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ON WEAKLY LOCAL RINGS

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Abstract. This article concerns a property of local rings and domains. A ring R is called *weakly local* if for every $a \in R$, a is regular or $1-a$ is regular, where a regular element means a non-zero-divisor. We study the structure of weakly local rings in relation to several kinds of factor rings and ring extensions that play roles in ring theory. We prove that the characteristic of a weakly local ring is either zero or a power of a prime number. It is also shown that the weakly local property can go up to polynomial (power series) rings and a kind of Abelian matrix rings.

Preliminary

Throughout this paper all rings are associative with identity unless otherwise stated. Let R be a ring. We use $C(R)$ and $U(R)$ to denote the monoid of regular elements and the group of units in R, respectively. Let $J(R)$, $I(R)$, $N^*(R)$ and $N(R)$ denote the Jacobson radical, the set of all idempotents, the upper nilradical and the set of all nilpotent elements in R, respectively. $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n), and let $\mathbb{Q}$ be the field of rational numbers.. Denote the n by n full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $T_n(R)$). I_n denotes the identity matrix in $Mat_n(R)$, and E_{ij} denotes the matrix with (i,j)-entry 1 and elsewhere 0, and write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn} =$

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$0\}$. For $S \subseteq R$, the right (resp., left) annihilator of S is R is denoted by $r_R(S)$ (resp., $l_R(S)$); that is $l_R(S) = \{r \in R \mid rs = 0 \text{ for all } s \in S\}$ and $r_R(S) = \{r \in R \mid sr = 0 \text{ for all } s \in S\}$. If $S = \{a\}$ then we write $l_R(a)$ (resp., $r_R(a)$). An element $a \in R$ is said to be right (resp., left) regular in R if $r_R(a) = 0$ (resp., $l_R(a) = 0$). An element is called regular if it is both right and left regular. The characteristic of R is denoted by $ch(R)$. Given a set S, the cardinality of S is expressed by $|S|$.

A ring is usually called *reduced* if it has no nonzero nilpotent elements. Due to Feller [1], a ring is called *right* (resp. *left*) duo if every right (resp. left) ideal is an ideal; a ring is called *duo* if it is both right and left duo. A ring is called *Abelian* if every idempotent is central. It is easily shown that both one-side duo rings and reduced rings are Abelian. Following Hong et al. [3], a ring R is called *right* (resp., *left*) *DR*) if $C(R)a \subseteq aC(R)$ (resp., $aC(R) \subseteq C(R)a$) for all $a \in R$; and a ring is called *DR* if it is both left and right DR. So a ring R is DR if and only if $C(R)a = aC(R)$ for all $a \in R$. A ring R is clearly DR when $C(R) \subseteq Z(R)$. Right DR rings are also Abelian by [3, Lemma 2.1(1)]. A ring R is called *local* if $R/J(R)$ is a division ring. Every local ring R is Abelian because for every $a \in R$, $a \in U(R)$ of $1 - a \in U(R)$.

1. Weakly local rings

We first deal with a ring property that is satisfied by domains and local rings. A ring R shall be called *weakly local* if for every $a \in R$, $a \in C(R)$ or $1 - a \in C(R)$. The following does an important role throughout this article.

LEMMA 1.1. (1) *The class of weakly local rings contains domains and local rings.*

(2) *If R is a weakly local ring then $I(R) = \{0, 1\}$.*

(3) *Every weakly local ring is Abelian.*

(4) *The class of weakly local rings is closed under subring with the inherited identity.*

(5) *Commutative rings need not be weakly local.*

Proof. (1) Domains are clearly weakly local. Let R be a local ring and $a \in R$. Then $a \in U(R)$ or $1 - a \in U(R)$; hence R is weakly local, since $U(R) \subseteq C(R)$.

(2) Let R be a weakly local ring and $e \in I(R)$. Then $e \in C(R)$ or $1 - e \in C(R)$; hence $e = 0$ or $e = 1$, noting that 1 is the only regular idempotent.

(3) is an immediate consequence of (2).

(4) Let R be a weakly local ring and S be a subring of R with the identity of R. Suppose $a \notin C(S)$. Then $a \notin C(R)$. Since R is weakly local, $1 - a \in C(R)$ and $1 - a \in C(S)$ follows. Thus S is weakly local.

