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RELATIVE φ-TYPE AND RELATIVE φ-WEAK TYPE BASED SOME GROWTH PROPERTIES OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Abstract. The principal objective of this paper is to introduce the ideas of relative φ-type, relative φ-weak type of entire functions of several complex variables and study some growth properties concerning them.

1. Introduction, Definitions and Notations

Let f be a non-constant entire function of two complex variables holomorphic in the closed polydisc

$$U = \{(z_1, z_2) : \ |z_i| \le r_i, \ i = 1, 2 \text{ for all } \ r_1 \ge 0, r_2 \ge 0\}$$

and $M_f(r_1, r_2) = \max\{|f(z_1, z_2)| : |z_i| \le r_i, \ i = 1, 2\}$. Then in view of maximum principal and Hartogs theorem [9, p. 2, p. 51], $M_f(r_1, r_2)$ is an increasing functions of r_1, r_2. In this connection the following definition is well known:

Definition 1. {[9, p. 339] (see also [1])} The order ${}_{v_2}\rho(f)$ and the lower order ${}_{v_2}\lambda(f)$ of an entire function f of two complex variables are

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defined as

$$\begin{matrix} {}_{v_2}\rho\left(f\right) \\ {}_{v_2}\lambda\left(f\right) \end{matrix} = \lim_{r_1,r_2\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log\log M_f\left(r_1,r_2\right)}{\log\left(r_1r_2\right)}.$$

The equivalent formula for ${}_{v_2}\rho\left(f\right)$ is [9, p. 338] is

$${}_{v_2}\rho\left(f\right) = \inf \mu > 0 : M_f\left(r_1,r_2\right) < \exp\left[\left(r_1r_2\right)^{\mu}\right], \text{for all } r_1 \geq R\left(\mu\right), r_2 \geq R\left(\mu\right).$$

Similarly, one can define ${}_{v_2}\lambda\left(f\right)$ as

$${}_{v_2}\lambda\left(f\right) = \sup \mu > 0 : M_f\left(r_1,r_2\right) > \exp\left[\left(r_1r_2\right)^{\mu}\right], \text{for all } r_1 \geq R\left(\mu\right), r_2 \geq R\left(\mu\right).$$

The rate of growth of entire function of two complex variables normally depends upon the order of it. The entire function of two complex variables with higher order is of faster growth than that of lesser order. But if orders of two entire functions of two complex variables are the same, then it is impossible to detect the function with faster growth. In that case, it is necessary to compute another class of growth indicators of entire functions of two complex variables called their type and lower type and thus one can define type and lower type of an entire function f of two complex variables denoted by ${}_{v_2}\sigma\left(f\right)$ and ${}_{v_2}\overline{\sigma}\left(f\right)$ respectively in the following way:

Definition 2. [12, p. 339] The type ${}_{v_2}\sigma\left(f\right)$ and the lower type ${}_{v_2}\overline{\sigma}\left(f\right)$ of an entire function f of two complex variables are defined as

$$\begin{matrix} {}_{v_2}\sigma\left(f\right) \\ {}_{v_2}\overline{\sigma}\left(f\right) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M_f\left(r_1,r_2\right)}{r_1^{{}_{v_2}\rho(f)} + r_2^{{}_{v_2}\rho(f)}} \text{ where } 0 < {}_{v_2}\rho\left(f\right) < \infty.$$

The above can alternatively be written as

$$\begin{aligned} {}_{v_2}\sigma\left(f\right) &= \inf\Big\{\mu > 0 : M_f\left(r_1,r_2\right) < \exp\left(\mu r_1^{{}_{v_2}\rho(f)} + \mu r_2^{{}_{v_2}\rho(f)}\right) \\ &\qquad \text{for all } r_1 \geq R\left(\mu\right), r_2 \geq R\left(\mu\right)\Big\} \end{aligned}$$

and

$$\begin{aligned} {}_{v_2}\overline{\sigma}\left(f\right) &= \sup\Big\{\mu > 0 : M_f\left(r_1,r_2\right) > \exp\left(\mu r_1^{{}_{v_2}\rho(f)} + \mu r_2^{{}_{v_2}\rho(f)}\right) \\ &\qquad \text{for all } r_1 \geq R\left(\mu\right), r_2 \geq R\left(\mu\right)\Big\}. \end{aligned}$$

Similarly one may define the following growth indicators:

Definition 3. The weak type ${}_{v_2}\tau(f)$ and the lower weak type ${}_{v_2}\overline{\tau}(f)$ of an entire function f of two complex variables are defined as

$$\begin{aligned} {}_{v_2}\tau(f) &= \inf\Big\{\mu > 0 : M_f(r_1, r_2) < \exp\Big(\mu r_1^{{}_{v_2}\lambda(f)} + \mu r_2^{{}_{v_2}\lambda(f)}\Big) \\ &\qquad \text{for all } r_1 \geq R(\mu), r_2 \geq R(\mu)\} \end{aligned}$$

and

$$\begin{aligned} {}_{v_2}\overline{\tau}(f) &= \sup\Big\{\mu > 0 : M_f(r_1, r_2) > \exp\Big(\mu r_1^{{}_{v_2}\lambda(f)} + \mu r_2^{{}_{v_2}\lambda(f)}\Big) \\ &\qquad \text{for all } r_1 \geq R(\mu), r_2 \geq R(\mu)\}. \end{aligned}$$

In [7], Chyzhykov et al. introduced the definition of φ-order of a meromorphic function on single variable in the unit disc. For details about φ-order, one may see [7]. Consequently the definition of φ-order of entire function holomorphic in the closed polydisc $\{(z_1, z_2) : |z_i| \leq r_i, i = 1, 2 \text{ for all } r_1 \geq 0, r_2 \geq 0\}$ is established in [5] which is as follows:

Definition 4. [5] Let $\varphi_i(r_1, r_2) \mid i = 1, 2 : [0, +\infty) \times [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function of two variables r_1 and r_2. The φ-order of an entire function f of two complex variables denoted by ${}_{v_2}\rho(f, \varphi)$ is defined as:

$$\begin{aligned} {}_{v_2}\rho(f, \varphi) = \inf\{\mu > 0 : M_f(r_1, r_2) < \exp[(\varphi_1(r_1, r_2)\varphi_2(r_1, r_2))^{\mu}]; \\ r_1 \geq R(\mu), r_2 \geq R(\mu)\}. \end{aligned}$$

Analogously, one can define the φ-lower order of f of two complex variables denoted by ${}_{v_2}\lambda(f, \varphi)$ as follows :

$$\begin{aligned} {}_{v_2}\lambda(f, \varphi) = \sup\{\mu > 0 : M_f(r_1, r_2) > \exp[(\varphi_1(r_1, r_2)\varphi_2(r_1, r_2))^{\mu}]; \\ r_1 \geq R(\mu), r_2 \geq R(\mu)\}, \end{aligned}$$

where $\varphi_i(r_1, r_2) \mid i = 1, 2 : [0, +\infty) \times [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function of two variables r_1 and r_2.

