thus, for each $0 \leq j \leq \ell - 1$,

$$I^k H_I^{\dim R - j}(\omega_R) = I^k \left(\varinjlim_{n \geq 1} \operatorname{Ext}_R^{\dim R - j}(R/I^n, \omega_R) \right) = 0.$$

Also, by the *Grothendieck's Vanishing Theorem*, $H_I^{\dim R + i}(\omega_R) = 0$ for each $i \geq 1$. Hence, by Lemma 2.1, $\operatorname{cd}(I, \omega_R) \leq \dim R - \ell$.

Since

$$\operatorname{Supp} \omega_R = \operatorname{Spec}(R) = \operatorname{Supp} R,$$

it follows from [7, Theorem 2.2] that $\operatorname{cd}(I, R) = \operatorname{cd}(I, \omega_R)$ and so

$$\operatorname{cd}(I, R) \leq \dim R - \ell = \dim R - \gamma_R(I, R).$$

Now, it is clear that $\gamma_R(I, R) \leq \dim R - \operatorname{cd}(I, R)$.

On the other hand, in view of [14, Lemma 2.10],

$$\dim R - \operatorname{cd}(I, R) \leq \dim R/I.$$

Therefore,

$$\beta_R(I, R) \leq \gamma_R(I, R) \leq \dim R - \operatorname{cd}(I, R) \leq \dim R/I.$$

Now, assume that $\gamma_R(I, R) = \dim R/I$. Then it is clear that

$$\gamma_R(I, R) = \dim R - \operatorname{cd}(I, R) \text{ and } \dim R/I = \dim R - \operatorname{cd}(I, R).$$

So, by [13, Theorem 17.4],

$$\operatorname{cd}(I, R) = \dim R - \dim R/I = \operatorname{height} I.$$

□

Corollary 2.3. *Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring of dimension d and I be an ideal of R with $\beta_R(I, R) = d - 1$. Then* $\operatorname{cd}(I, R) = 1$.

Proof. In view of Theorem 2.2, $\operatorname{cd}(I, R) \leq \dim R - \beta_R(I, R) = 1$.

Now, if $\operatorname{cd}(I, R) = 0$, then I is nilpotent and hence $\beta_R(I, R) = \dim R = d$, which is a contradiction. So, $\operatorname{cd}(I, R) = 1$. □

The following consequence of Theorem 2.2 presents a partially affirmative answer to Question 2 in an special case.

Corollary 2.4. *Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring of dimension d and I be an ideal of R with $\gamma_R(I, R) = d - 1$. Then* $\operatorname{cd}(I, R) = 1$.

Proof. In view of Theorem 2.2, $\operatorname{cd}(I, R) \leq \dim R - \gamma_R(I, R) = 1$.

Now, if $\operatorname{cd}(I, R) = 0$, then I is a nilpotent ideal and so

$$\gamma_R(I, R) \geq \beta_R(I, R) = \dim R = d,$$

which is a contradiction. Therefore, $\operatorname{cd}(I, R) = 1$. □

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