(5) Consider the ring $\mathbb{Z}_6$ and take $3 \in \mathbb{Z}_6$. Then $3 \notin C(\mathbb{Z}_6)$ and $1 - 3 = 4 \notin C(\mathbb{Z}_6)$. □

From Lemma 1.1, we obtain that $I(R) = \{0, 1\}$ (hence R is Abelian) for a local ring R, and that commutative rings need not be local. In the following example we see relations between weakly local rings and related ring properties.

Example 1.2. (1) We refer to [9, Theorem 1.3.5, Corollary 2.1.14, and Theorem 2.1.15]. Let K be a field of characteristic zero and $A = K\langle x, y\rangle$ be the free algebra with noncommuting indetermiantes x, y over K. Let I be the ideal of A generated by $yx - xy - 1$, and set $R = A/I$. Then R is a noncommutative domain and so R is weakly local by Lemma 1.1(1). But R is neither right nor left duo as can be seen by the one-sided ideals xR and Rx which cannot be two-sided. Moreover R is neither right nor left DR by [3, Lemma 2.1(3)].

(2) The weakly local domain R in (1) is clearly not local, noting $J(R) = 0$ by [9, Theorem 1.3.8].

(3) The weakly local ring $D_n(R)$ over a weakly local ring R ($n \geq 2$), in Theorem 2.2 to follow, is clearly not a domain.

(4) Let $|I| \geq 2$ and R_i be rings for every $i \in I$. Then the direct product of R_i's cannot be weakly local by Lemma 1.1(2).

Right duo rings need not be right DR by [3, Example 1.4(2)]. It is not proved yet that right DR rings are right duo. The weakly local property can be a condition under which right DR can be right duo.

Proposition 1.3. *Let R be a weakly local ring. If R is right DR then R is right duo.*

Proof. We apply the proof of [3, Proposition 1.6]. Let $r, a \in R$ and consider ra. Since R is weakly local, $r \in C(R)$ or $1-r \in C(R)$. Suppose that R is right DR. If $r \in C(R)$ then $ra = ar_1 \in aR$ for some $r_1 \in C(R)$.

If $1-r \in C(R)$ then $(1-r)a = ar_2$ for some $r_2 \in C(R)$. This yields $ra = a - ar_2 = a(1-r_2) \in aR$. Thus R is right duo. □

In the following we see a more general situation of the proof of Lemma 1.1(5).

THEOREM 1.4. (1) *Let $n = p^k$ such that p is a prime number and $k \geq 1$. Then $\mathbb{Z}_n$ is (weakly) local.*

(2) *Let $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ such that $k \geq 2$, p_i is a prime number for all i, $p_i \neq p_j$ if $i \neq j$, and $m_i \geq 1$ for all i. Then $\mathbb{Z}_n$ is not weakly local.*

(3) *Let R be a ring of $ch(R) = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ such that $k \geq 2$, p_i is a prime number for all i, $p_i \neq p_j$ if $i \neq j$, and $m_i \geq 1$ for all i. Then R is not weakly local.*

Proof. (1) is easily verified.

(2) $\mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \cdots \mathbb{Z}_{p_k^{m_k}}$. So $\mathbb{Z}_n$ contains an idempotent that is neither zero not the identity. Thus $\mathbb{Z}_n$ is not weakly local by Lemma 1.1(2).

(2) By hypothesis, R contains $\mathbb{Z}_n$ as a subring with the same identity, where $n = ch(R)$. But $\mathbb{Z}_n$ is not weakly local by (2), therefore R is not weakly local by Lemma 1.1(4). □

From Theorem 1.4, we obtain the following.

COROLLARY 1.5. (1) *The class of weakly local rings is not closed under factor rings.*

(2) *Let R be a weakly local ring. Then $ch(R)$ is either zero or a power of a prime number.*

Proof. (1) First note that $\mathbb{Z}$ is weakly local by Lemma 1.1(1). Let $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ such that $k \geq 2$, p_i is prime for all i, $p_i \neq p_j$ if $i \neq j$, and $m_i \geq 1$ for all i. Consider the ideal $n\mathbb{Z}$ and set $R = \mathbb{Z}/n\mathbb{Z}$. Then R is not weakly local by Theorem 1.4(2).