Extending this notion, it is natural for us to give the definitions of φ-type and φ-lower type of entire functions holomorphic in the closed polydisc $\{(z_1, z_2) : |z_i| \leq r_i, i = 1, 2 \text{ for all } r_1 \geq 0, r_2 \geq 0\}$ which are as follows:

Definition 5. Let $\varphi_i(r_1, r_2) \mid i = 1, 2 : [0, +\infty) \times [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function of two variables r_1 and

r_2. The φ-type and φ-lower type of an entire function f of two complex variables denoted respectively by ${}_{v_2}\sigma(f,\varphi)$ and ${}_{v_2}\overline{\sigma}(f,\varphi)$ are defined as:

$$\begin{aligned} {}_{v_2}\sigma(f,\varphi) &= \inf\{\mu > 0 : M_f(r_1,r_2) \\ &\quad < \exp\left(\mu\varphi_1(r_1,r_2)^{{}_{v_2}\rho(f,\varphi)} + \mu\varphi_2(r_1,r_2)^{{}_{v_2}\rho(f,\varphi)}\right) \\ &\qquad \text{for all } r_1 \geq R(\mu), r_2 \geq R(\mu)\}. \end{aligned}$$

and

$$\begin{aligned} {}_{v_2}\overline{\sigma}(f,\varphi) &= \sup\{\mu > 0 : M_f(r_1,r_2) \\ &\quad > \exp\left(\mu\varphi_1(r_1,r_2)^{{}_{v_2}\rho(f,\varphi)} + \mu\varphi_2(r_1,r_2)^{{}_{v_2}\rho(f,\varphi)}\right) \\ &\qquad \text{for all } r_1 \geq R(\mu), r_2 \geq R(\mu)\}. \end{aligned}$$

Similarly one may define the following growth indicators:

Definition 6. Let $\varphi_i(r_1,r_2) \mid i = 1,2 : [0,+\infty) \times [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function of two variables r_1 and r_2. The φ-weak type ${}_{v_2}\tau(f,\varphi)$ and φ-lower weak type ${}_{v_2}\overline{\tau}(f,\varphi)$ of an entire function f of two complex variables are defined as:

$$\begin{aligned} {}_{v_2}\tau(f,\varphi) &= \inf\{\mu > 0 : M_f(r_1,r_2) \\ &\quad < \exp\left(\mu\varphi_1(r_1,r_2)^{{}_{v_2}\lambda(f,\varphi)} + \mu\varphi_2(r_1,r_2)^{{}_{v_2}\lambda(f,\varphi)}\right) \\ &\qquad \text{for all } r_1 \geq R(\mu), r_2 \geq R(\mu)\}. \end{aligned}$$

and

$$\begin{aligned} {}_{v_2}\overline{\tau}(f,\varphi) &= \sup\{\mu > 0 : M_f(r_1,r_2) \\ &\quad > \exp\left(\mu\varphi_1(r_1,r_2)^{{}_{v_2}\lambda(f,\varphi)} + \mu\varphi_2(r_1,r_2)^{{}_{v_2}\lambda(f,\varphi)}\right) \\ &\qquad \text{for all } r_1 \geq R(\mu), r_2 \geq R(\mu)\}. \end{aligned}$$

Now if we consider Definition 1 for single variable, then the definition coincides with the classical definition of order (see [15]) which is as follows:

Definition 7. [15] The order $\rho(f)$ and the lower order $\lambda(f)$ of an entire function f are defined in the following way:

$$\begin{matrix} \rho(f) \\ \lambda(f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log\log M_f(r)}{\log r},$$

where $M_f(r) = \max\{|f(z)| : |z| = r\}$.

Further if f is non-constant then $M_f(r)$ is strictly increasing and continuous, and its inverse ${M_f}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim\limits_{s\to\infty} {M_f}^{-1}(s) = \infty$. Bernal {[2], [3]} introduced the definition of relative order of f with respect to g, denoted by $\rho_g(f)$ as follows :

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1}(M_f(r))}{\log r}.\end{aligned}$$

The definition coincides with the classical one [15] if $g(z) = \exp z$.

During the past decades, several authors (see [6],[10],[11],[12],[13],[14]) made close investigations on the properties of relative order of entire functions of single variable. In the case of relative order, it was then natural for Banerjee and Dutta [4] to define the relative order of entire functions of two complex variables as follows:

Definition 8. [4] The relative order between two entire functions of two complex variables denoted by ${}_{v_2}\rho_g(f)$ is defined as:

$$ {}_{v_2}\rho_g(f) = \inf\{\mu > 0 : M_f(r_1, r_2) < M_g(r_1^\mu, r_2^\mu)\,; r_1 \geq R(\mu), r_2 \geq R(\mu)\}$$

where f and g are entire functions holomorphic in the closed polydisc

$$U = \{(z_1, z_2) : |z_i| \leq r_i,\ i = 1, 2 \text{ for all } r_1 \geq 0, r_2 \geq 0\}$$

and the definition coincides with Definition 1 {see [4]} if $g(z) = \exp(z_1 z_2)$.

Extending this notion, Dutta [8] introduced the idea of relative order of entire functions of several complex variables in the following way:

Definition 9. [8] Let $f(z_1, z_2, ..., z_n)$ and $g(z_1, z_2, ..., z_n)$ be any two entire functions of n variables $z_1, z_2, ..., z_n$ with maximum modulus functions

$M_f(r_1, r_2, ..., r_n)$ and $M_g(r_1, r_2, ..., r_n)$ respectively then the relative order of f with respect to g, denoted by ${}_{v_n}\rho_g(f)$ is defined by

$$\begin{aligned} {}_{v_n}\rho_g(f) = \inf\{\mu > 0 : M_f(r_1, r_2, ..., r_n) < M_g(r_1^\mu, r_2^\mu, ..., r_n^\mu)\,; \\ \text{for } r_i \geq R(\mu), i = 1, 2, ..., n\}.\end{aligned}$$

Similarly, one can define the relative lower order of f with respect to g denoted by ${}_{v_n}\lambda_g(f)$ as follows :

$$\begin{aligned} {}_{v_n}\lambda_g(f) = \sup\{\mu > 0 : M_f(r_1, r_2, ..., r_n) > M_g(r_1^\mu, r_2^\mu, ..., r_n^\mu)\,; \\ \text{for } r_i \geq R(\mu), i = 1, 2, ..., n\}.\end{aligned}$$

Now in order to refine the above growth scale, one may introduce the definitions of other growth indicators, such as relative type and relative lower type between two entire functions of severable complex variables which are as follows:

Definition 10. Let $f(z_1, z_2, ..., z_n)$ and $g(z_1, z_2, ..., z_n)$ be any two entire functions of n variables $z_1, z_2, ..., z_n$ with maximum modulus functions

