Korean J. Math. **28** (2020), No. 1, pp. 49–64
http://dx.doi.org/10.11568/kjm.2020.28.1.49

FUZZY PRIME SPECTRUM OF C-ALGEBRAS

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Abstract. In this paper, we define fuzzy prime ideals of $C-$algebras and investigate some of their properties. Furthermore, we study the topological properties of the space of fuzzy prime ideals of $C-$algebra equipped with the hull-kernel topology.

1. Introduction

Guzman and Squier [9] introduced the variety of C-algebras as the variety generated by the three-element algebra $C = \{T, F, U\}$ with the operations "$\wedge$"; "$\vee$" and "$'$" of type $(2, 2, 1)$, which is the algebraic form of the three-valued conditional logic. They proved that C and the two element Boolean algebra $B = \{T, F\}$ are the only subdirectly irreducible $C-$algebras and that the variety of $C-$algebras is a minimal cover of the variety of Boolean algebras. Many more results on the structure of $C-$algebras can be found in literature (see [12, 13, 15, 16, 18–20]).

The concept of fuzzy sets was first introduced by Zadeh [22] and this concept was adapted by Rosenfeld [14] to define fuzzy subgroups. Since then, many authors have been studying fuzzy subalgebras of several algebraic structures (see [1–6, 10, 11, 17]). In [1], we have introduced the notion of fuzzy ideals of $C-$algebras and investigate some of their properties. In the present paper, we continue our study and define fuzzy

Received October 2, 2019. Revised December 30, 2019. Accepted January 2, 2020.

2010 Mathematics Subject Classification: 03E72, 08A72, 20A99.

Key words and phrases: C-algebras; fuzzy ideals; fuzzy fuzzy prime ideals; fuzzy prime spectrum.

prime ideals in $C-$algebras. Mainly, we give an internal characterization for fuzzy prime ideals of $C-$algebras analogous to the well known characterization theorem of Swamy and Swamy [17] in the case of rings. In addition, we study the topological properties of the space of fuzzy prime ideals of $C-$algebra equipped with the hull-kernel topology, which we call it the fuzzy prime spectrum.

2. Preliminaries

In this section, we recall some definitions and basic results which will be used in the paper.

DEFINITION 2.1. [9] An algebra $(A, \vee, \wedge, ')$ of type $(2, 2, 1)$ is called a C-algebra, if it satisfies the following axioms:
(1) $a'' = a$,
(2) $(a \wedge b)' = a' \vee b'$,
(3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$,
(4) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,
(5) $(a \vee b) \wedge c = (a \wedge c) \vee (a' \wedge b \wedge c)$,
(6) $a \vee (a \wedge b) = a$,
(7) $(a \wedge b) \vee (b \wedge a) = (b \wedge a) \vee (a \wedge b)$ for all $a, b, c \in A$.

Throughout this paper A denotes a $C-$algebra $(A, \vee, \wedge, ')$ unless and otherwise stated.

EXAMPLE 2.2. [9] The three element algebra $C = \{T, F, U\}$ with the operations given by the following tables is a C-algebra.

$\vee$	T	F	U
T	T	T	T
F	T	F	U
U	U	U	U

$\wedge$	T	F	U
T	T	F	U
F	F	F	F
U	U	U	U

x	x'
T	F
F	T
U	U

Note: [9] The identities 2.1(1) and 2.1(2) imply that the variety of C-algebras satisfies all the dual statements of 2.1(2) to 2.1(7).

DEFINITION 2.3. [9] An element z of a C-algebra A is called a left zero for $\wedge$ if $z \wedge x = z$ for all $x \in A$.

DEFINITION 2.4. [18] A nonempty subset I of a C-algebra A is called an ideal of A, if

1. $a, b \in I \Rightarrow a \vee b \in I$ and

2. $a \in I \Rightarrow x \wedge a \in I$, for each $x \in A$.

It is observed that $a \wedge b \in I$ if and only if $b \wedge a \in I$ for all $a, b \in A$. For any subset $S \subseteq A$, the smallest ideal of A containing S is called the ideal of A generated by S and is denoted by $\langle S \rangle$. Note that:

$$\langle S \rangle = \{\bigvee (y_i \wedge x_i) : y_i \in A, x_i \in S, i = 1, ..., n \text{ for some } n \in Z_+\}$$

If $S = \{a\}$ then we write $\langle a \rangle$ for $\langle S \rangle$. In this case $\langle a \rangle = \{x \wedge a : x \in A\}$. Moreover, it is observed in [18] that the set $I_0 = \{x \wedge x' : x \in A\}$ is the smallest ideal in A.

Definition 2.5. [20] A proper ideal P of A is called a prime ideal, if for any $a, b \in A$,

$$a \wedge b \in P \Rightarrow a \in P \text{ or } b \in P$$

It is observed in [20] that a proper ideal P of A is prime if and only if

$$I \cap J \subseteq P \Rightarrow I \subseteq P \text{ or } J \subseteq P$$

for all ideals I and J of A.