(2) Suppose $ch(R) \neq 0$. Then the proof is done by Theorem 1.4. □

As another example of Corollary 1.5(1), let R_0 be the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$, where p is an odd prime; and next set R be the quaternions over R_0. Then R is clearly a domain (hence weakly local) and $J(R) = pR$. But $R/J(R)$ is isomorphic to $Mat_2(\mathbb{Z}_p)$ by the argument in [2, Exercise 2A]. But $Mat_2(\mathbb{Z}_p)$ is not weakly local by Lemma 1.1(2). Thus $R/J(R))$ is not weakly local.

Compare Corollary 1.5(2) with the fact that the characteristic of a domain is either zero or a prime number. The following elaborates upon Theorem 1.4.

Remark 1.6. For a given ring R, identify $s \in \mathbb{Z}$ ($\mathbb{Z}_n$) with $s \cdot 1$ in R, where 1 is the identity of R.

(1) Let R be the ring in Theorem 1.4(2), i.e., $R = \mathbb{Z}/n\mathbb{Z}$ with $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ such that $k \geq 2$, p_i is a prime number for all i, $p_i \neq p_j$ if $i \neq j$, and $m_i \geq 1$ for all i. Consider $a = p_i^{m_i}$ for $i \in \{1, 2, \ldots, k\}$. Since R is finite, there exist $s, t \geq 1$ such that $a^s = a^{s+t}$. Then $a^{st} \in I(R)$ by the proof of [5, Proposition 16]. Note that $a^{st} = p_i^{m_i st}$ is nonzero since $k \geq 2$, and moreover $a^{st} \neq 1$ since $a^{st}(p_1^{m_1} \cdots p_{i-1}^{m_{i-1}} p_{i+1}^{m_{i+1}} \cdots p_k^{m_k}) = 0$, noting $p_1^{m_1} \cdots p_{i-1}^{m_{i-1}} p_{i+1}^{m_{i+1}} \cdots p_k^{m_k} \neq 0$.

(2) Let R be a ring of $ch(R) = m_1 m_2 \cdots m_k$ with $k \geq 2$ and $m_i \geq 2$ for all i. Suppose $m_{j+1} = m_j + 1$ for some $1 \leq j \leq k-1$. Note $m_{j+1} \notin C(R)$. Consider $l = 1 - m_{j+1}$. Since $l = l + 0 = l + m_1 m_2 \cdots m_k$, we get

$$l = 1-(m_j+1) = -m_j + m_1 m_2 \cdots m_k = m_j(m_1 \cdots m_{j-1} m_{j+1} \cdots m_k - 1)$$

and hence we have

$$\begin{aligned} & l(m_1 \cdots m_{j-1} m_{j+1} \cdots m_k) \\ =& [m_j(m_1 \cdots m_{j-1} m_{j+1} \cdots m_k - 1)](m_1 \cdots m_{j-1} m_{j+1} \cdots m_k) \\ =& [m_j(m_1 \cdots m_{j-1} m_{j+1} \cdots m_k)](m_1 \cdots m_{j-1} m_{j+1} \cdots m_k - 1) \\ =& n(m_1 \cdots m_{j-1} m_{j+1} \cdots m_k - 1) = 0, \end{aligned}$$

where $n = m_1 m_2 \cdots m_k$. So $1 - m_{j+1} \notin C(R)$ since noting $m_1 \cdots m_{j-1} m_{j+1} \cdots m_k \neq 0$, and therefore R is not weakly local.

Recall that a ring R is called *semilocal* if $R/J(R)$ is semisimple Artinian, and that a ring R is called *semiperfect* if R is semilocal and idempotents can be lifted modulo $J(R)$. Local rings are semiperfect by [8, Corollary 3.7.1].

Proposition 1.7. *A ring R is weakly local and semiperfect if and only if R is local.*

Proof. Let R be weakly local and semiperfect. Since R is semiperfect, R has a finite orthogonal set $\{e_1, e_2, \ldots, e_n\}$ of local idempotents whose sum is 1 by [8, Proposition 3.7.2], say $R = \prod_{i=1}^{n} e_i R$ such that each $e_i R e_i$ is a local ring. But, Lemma 1.1(2), we must get $\{e_1, \ldots, e_n\} = \{1\}$ since

R is weakly local, so that R is local. The converse is shown by [8, Corollary 3.7.1]. □

It is easily checked that for a local ring R, $J(R)$ contains $N(R)$. So one may ask whether for a weakly local ring R, $J(R)$ contains $N(R)$. But the answer is negative by the following.