$M_f\left(r_1, r_2, ..., r_n\right)$ and $M_g\left(r_1, r_2, ..., r_n\right)$ respectively. Then the relative type ${}_{v_n}\sigma_g\left(f\right)$ and the relative lower type ${}_{v_n}\overline{\sigma}_g\left(f\right)$ of f with respect to g with non-zero finite relative order ${}_{v_n}\rho_g\left(f\right)$ are defined as:

$$\begin{aligned} {}_{v_n}\sigma_g\left(f\right) = \inf\{\mu > 0 : & M_f\left(r_1, r_2, ..., r_n\right) \\ & < M_g\left(\mu r_1^{{}_{v_n}\rho_g(f)}, \mu r_2^{{}_{v_n}\rho_g(f)}, ..., \mu r_n^{{}_{v_n}\rho_g(f)}\right); \\ & \text{for } r_i \geq R\left(\mu\right), i = 1, 2, ..., n\}. \end{aligned}$$

and

$$\begin{aligned} {}_{v_n}\overline{\sigma}_g\left(f\right) = \sup\{\mu > 0 : & M_f\left(r_1, r_2, ..., r_n\right) \\ & > M_g\left(\mu r_1^{{}_{v_n}\rho_g(f)}, \mu r_2^{{}_{v_n}\rho_g(f)}, ..., \mu r_n^{{}_{v_n}\rho_g(f)}\right); \\ & \text{for } r_i \geq R\left(\mu\right), i = 1, 2, ..., n\}. \end{aligned}$$

Analogously, to determine the relative growth of f of two complex variables having same non zero finite relative lower order with respect to another entire function g of severable complex variables, one can introduce the definition of relative weak type ${}_{v_n}\tau_g\left(f\right)$ and relative lower weak type ${}_{v_n}\overline{\tau}_g\left(f\right)$ of f with respect to g of finite positive relative lower order ${}_{v_n}\lambda_g\left(f\right)$ in the following way:

Definition 11. Let $f(z_1, z_2, ..., z_n)$ and $g(z_1, z_2, ..., z_n)$ be any two entire functions of n variables $z_1, z_2, ..., z_n$ with maximum modulus functions

$M_f\left(r_1, r_2, ..., r_n\right)$ and $M_g\left(r_1, r_2, ..., r_n\right)$ respectively. Then the relative weak type ${}_{v_n}\tau_g\left(f\right)$ and the relative lower weak type ${}_{v_n}\overline{\tau}_g\left(f\right)$ of f with respect to g with non-zero finite relative lower order ${}_{v_n}\lambda_g\left(f\right)$ are defined as:

$$
\begin{aligned}
{}_{v_n}\tau_g(f) = \inf\{\mu > 0 : M_f(r_1, r_2, ..., r_n) \\
< M_g\left(\mu r_1^{{}_{v_n}\lambda_g(f)}, \mu r_2^{{}_{v_n}\lambda_g(f)}, ..., \mu r_n^{{}_{v_n}\lambda_g(f)}\right); \\
\text{for } r_i \geq R(\mu), i = 1, 2, ..., n\}.
\end{aligned}
$$

and

$$
\begin{aligned}
{}_{v_n}\overline{\tau}_g(f) = \sup\{\mu > 0 : M_f(r_1, r_2, ..., r_n) \\
> M_g\left(\mu r_1^{{}_{v_n}\lambda_g(f)}, \mu r_2^{{}_{v_n}\lambda_g(f)}, ..., \mu r_n^{{}_{v_n}\lambda_g(f)}\right); \\
\text{for } r_i \geq R(\mu), i = 1, 2, ..., n\}.
\end{aligned}
$$

Now in order to make some progress in the study of relative order of entire functions of several complex variables,in [5], the definition of relative φ-order between two entire functions of several complex variables is given which is as follows:

Definition 12. Let $\varphi_i(r_1, r_2, ..., r_n) \mid i = 1, 2, .., n : [0, +\infty) \times [0, +\infty) \times ... \times [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function of n variables $r_1, r_2, ..., r_n$. Also let f and g be any two entire functions of n complex variables with maximum modulus functions $M_f(r_1, r_2, ..., r_n)$ and $M_g(r_1, r_2, ..., r_n)$ respectively then the relative φ-order of f with respect to g, denoted by ${}_{v_n}\rho_g(f, \varphi)$ is defined by

$$
\begin{aligned}
{}_{v_n}\rho_g(f, \varphi) = \inf\{\mu > 0 : M_f(r_1, r_2, ..., r_n) < M_g(\varphi_1^\mu, \varphi_2^\mu, ..., \varphi_n^\mu); \\
\text{for } r_i \geq R(\mu), i = 1, 2, ..., n\},
\end{aligned}
$$

where $\varphi_i \mid i = 1, 2, .., n$ stand for $\varphi_i(r_1, r_2, ..., r_n) \mid i = 1, 2, .., n$.

Likewise, one can define the relative φ-lower order of f with respect to g denoted by ${}_{v_n}\lambda_g(f, \varphi)$ as follows :

$$
\begin{aligned}
{}_{v_n}\lambda_g(f, \varphi) = \sup\{\mu > 0 : M_f(r_1, r_2, ..., r_n) > M_g(\varphi_1^\mu, \varphi_2^\mu, ..., \varphi_n^\mu); \\
\text{for } r_i \geq R(\mu), i = 1, 2, ..., n\},
\end{aligned}
$$

where $\varphi_i \mid i = 1, 2, .., n : [0, +\infty) \times [0, +\infty) \times ... \times [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function of n variables $r_1, r_2, ..., r_n$.

Further an entire function f of several complex variables for which relative φ-order and relative φ-lower order with respect to another entire function g of several complex variables are the same is called a function of regular relative φ-growth with respect to g. Otherwise, f is said to be irregular relative φ-growth.with respect to g.

Moreover in order to refine the above growth scale, one may introduce the definitions of other growth indicators, such as relative φ-type and relative φ-lower type between two entire functions of severable complex variables which are as follows:

Definition 13. Let f and g be two entire functions of n variables $r_1, r_2, ..., r_n$ and $\varphi_i(r_1, r_2, ..., r_n)|i = 1, 2, ..., n : [0, +\infty) \times [0, \infty) \times ... \times [0, \infty) \to [0, \infty)$ be a non-decreasing unbounded functions of n variables $r_1, r_2, ..., r_n$. Also let $0 < {}_{v_n}\rho_g(f, \varphi) < \infty$. Then we can define the relative φ-type of the function f with respect to g, denoted by ${}_{v_n}\sigma_g(f, \varphi)$, in the following manner:

$$\begin{aligned} {}_{v_n}\sigma_g(f,\varphi) = \inf \{\mu > 0 : & M_f(r_1, r_2, ..., r_n) \\ & < M_g(\mu\varphi_1^{{}_{v_n}\rho_g(f,\varphi)}, \mu\varphi_2^{{}_{v_n}\rho_g(f,\varphi)}, ..., \mu\varphi_n^{{}_{v_n}\rho_g(f,\varphi)}); \\ & \text{for } r_i \geq R(\mu), i = 1, 2, ..., n\}, \end{aligned}$$

Similarly, one can introduce the relative φ-lower type of f with respect to g, denoted by ${}_{v_n}\overline{\sigma}_g(f, \varphi)$ as

$$\begin{aligned} {}_{v_n}\overline{\sigma}_g(f,\varphi) = \sup \{\mu > 0 : & M_f(r_1, r_2, ..., r_n) \\ & > M_g(\mu\varphi_1^{{}_{v_n}\rho_g(f,\varphi)}, \mu\varphi_2^{{}_{v_n}\rho_g(f,\varphi)}, ..., \mu\varphi_n^{{}_{v_n}\rho_g(f,\varphi)}); \\ & \text{for } r_i \geq R(\mu), i = 1, 2, ..., n\}. \end{aligned}$$

In the like manner, to measure the relative growth of an entire function f of n variables having the relative φ-lower order with respect to another one, say g, the notion of relative φ-weak type ${}_{v_n}\overline{\tau}_g(f, \varphi)$ and the growth-indicator ${}_{v_n}\tau_g(f, \varphi)$ can be defined as follows.