Remember that, for any set A, a function $\mu : A \to [0, 1]$ is called a fuzzy subset of A. For each $t \in [0, 1]$ the set

$$\mu_t = \{x \in A : \mu(x) \geq t\}$$

is called the level subset of μ at t [22]. For numbers α and β in $[0, 1]$ we write $\alpha \wedge \beta$ instead of $min\{\alpha, \beta\}$ and $\alpha \vee \beta$ for $max\{\alpha, \beta\}$. We call a fuzzy subset μ of A, nonzero if there is some $x \in A$ such that $\mu(x) \neq 0$. We denote by 0_A and 1_A; fuzzy subsets of A defined by:

$$0_A(x) = 0 \text{ and } 1_A(x) = 1 \text{ for all } x \in A.$$

Definition 2.6. [1] A fuzzy subset μ of A is called a fuzzy ideal of A if:

1. $\mu(z) = 1$, for all $z \in I_0$
2. $\mu(a \vee b) \geq \mu(a) \wedge \mu(b)$
3. $\mu(a \wedge b) \geq \mu(b)$

for all $a, b \in A$.

We denote the class of all fuzzy ideals of A by $FI(A)$.

Lemma 2.7. [1] *Let μ be a fuzzy ideal of A. Then the following hold for all $a, b \in A$.*

1. $\mu(a \wedge b) \geq \mu(a)$
2. $\mu(a \wedge b) = \mu(b \wedge a)$
3. $\mu(a \wedge x \wedge b) \geq \mu(a \wedge b)$ *for each* $x \in A$
4. $\mu(a) \geq \mu(a \vee b)$ *and hence* $\mu(a) \wedge \mu(b) = \mu(a \vee b) \wedge \mu(b \vee a)$
5. If $x \in \langle a \rangle$, *then* $\mu(x) \geq \mu(a)$.

It is also observed that the fuzzy subset μ^0 of A defined by:

$$\mu^0(x) = \begin{cases} 1 & \text{if } x \in I_0 \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in A$, is the smallest fuzzy ideal of A. We refer to [7, 8] for the standard concepts on lattice theory and general universal algebras.

3. Fuzzy Prime Ideals

In this section, we define fuzzy prime ideals in $C-$algebras and investigate some of their properties.

DEFINITION 3.1. A fuzzy ideal μ of A is called a fuzzy prime ideal of A if the following holds for all fuzzy ideals σ and θ of A:

$$\sigma \cap \theta \subseteq \mu \Rightarrow \sigma \subseteq \mu \text{ or } \theta \subseteq \mu.$$

We denote the class of all fuzzy prime ideals of A by $FP(A)$. The following lemmas immediately follow from the definition.

LEMMA 3.2. *A non-constant fuzzy ideal* μ *of* A *is fuzzy prime if and only if it satisfies the following:*

$$\sigma \cap \theta = \mu \Rightarrow \sigma = \mu \text{ or } \theta = \mu$$

for all $\sigma, \theta \in FI(A)$.

LEMMA 3.3. *Let* μ *and* θ *be fuzzy prime ideals of* A*. Then* $\mu \cap \theta$ *is a prime fuzzy ideal of* A *if and only if either* $\mu \subseteq \theta$ *or* $\theta \subseteq \mu$.

For each $x \in A$ and $\alpha \in (0, 1]$ remember from [21] that, the fuzzy subset x_α of A given by:

$$x_\alpha(z) = \begin{cases} \alpha & \text{if } z = x \\ 0 & \text{otherwise} \end{cases}$$

for all $z \in A$, is called a fuzzy point of A. In this case x is called the support of x_α and α its value. For a fuzzy point x_α of A and a fuzzy subset μ of A we write $x_\alpha \in \mu$ to say that $\mu(x) \geq \alpha$. Moreover, for fuzzy subsets μ and σ of A, $\mu \wedge \sigma$ is a fuzzy subset of A given by:

$$(\mu \wedge \sigma)(x) = Sup\{\mu(y) \wedge \sigma(z) : y \wedge z = x\}$$

for all $x \in A$. If μ and σ are fuzzy ideals, then $\mu \wedge \sigma = \mu \cap \sigma$. In the following we give an equivalent characterization for prime fuzzy ideals in terms of fuzzy points.

THEOREM 3.4. *A non-constant fuzzy ideal μ of A is a fuzzy prime ideal if and only if for any fuzzy points x_α and y_β of A:*

$$x_\alpha \wedge y_\beta \in \mu \Rightarrow x_\alpha \in \mu \text{ or } y_\beta \in \mu.$$

Proof. Suppose that μ is fuzzy prime and let x_α and y_β be fuzzy points of A such that $x_\alpha \wedge y_\beta \in \mu$. Then $\langle x_\alpha \wedge y_\beta \rangle \subseteq \langle \mu \rangle = \mu$. For each$z \in A$, we have

$$(x_\alpha \wedge y_\beta)(z) = \begin{cases} \alpha \wedge \beta & \text{if } z = x \wedge y \\ 0 & \text{otherwise} \end{cases}$$

and

$$\langle x_\alpha \wedge y_\beta \rangle(z) = \begin{cases} 1 & \text{if } z \in I_0 \\ \alpha \wedge \beta & \text{if } z \in \langle x \wedge y \rangle - I_0 \\ 0 & \text{otherwise} \end{cases}$$