EXAMPLE 1.8. We deal with a subring of $Mat_n(\mathbb{Z}_2)[x]$ for $n \geq 2$. Set

$$R = \mathbb{Z}_2 + xMat_n(\mathbb{Z}_2)[x].$$

Let $f(x) \in R$. Note that $1+g(x) \in C(R)$ for all $g(x) \in xMat_n(\mathbb{Z}_2)[x]$ by the argument in the proof of Theorem 2.3 to follow. So $f(x) \in C(R)$ or $1 - f(x) \in C(R)$. Thus R is weakly local.

Next we claim $J(R) = 0$. Letting $f_1(x) = 1 + g_1(x)$ with $g_1(x) \in xMat_n(\mathbb{Z}_2)[x]$, $1 - f_1(x) = -g_1(x) \notin U(R)$, so that $f_1(x) \notin J(R)$. This yields $J(R) \subseteq xMat_n(\mathbb{Z}_2)[x]$. Let $h(x) = \sum_{i=1}^{m} a_i x^i \in xMat_n(\mathbb{Z}_2)[x]$ with $a_m \neq 0$. Since $Mat_n(\mathbb{Z}_2)$ is simple, we have $Rh(x)R$ contains

$$\{b_1x + \cdots + b_{m+1}x^{m+1} + x^{m+2} \mid b_i \in Mat_n(\mathbb{Z}_2)\}$$

because $R_0a_mR_0 = Mat_n(\mathbb{Z}_2)x^2$, where $R_0 = Mat_n(\mathbb{Z}_2)x$. But

$$1 - (b_1x + \cdots + b_{m+1}x^{m+1} + x^{m+2}) \notin U(R),$$

entailing that $b_1x+\cdots+b_{m+1}x^{m+1}+x^{m+2}$ is not contained in $J(R)$. Thus result implies $h(x) \notin J(R)$, concluding $J(R) = 0$. However $N(R) \neq 0$ as can be seen by the nilpotent polynomial $E_{12}x$ in R.

2. Extensions of weakly local rings

In this section we study the weakly local property of some kinds of ring extensions that play important roles in ring theory.

LEMMA 2.1. [6, Lemma 2.1] *Let R be a ring and $0 \neq a \in R$. Then a is right (resp., left) regular if and only if $(a_{ij}) \in D_n(R)$ is right (resp., left) regular, where $a_{ii} = a$.*

We obtain the following useful information about the weakly local property.

THEOREM 2.2. *Let $n \geq 2$. A ring R is weakly local if and only if $D_n(R)$ is weakly local.*

Proof. Let R be a weakly local ring and $A = (a_{ij}) \in D_n(R)$. Assume $(a_{ij}) \notin C(D_n(R))$. Then $a_{ii} \notin C(R)$ by Lemma 2.1. Since R is weakly local, $1-a_{ii} \in C(R)$. This yields $I_n - A = (b_{ij})$ such that $b_{ii} = 1-a_{ii}$ and $b_{ij} = -a_{ij}$ for all i, j with $i < j$. Then $I_n - A$ is regular by Lemma 2.1. So $D_n(R)$ is weakly local. The converse is proved by Lemma 1.1(4). □

Note that $I(D_n(R)) = \{0, I_n\}$ over the weakly local ring R by Lemma 1.1(2) and [4, Lemma 2]. However neither $Mat_n(R)$ nor $T_n(R)$ cannot be weakly local over any ring R by Lemma 1.1(3) for $n \geq 2$.