Definition 14. Let f and g be two entire functions of n variables $r_1, r_2, ..., r_n$ and $\varphi_i(r_1, r_2, ..., r_n)|i = 1, 2, ..., n : [0, +\infty) \times [0, \infty) \times ... \times [0, \infty) \to [0, \infty)$ be a non-decreasing unbounded functions of n variables $r_1, r_2, ..., r_n$. Then the relative φ-weak type ${}_{v_n}\tau_g(f, \varphi)$ and the relative φ-lower weak type ${}_{v_n}\overline{\tau}_g(f, \varphi)$ of an entire function f with non-zero finite relative φ-lower order ${}_{v_n}\lambda_g(f, \varphi)$ are defined as:

$$\begin{aligned} {}_{v_n}\tau_g(f,\varphi) = \inf \{\mu > 0 : & M_f(r_1, r_2, ..., r_n) \\ & < M_g(\mu\varphi_1^{{}_{v_n}\lambda_g(f,\varphi)}, \mu\varphi_2^{{}_{v_n}\lambda_g(f,\varphi)}, ..., \mu\varphi_n^{{}_{v_n}\lambda_g(f,\varphi)}); \\ & \text{for } r_i \geq R(\mu), i = 1, 2, ..., n\}, \end{aligned}$$

and

$$\begin{aligned}{}_{v_n}\overline{\tau}_g(f,\varphi) = \sup\{\mu > 0 : & \, M_f(r_1, r_2, ..., r_n) \\ & > M_g(\mu\varphi_1^{{}_{v_n}\lambda_g(f,\varphi)}, \mu\varphi_2^{{}_{v_n}\lambda_g(f,\varphi)}, ..., \mu\varphi_n^{{}_{v_n}\lambda_g(f,\varphi)}); \\ & \qquad \text{for } r_i \geq R(\mu), i = 1, 2, ..., n\}.\end{aligned}$$

Here, in this paper, we study some basic properties of relative φ-type and relative φ-weak type of entire functions of several complex variables with respect to another one. We do not explain the standard definitions and notations in the theory of entire function of several complex variables as those are available in [9].

2. Main Results

In this section we present the main results of the paper. First of all, we recall one related known property which will be needed in order to prove our results, as we see in the following theorem.

THEOREM 1. [5] *Let f, g and h be any three entire functions of several complex variables such that* $0 < {}_{v_n}\lambda_h(f,\varphi) \leq {}_{v_n}\rho_h(f,\varphi) < \infty$ *and* $0 < {}_{v_n}\lambda_h(g) \leq {}_{v_n}\rho_h(g) < \infty$. *Then*

$$\begin{aligned}\frac{{}_{v_n}\lambda_h(f,\varphi)}{{}_{v_n}\rho_h(g)} &\leq {}_{v_n}\lambda_g(f,\varphi) \leq \min\left\{\frac{{}_{v_n}\lambda_h(f,\varphi)}{{}_{v_n}\lambda_h(g)}, \frac{{}_{v_n}\rho_h(f,\varphi)}{{}_{v_n}\rho_h(g)}\right\} \\ &\leq \max\left\{\frac{{}_{v_n}\lambda_h(f,\varphi)}{{}_{v_n}\lambda_h(g)}, \frac{{}_{v_n}\rho_h(f,\varphi)}{{}_{v_n}\rho_h(g)}\right\} \leq {}_{v_n}\rho_g(f,\varphi) \leq \frac{{}_{v_n}\rho_h(f,\varphi)}{{}_{v_n}\lambda_h(g)}.\end{aligned}$$

REMARK 1. [5] From the conclusion of Theorem 1, one may write ${}_{v_n}\rho_g(f,\varphi) = \frac{{}_{v_n}\rho_h(f,\varphi)}{{}_{v_n}\rho_h(g)}$ and ${}_{v_n}\lambda_g(f,\varphi) = \frac{{}_{v_n}\lambda_h(f,\varphi)}{{}_{v_n}\lambda_h(g)}$ when ${}_{v_n}\lambda_h(g) = {}_{v_n}\rho_h(g)$. Similarly ${}_{v_n}\rho_g(f,\varphi) = \frac{{}_{v_n}\lambda_h(f,\varphi)}{{}_{v_n}\lambda_h(g)}$ and ${}_{v_n}\lambda_g(f,\varphi) = \frac{{}_{v_n}\rho_h(f,\varphi)}{{}_{v_n}\rho_h(g)}$ when ${}_{v_n}\lambda_h(f,\varphi) = {}_{v_n}\rho_h(f,\varphi)$.

THEOREM 2. *Let f, g and h be any three entire functions of several complex variables such that* $0 < {}_{v_n}\rho_h(f,\varphi) < \infty$ *and* $0 < {}_{v_n}\lambda_h(g) \leq$

${}_{v_n}\rho_h(g) < \infty$. *Then*

$$\max\left\{\left(\frac{{}_{v_n}\overline{\sigma}_h(f,\varphi)}{{}_{v_n}\overline{\tau}_h(g)}\right)^{\frac{1}{{}_{v_n}\lambda_h(g)}},\left(\frac{{}_{v_n}\sigma_h(f,\varphi)}{{}_{v_n}\tau_h(g)}\right)^{\frac{1}{{}_{v_n}\lambda_h(g)}}\right\}$$

$$\leq {}_{v_n}\sigma_g(f,\varphi) \leq \left(\frac{{}_{v_n}\sigma_h(f,\varphi)}{{}_{v_n}\overline{\sigma}_h(g)}\right)^{\frac{1}{{}_{v_n}\rho_h(g)}}.$$

Proof. Let us consider that $\varepsilon\,(>0)$ is arbitrary number. Now from the definitions of ${}_{v_n}\sigma_g(f,\varphi)$ and ${}_{v_n}\overline{\sigma}_g(f,\varphi)$, we have for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$(1)\quad M_f(r_1,r_2,...,r_n) < M_g\Big(\left({}_{v_n}\sigma_g(f,\varphi)+\varepsilon\right)\varphi_1^{{}_{v_n}\rho_g(f,\varphi)},$$
$$\left({}_{v_n}\sigma_g(f,\varphi)+\varepsilon\right)\varphi_2^{{}_{v_n}\rho_g(f,\varphi)},...,\left({}_{v_n}\sigma_g(f,\varphi)+\varepsilon\right)\varphi_n^{{}_{v_n}\rho_g(f,\varphi)}\Big),$$

$$(2)\quad M_f(r_1,r_2,...,r_n) > M_g\Big(\left({}_{v_n}\overline{\sigma}_g(f,\varphi)-\varepsilon\right)\varphi_1^{{}_{v_n}\rho_g(f,\varphi)},$$
$$\left({}_{v_n}\overline{\sigma}_g(f,\varphi)-\varepsilon\right)\varphi_2^{{}_{v_n}\rho_g(f,\varphi)},...,\left({}_{v_n}\overline{\sigma}_g(f,\varphi)-\varepsilon\right)\varphi_n^{{}_{v_n}\rho_g(f,\varphi)}\Big),$$

and also for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity, we get that

$$(3)\quad M_f(r_1,r_2,...,r_n) > M_g\Big(\left({}_{v_n}\sigma_g(f,\varphi)-\varepsilon\right)\varphi_1^{{}_{v_n}\rho_g(f,\varphi)},$$
$$\left({}_{v_n}\sigma_g(f,\varphi)-\varepsilon\right)\varphi_2^{{}_{v_n}\rho_g(f,\varphi)},...,\left({}_{v_n}\sigma_g(f,\varphi)-\varepsilon\right)\varphi_n^{{}_{v_n}\rho_g(f,\varphi)}\Big),$$

$$(4)\quad M_f(r_1,r_2,...,r_n) < M_g\Big(\left({}_{v_n}\overline{\sigma}_g(f,\varphi)+\varepsilon\right)\varphi_1^{{}_{v_n}\rho_g(f,\varphi)},$$
$$\left({}_{v_n}\overline{\sigma}_g(f,\varphi)+\varepsilon\right)\varphi_2^{{}_{v_n}\rho_g(f,\varphi)},...,\left({}_{v_n}\overline{\sigma}_g(f,\varphi)+\varepsilon\right)\varphi_n^{{}_{v_n}\rho_g(f,\varphi)}\Big).$$

Similarly from the definitions of ${}_{v_n}\sigma_h(g)$ and ${}_{v_n}\overline{\sigma}_h(g)$, it follows for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$(5)\quad M_g(r_1,r_2,...,r_n) < M_h\Big(\left({}_{v_n}\sigma_h(g)+\varepsilon\right)r_1^{{}_{v_n}\rho_h(g)},$$
$$\left({}_{v_n}\sigma_h(g)+\varepsilon\right)r_2^{{}_{v_n}\rho_h(g)},...,\left({}_{v_n}\sigma_h(g)+\varepsilon\right)r_n^{{}_{v_n}\rho_h(g)}\Big),$$

$$(6)\quad M_g(r_1,r_2,...,r_n) > M_h\Big(\left({}_{v_n}\overline{\sigma}_h(g)-\varepsilon\right)r_1^{{}_{v_n}\rho_h(g)},$$
$$\left({}_{v_n}\overline{\sigma}_h(g)-\varepsilon\right)r_2^{{}_{v_n}\rho_h(g)},...,\left({}_{v_n}\overline{\sigma}_h(g)-\varepsilon\right)r_n^{{}_{v_n}\rho_h(g)}\Big).$$

Also for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity, we obtain that

$$M_g(r_1, r_2, ..., r_n) > M_h\Big(({}_{v_n}\sigma_h(g) - \varepsilon)\, r_1^{{}_{v_n}\rho_h(g)}, ({}_{v_n}\sigma_h(g) - \varepsilon)\, r_2^{{}_{v_n}\rho_h(g)}, ..., ({}_{v_n}\sigma_h(g) - \varepsilon)\, r_n^{{}_{v_n}\rho_h(g)}\Big), \tag{7}$$

$$M_g(r_1, r_2, ..., r_n) < M_h\Big(({}_{v_n}\overline{\sigma}_h(g) + \varepsilon)\, r_1^{{}_{v_n}\rho_h(g)}, ({}_{v_n}\overline{\sigma}_h(g) + \varepsilon)\, r_2^{{}_{v_n}\rho_h(g)}, ..., ({}_{v_n}\overline{\sigma}_h(g) + \varepsilon)\, r_n^{{}_{v_n}\rho_h(g)}\Big). \tag{8}$$

Further from the definitions of ${}_{v_n}\tau_g(f, \varphi)$ and ${}_{v_n}\overline{\tau}_g(f, \varphi)$, we have for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_f(r_1, r_2, ..., r_n) < M_g\Big(({}_{v_n}\tau_g(f, \varphi) + \varepsilon)\, \varphi_1^{{}_{v_n}\lambda_g(f,\varphi)}, ({}_{v_n}\tau_g(f, \varphi) + \varepsilon)\, \varphi_2^{{}_{v_n}\lambda_g(f,\varphi)}, ..., ({}_{v_n}\tau_g(f, \varphi) + \varepsilon)\, \varphi_n^{{}_{v_n}\lambda_g(f,\varphi)}\Big), \tag{9}$$

$$M_f(r_1, r_2, ..., r_n) > M_g\Big(({}_{v_n}\overline{\tau}_g(f, \varphi) - \varepsilon)\, \varphi_1^{{}_{v_n}\lambda_g(f,\varphi)}, ({}_{v_n}\overline{\tau}_g(f, \varphi) - \varepsilon)\, \varphi_2^{{}_{v_n}\lambda_g(f,\varphi)}, ..., ({}_{v_n}\overline{\tau}_g(f, \varphi) - \varepsilon)\, \varphi_n^{{}_{v_n}\lambda_g(f,\varphi)}\Big). \tag{10}$$

and also for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity, we get that

$$M_f(r_1, r_2, ..., r_n) > M_g\Big(({}_{v_n}\tau_g(f, \varphi) - \varepsilon)\, \varphi_1^{{}_{v_n}\lambda_g(f,\varphi)}, ({}_{v_n}\tau_g(f, \varphi) - \varepsilon)\, \varphi_2^{{}_{v_n}\lambda_g(f,\varphi)}, ..., ({}_{v_n}\tau_g(f, \varphi) - \varepsilon)\, \varphi_n^{{}_{v_n}\lambda_g(f,\varphi)}\Big), \tag{11}$$

$$M_f(r_1, r_2, ..., r_n) < M_g\Big(({}_{v_n}\overline{\tau}_g(f, \varphi) + \varepsilon)\, \varphi_1^{{}_{v_n}\lambda_g(f,\varphi)}, ({}_{v_n}\overline{\tau}_g(f, \varphi) + \varepsilon)\, \varphi_2^{{}_{v_n}\lambda_g(f,\varphi)}, ..., ({}_{v_n}\overline{\tau}_g(f, \varphi) + \varepsilon)\, \varphi_n^{{}_{v_n}\lambda_g(f,\varphi)}\Big). \tag{12}$$