THEOREM 2.3. *For a ring R the following conditions are equivalent:*
(1) *R is weakly local;*
(2) *$R[[x]]$ is weakly local;*
(3) *$R[x]$ is weakly local.*

Proof. Let $f(x) = \sum_{i=s}^{\infty} a_i x^i \in R[[x]]$ with $s \geq 0$. If $a_s \in C(R)$ then $f(x) \in C(R[[x]])$. For, letting $f(x)g(x) = 0$ for $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x]]$, we first get $a_s b_0 = 0$ since $a_s \in C(R)$ and $b_0 = 0$ follows; and inductively we have $b_j = 0$ for all $j \geq 0$. Letting $g(x)f(x) = 0$, we get $g(x) = 0$ similarly. Thus

$$\{\sum_{i=s}^{\infty} a_i x^i \in R[[x]] \mid a_s \in C(R)\} \subseteq C(R[[x]]),$$

where $s \geq 0$. A similar argument yields

$$\{\sum_{i=s}^{m} a_i x^i \in R[x] \mid a_s \in C(R) \text{ or } a_m \in C(R)\} \subseteq C(R[x]),$$

where $0 \leq s \leq m$.

(1) ⇒ (2). Let R be weakly local and $f(x) = \sum_{i=s}^{\infty} a_i x^i \in R[[x]]$ with $a_s \neq 0$. Assume $f(x) = \sum_{i=s}^{\infty} a_i x^i \notin C(R[[x]])$. Then $a_s \notin C(R)$ by the above argument. Suppose $s = 0$. Since R is weakly local, $1 - a_s \in C(R)$ and so $1 - f(x) = (1-a_0) - \sum_{i=1}^{\infty} a_i x^i \in C(R[[x]]$ by the above argument. Suppose $s \geq 1$. Then $1 - f(x) = 1 - \sum_{i=s}^{\infty} a_i x^i \in U(R[[x]])$. Thus $R[[x]]$ is weakly local.

(2) ⇒ (3) and (3) ⇒ (1) are proved by Lemma 1.1(4). □

In Theorem 2.3, $I(R[[x]]) = I(R[x]) = I(R) = \{0, 1\}$ by Lemma 1.1(2) and [7, Lemma 8], implying that $R[[x]]$ and $R[x]$ are also Abelian. Recall that domains and local rings are both weakly local. Domains clearly pass to polynomial rings, but this is not valid for local rings. Letting D be a division ring, $D[x]$ is not local (since $J(D[x]) = 0$).

Let R be a ring. $R[x; x^{-1}]$ denotes the *Laurent polynomial* ring in x over R, i.e., every element of $R[x; x^{-1}]$ has a unique representation in the form $\sum_{i\in\mathbb{Z}} a_i x^i$ with all but finitely many coefficients being zero. $R[[x; x^{-1}]]$ denotes the *Laurent power series* ring in x over R, i.e., every element of $R[[x; x^{-1}]]$ has a unique representation in the form $\sum_{i\in\mathbb{Z}} a_i x^i$ with $a_{-n} = 0$ for all but finitely many $n \geq 1$.

Remark 2.4. Let R be a ring. Applying the argument in the proof of Theorem 2.3, we obtain

$$\{\sum_{i=s}^{\infty} a_i x^i \in R[[x; x^{-1}]] \mid a_s \in C(R)\} \subseteq C(R[[x; x^{-1}]])$$

and

$$\{\sum_{i=t}^{m} a_i x^i \in R[x; x^{-1}] \mid a_t \in C(R) \text{ or } a_m \in C(R)\} \subseteq C(R[x; x^{-1}]),$$

where $s \in \mathbb{Z}$ and $t \leq m \in \mathbb{Z}$.

Let R be weakly local and suppose that $f(x) = \sum_{i=s}^{\infty} a_i x^i \notin C(R[[x; x^{-1}]])$. Then $a_s \notin C(R)$ by the preceding argument. Since R is weakly local, $1 - a_s \in C(R)$. So $x^s - f(x) = (1 - a_s)x^s + \sum_{i=s+1}^{\infty}(-a_i)x^i \in C(R[[x; x^{-1}]])$. Next suppose that $g(x) = \sum_{i=t}^{m} a_i x^i \notin C(R[x; x^{-1}])$. Then $a_t, a_m \notin C(R)$ by the above argument. Since R is weakly local, $1 - a_t, 1 - a_m \in C(R)$. So $x^t - g(x) = (1 - a_t)x^t + \sum_{i=t+1}^{m}(-a_i)x^i, x^m - g(x) = \sum_{i=t}^{m-1}(-a_i)x^i + (1 - a_m)x^m \in C(R[x; x^{-1}])$.

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