Similarly from the definitions of ${}_{v_n}\tau_h(g)$ and ${}_{v_n}\overline{\tau}_h(g)$, it follows for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_g(r_1, r_2, ..., r_n) < M_h\Big(({}_{v_n}\tau_h(g) + \varepsilon)\, r_1^{{}_{v_n}\lambda_h(g)}, ({}_{v_n}\tau_h(g) + \varepsilon)\, r_2^{{}_{v_n}\lambda_h(g)}, ..., ({}_{v_n}\tau_h(g) + \varepsilon)\, r_n^{{}_{v_n}\lambda_h(g)}\Big), \tag{13}$$

$$M_g(r_1, r_2, ..., r_n) > M_h\Big(({}_{v_n}\overline{\tau}_h(g) - \varepsilon)\, r_1^{{}_{v_n}\lambda_h(g)}, \tag{14}$$

$$\left({}_{v_n}\overline{\tau}_h(g)-\varepsilon\right) r_2^{{}_{v_n}\lambda_h(g)},\ldots,\left({}_{v_n}\overline{\tau}_h(g)-\varepsilon\right) r_n^{{}_{v_n}\lambda_h(g)}\Big).$$

Also for a sequence of values of $r_1, r_2, \ldots, r_n$ tending to infinity, we obtain that

$$M_g(r_1,r_2,\ldots,r_n) > M_h\Big(\left({}_{v_n}\tau_h(g)-\varepsilon\right) r_1^{{}_{v_n}\lambda_h(g)},$$
$$\left({}_{v_n}\tau_h(g)-\varepsilon\right) r_2^{{}_{v_n}\lambda_h(g)},\ldots,\left({}_{v_n}\tau_h(g)-\varepsilon\right) r_n^{{}_{v_n}\lambda_h(g)}\Big), \tag{15}$$

$$M_g(r_1,r_2,\ldots,r_n) < M_h\Big(\left({}_{v_n}\overline{\tau}_h(g)+\varepsilon\right) r_1^{{}_{v_n}\lambda_h(g)},$$
$$\left({}_{v_n}\overline{\tau}_h(g)+\varepsilon\right) r_2^{{}_{v_n}\lambda_h(g)},\ldots,\left({}_{v_n}\overline{\tau}_h(g)+\varepsilon\right) r_n^{{}_{v_n}\lambda_h(g)}\Big). \tag{16}$$

Therefore from (1) and in view of (13), we get for all sufficiently large values of $r_1, r_2, \ldots, r_n$ that

$$M_f(r_1,r_2,\ldots,r_n) <$$
$$M_h\Big(\left({}_{v_n}\tau_h(g)+\varepsilon\right)\left({}_{v_n}\sigma_g(f,\varphi)+\varepsilon\right)^{{}_{v_n}\lambda_h(g)}\varphi_1^{{}_{v_n}\lambda_h(g)\,{}_{v_n}\rho_g(f,\varphi)},$$
$$\left({}_{v_n}\tau_h(g)+\varepsilon\right)\left({}_{v_n}\sigma_g(f)+\varepsilon\right)^{{}_{v_n}\lambda_h(g)}\varphi_2^{{}_{v_n}\lambda_h(g)\,{}_{v_n}\rho_g(f,\varphi)},\ldots,$$
$$\left({}_{v_n}\tau_h(g)+\varepsilon\right)\left({}_{v_n}\sigma_g(f)+\varepsilon\right)^{{}_{v_n}\lambda_h(g)}\varphi_n^{{}_{v_n}\lambda_h(g)\,{}_{v_n}\rho_g(f,\varphi)}\Big).$$

Since in view of Theorem 1 $\frac{{}_{v_n}\rho_h(f,\varphi)}{{}_{v_n}\lambda_h(g)} \geq {}_{v_n}\rho_g(f,\varphi)$ and $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\begin{aligned} {}_{v_n}\sigma_h(f,\varphi) &\leq {}_{v_n}\tau_h(g)\ {}_{v_n}\sigma_g(f,\varphi)^{{}_{v_n}\lambda_h(g)} \\ i.e.,\ {}_{v_n}\sigma_g(f,\varphi) &\geq \left(\frac{{}_{v_n}\sigma_h(f,\varphi)}{{}_{v_n}\tau_h(g)}\right)^{\frac{1}{{}_{v_n}\lambda_h(g)}}. \end{aligned} \tag{17}$$

Analogously from (1) and (16), we get that

$${}_{v_n}\sigma_g(f,\varphi) \geq \left(\frac{{}_{v_n}\overline{\sigma}_h(f,\varphi)}{{}_{v_n}\overline{\tau}_h(g)}\right)^{\frac{1}{{}_{v_n}\lambda_h(g)}}, \tag{18}$$

as in view of Theorem 1 it follows that $\frac{{}_{v_n}\rho_h(f,\varphi)}{{}_{v_n}\lambda_h(g)} \geq {}_{v_n}\rho_g(f,\varphi)$. Further in view of Theorem 1, since $\frac{{}_{v_n}\rho_h(f,\varphi)}{{}_{v_n}\rho_h(g)} \leq {}_{v_n}\rho_g(f,\varphi)$, we obtain from (3) and (6) for a sequence of values of $r_1, r_2, \ldots, r_n$ tending to infinity that

$$M_f(r_1,r_2,\ldots,r_n) > M_h\Big(\left({}_{v_n}\overline{\sigma}_h(g)-\varepsilon\right)\left({}_{v_n}\sigma_g(f,\varphi)-\varepsilon\right)^{{}_{v_n}\rho_h(g)}\varphi_1^{{}_{v_n}\rho_h(f,\varphi)},$$
$$\left({}_{v_n}\overline{\sigma}_h(g)-\varepsilon\right)\left({}_{v_n}\sigma_g(f,\varphi)-\varepsilon\right)^{{}_{v_n}\rho_h(g)}\varphi_2^{{}_{v_n}\rho_h(f,\varphi)},\ldots,$$

$$\left({}_{v_n}\overline{\sigma}_h(g)-\varepsilon\right)\left({}_{v_n}\sigma_g(f,\varphi)-\varepsilon\right)^{{}_{v_n}\rho_h(g)}\varphi_n^{{}_{v_n}\rho_h(f,\varphi)}\Big).$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\begin{aligned} {}_{v_n}\sigma_h(f,\varphi) &\geq {}_{v_n}\overline{\sigma}_h(g)\ {}_{v_n}\sigma_g(f,\varphi)^{{}_{v_n}\rho_h(g)} \\ \text{i.e., } {}_{v_n}\sigma_g(f,\varphi) &\leq \left(\frac{{}_{v_n}\sigma_h(f,\varphi)}{{}_{v_n}\overline{\sigma}_h(g)}\right)^{\frac{1}{{}_{v_n}\rho_h(g)}}. \end{aligned} \tag{19}$$

Thus the theorem follows from (17), (18) and (19). □

Since in view of Theorem 1, it follows that $\frac{{}_{v_n}\lambda_h(f,\varphi)}{{}_{v_n}\rho_h(g)}\leq {}_{v_n}\rho_g(f,\varphi)$ and $\frac{{}_{v_n}\lambda_h(f,\varphi)}{{}_{v_n}\lambda_h(g)}\leq {}_{v_n}\rho_g(f,\varphi)$, therefore the conclusion of the following theorem can be carried out from (3) and (6); (3) and (14) respectively after applying the same technique of Theorem 2. So its proof is omitted.

THEOREM 3. *Let f, g and h be any three entire functions of several complex variables such that $0<{}_{v_n}\lambda_h(f,\varphi)<\infty$ and $0<{}_{v_n}\lambda_h(g)\leq {}_{v_n}\rho_h(g)<\infty$. Then*

$${}_{v_n}\sigma_g(f,\varphi)\leq\min\left\{\left(\frac{{}_{v_n}\tau_h(f,\varphi)}{{}_{v_n}\overline{\tau}_h(g)}\right)^{\frac{1}{{}_{v_n}\lambda_h(g)}},\left(\frac{{}_{v_n}\tau_h(f,\varphi)}{{}_{v_n}\overline{\sigma}_h(g)}\right)^{\frac{1}{{}_{v_n}\rho_h(g)}}\right\}.$$

Similarly in the line of Theorem 2 and with the help of Theorem 1, one may easily carry out the following theorem from pairwise inequalities numbers (12) and (13); (6) and (10); (7) and (10); respectively and therefore its proof is omitted:

THEOREM 4. *Let f, g and h be any three entire functions of several complex variables such that $0<{}_{v_n}\lambda_h(f,\varphi)\leq{}_{v_n}\rho_h(f,\varphi)<\infty$ and $0<{}_{v_n}\lambda_h(g)\leq {}_{v_n}\rho_h(g)<\infty$. Then*

$$\begin{aligned} \left(\frac{{}_{v_n}\overline{\tau}_h(f,\varphi)}{{}_{v_n}\tau_h(g)}\right)^{\frac{1}{{}_{v_n}\lambda_h(g)}} &\leq {}_{v_n}\overline{\tau}_g(f,\varphi) \\ &\leq \min\left\{\left(\frac{{}_{v_n}\overline{\tau}_h(f,\varphi)}{{}_{v_n}\overline{\sigma}_h(g)}\right)^{\frac{1}{{}_{v_n}\rho_h(g)}},\left(\frac{{}_{v_n}\tau_h(f,\varphi)}{{}_{v_n}\sigma_h(g)}\right)^{\frac{1}{{}_{v_n}\rho_h(g)}}\right\}. \end{aligned}$$

THEOREM 5. *Let f, g and h be any three entire functions of several complex variables such that $0<{}_{v_n}\rho_h(f,\varphi)<\infty$ and $0<{}_{v_n}\lambda_h(g)\leq {}_{v_n}\rho_h(g)<\infty$. Then*

$${}_{v_n}\overline{\tau}_g(f,\varphi)\geq\max\left\{\left(\frac{{}_{v_n}\overline{\sigma}_h(f,\varphi)}{{}_{v_n}\sigma_h(g)}\right)^{\frac{1}{{}_{v_n}\rho_h(g)}},\left(\frac{{}_{v_n}\overline{\sigma}_h(f,\varphi)}{{}_{v_n}\tau_h(g)}\right)^{\frac{1}{{}_{v_n}\lambda_h(g)}}\right\}.$$

With the help of Theorem 1, the conclusion of the above theorem can be carried out from (5), (12) and (12), (13) respectively after applying the same technique of Theorem 2 and therefore its proof is omitted.

Similarly in view of Theorem 1, the conclusion of the following theorem can be carried out from pairwise inequalities numbered (4) and (13); (2) and (7); (2) and (6) respectively after applying the same technique of Theorem 2 and therefore its proof is omitted.

Theorem 6. *Let f, g and h be any three entire functions of several complex variables such that $0 < {}_{v_n}\rho_h(f,\varphi) < \infty$ and $0 < {}_{v_n}\lambda_h(g) \leq {}_{v_n}\rho_h(g) < \infty$. Then*

$$\begin{aligned}\left(\frac{{}_{v_n}\overline{\sigma}_h(f,\varphi)}{{}_{v_n}\tau_h(g)}\right)^{\frac{1}{{}_{v_n}\lambda_h(g)}} &\leq {}_{v_n}\overline{\sigma}_g(f,\varphi) \\ &\leq \min\left\{\left(\frac{{}_{v_n}\overline{\sigma}_h(f,\varphi)}{{}_{v_n}\overline{\sigma}_h(g)}\right)^{\frac{1}{{}_{v_n}\rho_h(g)}}, \left(\frac{{}_{v_n}\sigma_h(f,\varphi)}{{}_{v_n}\sigma_h(g)}\right)^{\frac{1}{{}_{v_n}\rho_h(g)}}\right\}.\end{aligned}$$

Theorem 7. *Let f, g and h be any three entire functions of several complex variables such that $0 < {}_{v_n}\lambda_h(f,\varphi) < \infty$ and $0 < {}_{v_n}\lambda_h(g) \leq {}_{v_n}\rho_h(g) < \infty$. Then*

$$\begin{aligned}{}_{v_n}\overline{\sigma}_g(f,\varphi) \leq \min\Bigg\{&\left(\frac{{}_{v_n}\overline{\tau}_h(f,\varphi)}{{}_{v_n}\overline{\tau}_h(g)}\right)^{\frac{1}{{}_{v_n}\lambda_h(g)}}, \left(\frac{{}_{v_n}\tau_h(f,\varphi)}{{}_{v_n}\tau_h(g)}\right)^{\frac{1}{{}_{v_n}\lambda_h(g)}}, \\ &\left(\frac{{}_{v_n}\tau_h(f,\varphi)}{{}_{v_n}\sigma_h(g)}\right)^{\frac{1}{{}_{v_n}\rho_h(g)}}, \left(\frac{{}_{v_n}\overline{\tau}_h(f,\varphi)}{{}_{v_n}\overline{\sigma}_h(g)}\right)^{\frac{1}{{}_{v_n}\rho_h(g)}}\Bigg\}.\end{aligned}$$

The conclusion of the above theorem can be carried out from pairwise inequalities numbered (2) and (14); (2) and (15); (2) and (7); (2) and (6) respectively after applying the same technique of Theorem 2 and with the help of Theorem 1. Therefore its proof is omitted.

Similarly in the line of Theorem 2 and with the help of Theorem 1, one may easily carry out the following theorem from pairwise inequalities numbered (9) and (13); (9) and (16); (6) and (11) respectively and therefore its proof is omitted:

Theorem 8. *Let f, g and h be any three entire functions of several complex variables such that $0 < {}_{v_n}\lambda_h(f,\varphi) < \infty$ and $0 < {}_{v_n}\lambda_h(g) \leq {}_{v_n}\rho_h(g) < \infty$. Then*

$$\max\left\{\left(\frac{{}_{v_n}\tau_h(f,\varphi)}{{}_{v_n}\tau_h(g)}\right)^{\frac{1}{{}_{v_n}\lambda_h(g)}}, \left(\frac{{}_{v_n}\overline{\tau}_h(f,\varphi)}{{}_{v_n}\overline{\tau}_h(g)}\right)^{\frac{1}{{}_{v_n}\lambda_h(g)}}\right\}$$

$$\leq {}_{v_n}\tau_g(f,\varphi) \leq \left(\frac{{}_{v_n}\tau_h(f,\varphi)}{{}_{v_n}\overline{\sigma}_h(g)}\right)^{\frac{1}{{}_{v_n}\rho_h(g)}} .$$

Theorem 9. *Let f, g and h be any three entire functions of several complex variables such that $0 < {}_{v_n}\lambda_h(f,\varphi) \leq {}_{v_n}\rho_h(f,\varphi) < \infty$ and $0 < {}_{v_n}\lambda_h(g) \leq {}_{v_n}\rho_h(g) < \infty$. Then*

$$ {}_{v_n}\tau_g(f,\varphi) \geq \max\left\{ \left(\frac{{}_{v_n}\overline{\sigma}_h(f,\varphi)}{{}_{v_n}\overline{\sigma}_h(g)}\right)^{\frac{1}{{}_{v_n}\rho_h(g)}}, \left(\frac{{}_{v_n}\sigma_h(f,\varphi)}{{}_{v_n}\sigma_h(g)}\right)^{\frac{1}{{}_{v_n}\rho_h(g)}}, \right.$$
$$\left. \left(\frac{{}_{v_n}\sigma_h(f,\varphi)}{{}_{v_n}\tau_h(g)}\right)^{\frac{1}{{}_{v_n}\lambda_h(g)}}, \left(\frac{{}_{v_n}\overline{\sigma}_h(f,\varphi)}{{}_{v_n}\overline{\tau}_h(g)}\right)^{\frac{1}{{}_{v_n}\lambda_h(g)}} \right\} .$$

The conclusion of the above theorem can be carried out from pairwise inequalities numbered (8) and (9); (5) and (9); (9) and (13); (9) and (16) respectively after applying the same technique of Theorem 2 and with the help of Theorem 1. Therefore its proof is omitted.

Now we state the following two theorems without their proofs as because those can be derived easily using the same technique or with some easy reasoning with the help of Remark 1 and therefore left to the readers.

Theorem 10. *Let f, g and h be any three entire functions of several complex variables such that $0 < {}_{v_n}\rho_h(f,\varphi) < \infty$ and $0 < {}_{v_n}\rho_h(g)\,(= {}_{v_n}\lambda_h(g)) < \infty$. Then*

$$\left(\frac{{}_{v_n}\overline{\sigma}_h(f,\varphi)}{{}_{v_n}\sigma_h(g)}\right)^{\frac{1}{{}_{v_n}\rho_h(g)}} \leq {}_{v_n}\overline{\sigma}_g(f,\varphi)$$
$$\leq \min\left\{ \left(\frac{{}_{v_n}\overline{\sigma}_h(f,\varphi)}{{}_{v_n}\overline{\sigma}_h(g)}\right)^{\frac{1}{{}_{v_n}\rho_h(g)}}, \left(\frac{{}_{v_n}\sigma_h(f,\varphi)}{{}_{v_n}\sigma_h(g)}\right)^{\frac{1}{{}_{v_n}\rho_h(g)}} \right\}$$
$$\leq \max\left\{ \left(\frac{{}_{v_n}\overline{\sigma}_h(f,\varphi)}{{}_{v_n}\overline{\sigma}_h(g)}\right)^{\frac{1}{{}_{v_n}\rho_h(g)}}, \left(\frac{{}_{v_n}\sigma_h(f,\varphi)}{{}_{v_n}\sigma_h(g)}\right)^{\frac{1}{{}_{v_n}\rho_h(g)}} \right\}$$
$$\leq {}_{v_n}\sigma_g(f,\varphi) \leq \left(\frac{{}_{v_n}\sigma_h(f,\varphi)}{{}_{v_n}\overline{\sigma}_h(g)}\right)^{\frac{1}{{}_{v_n}\rho_h(g)}} .$$

Remark 2. In Theorem 10, if we will replace the conditions " $0 < {}_{v_n}\rho_h(f,\varphi) < \infty$ and $0 < {}_{v_n}\rho_h(g)\,(= {}_{v_n}\lambda_h(g)) < \infty$" by " $0 < {}_{v_n}\rho_h(f,\varphi)\,(= {}_{v_n}\lambda_h(f,\varphi)) < \infty$ and $0 < {}_{v_n}\rho_h(g) < \infty$" respectively, then Theorem 10 remains valid with ${}_{v_n}\overline{\tau}_g\left(f,\varphi\right)$ and ${}_{v_n}\tau_g\left(f,\varphi\right)$ replacing ${}_{v_n}\overline{\sigma}_g\left(f,\varphi\right)$ and ${}_{v_n}\sigma_g\left(f,\varphi\right)$ respectively.

Theorem 11. *Let f, g and h be any three entire functions of several complex variables such that* $0 < {}_{v_n}\rho_h\left(f,\varphi\right)\left(= {}_{v_n}\lambda_h\left(f,\varphi\right)\right) < \infty$ *and* $0 < {}_{v_n}\lambda_h\left(g\right) < \infty$. *Then*

$$\begin{aligned}\left(\frac{{}_{v_n}\overline{\tau}_h\left(f,\varphi\right)}{{}_{v_n}\tau_h\left(g\right)}\right)^{\frac{1}{{}_{v_n}\lambda_h(g)}} &\leq {}_{v_n}\overline{\sigma}_g\left(f,\varphi\right)\\ &\leq \min\left\{\left(\frac{{}_{v_n}\overline{\tau}_h\left(f,\varphi\right)}{{}_{v_n}\overline{\tau}_h\left(g\right)}\right)^{\frac{1}{{}_{v_n}\lambda_h(g)}}, \left(\frac{{}_{v_n}\tau_h\left(f,\varphi\right)}{{}_{v_n}\tau_h\left(g\right)}\right)^{\frac{1}{{}_{v_n}\lambda_h(g)}}\right\}\\ &\leq \max\left\{\left(\frac{{}_{v_n}\overline{\tau}_h\left(f,\varphi\right)}{{}_{v_n}\overline{\tau}_h\left(g\right)}\right)^{\frac{1}{{}_{v_n}\lambda_h(g)}}, \left(\frac{{}_{v_n}\tau_h\left(f,\varphi\right)}{{}_{v_n}\tau_h\left(g\right)}\right)^{\frac{1}{{}_{v_n}\lambda_h(g)}}\right\}\\ &\leq {}_{v_n}\sigma_g\left(f,\varphi\right) \leq \left(\frac{{}_{v_n}\tau_h\left(f,\varphi\right)}{{}_{v_n}\overline{\tau}_h\left(g\right)}\right)^{\frac{1}{{}_{v_n}\lambda_h(g)}}.\end{aligned}$$

Remark 3. In Theorem 11, if we will replace the conditions " $0 < {}_{v_n}\rho_h\left(f,\varphi\right)$ $\left(= {}_{v_n}\lambda_h\left(f,\varphi\right)\right) < \infty$ and $0 < {}_{v_n}\lambda_h\left(g\right) < \infty$" by " $0 < {}_{v_n}\lambda_h\left(f,\varphi\right) < \infty$ and $0 < {}_{v_n}\rho_h\left(g\right)\left(= {}_{v_n}\lambda_h\left(g\right)\right) < \infty$" respectively, then Theorem 11 remains valid with ${}_{v_n}\overline{\tau}_g\left(f,\varphi\right)$ and ${}_{v_n}\tau_g\left(f,\varphi\right)$ replacing ${}_{v_n}\overline{\sigma}_g\left(f,\varphi\right)$ and ${}_{v_n}\sigma_g\left(f,\varphi\right)$ respectively.

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