Since $\langle x \rangle \cap \langle y \rangle = \langle x \wedge y \rangle$ (see Lemma 3.4 of [18]), we get $\langle x_\alpha \rangle \cap \langle y_\beta \rangle = \langle x_\alpha \wedge y_\beta \rangle$; that is, $\langle x_\alpha \rangle \cap \langle y_\beta \rangle \subseteq \mu$. Since μ is fuzzy prime either $\langle x_\alpha \rangle \subseteq \mu$ or $\langle y_\beta \rangle \subseteq \mu$ which gives that either $x_\alpha \in \mu$ or $y_\beta \in \mu$. Conversely, suppose that for any fuzzy points x_α and y_β of A:

$$x_\alpha \wedge y_\beta \in \mu \Rightarrow x_\alpha \in \mu \text{ or } y_\beta \in \mu$$

Let σ and θ be fuzzy ideals of A such that $\sigma \cap \theta \subseteq \mu$. Suppose on the contrary that $\sigma \nsubseteq \mu$ and $\theta \nsubseteq \mu$. Then there exist $x, y \in A$ such that $\sigma(x) > \mu(x)$ and $\theta(y) > \mu(y)$. If we put $\alpha = \sigma(x)$ and $\beta = \theta(y)$, then x_α and y_β are fuzzy points of A such that $x_\alpha \notin \mu$ and $y_\beta \notin \mu$. By our assumption we get $x_\alpha \wedge y_\beta \notin \mu$; that is, $\mu(x \wedge y) < \alpha \wedge \beta$. Now consider the following:

$$\begin{aligned} \mu(x \wedge y) &\geq (\sigma \cap \theta)(x \wedge y) \\ &= \sigma(x \wedge y) \wedge \theta(x \wedge y) \\ &\geq \sigma(x) \wedge \theta(y) \\ &= \alpha \wedge \beta \end{aligned}$$

which is a contradiction. Therefore μ is fuzzy prime. $\square$

In the following theorem we give an internal characterization for fuzzy prime ideals in $C-$algebra analogous to the well known characterization of Swamy and Swamy [17] in the case of rings.

THEOREM 3.5. *A non-constant fuzzy ideal μ of A is a prime fuzzy ideal if and only if $Img(\mu) = \{1, \alpha\}$, where $\alpha \in [0, 1)$ and the set $\mu_* = \{x \in A : \mu(x) = 1\}$ is a prime ideal of A.*

Proof. Suppose that μ is a prime fuzzy ideal. Clearly $1 \in Img(\mu)$ and since μ is non-constant there is some $a \in A$ such that $\mu(a) < 1$. We show that $\mu(a) = \mu(b)$ for all $a, b \in A - \mu_*$. Suppose not, then there exist $a, b \in A - \mu_*$ such that $\mu(a) \neq \mu(b)$. Without loss of generality we can assume that $\mu(b) < \mu(a) < 1$. Define fuzzy subsets σ and θ as follows:

$$\sigma(x) = \begin{cases} 1 & \textit{if } x \in \langle a\rangle \\ 0 & \textit{otherwise} \end{cases}$$

and

$$\theta(x) = \begin{cases} 1 & \textit{if } x \in I_0 \\ \mu(a) & \textit{otherwise} \end{cases}$$

for all $x \in A$. Then it can be easily verified that both σ and θ are fuzzy ideals of A. Let $z \in A$. If $z \in I_0$, then $(\sigma \cap \theta)(z) = 1 = \mu(z)$. If $z \in \langle a] - I_0$, then $z = x \wedge a$ for some $x \in A$ and we have:

$$\begin{aligned} (\sigma \cap \theta)(z) &= \sigma(z) \wedge \theta(z) \\ &= \mu(a) \\ &\leq \mu(z) \end{aligned}$$

Also if $z \notin \langle a]$, then $\sigma(z) = 0$. So that $\sigma \cap \theta(z) = 0 \leq \mu(z)$. Therefore $(\sigma \cap \theta) \subseteq \mu$. But we have $\sigma(a) = 1 > \mu(a)$ and $\theta(b) = \mu(a) > \mu(b)$ which is a contradiction. Thus $\mu(a) = \mu(b)$ for all $a, b \in A - \mu_*$ and hence $Img(\mu) = \{1, \alpha\}$, for some $\alpha \in [0, 1)$. Next we show that the set $\mu_* = \{x \in A : \mu(x) = 1\}$ is a prime ideal. Clearly, it is a proper ideal. Now for any ideals I and J of A consider the following:

$$\begin{aligned} I \cap J \subseteq \mu_* &\Rightarrow \chi_{(I\cap J)} \subseteq \chi_{\mu_*} \subseteq \mu \\ &\Rightarrow \chi_I \cap \chi_J \subseteq \mu \\ &\Rightarrow \chi_I \subseteq \mu \text{ or } \chi_J \subseteq \mu \\ &\Rightarrow I \subseteq \mu_* \text{ or } J \subseteq \mu_* \end{aligned}$$

Therefore μ_* is prime. Conversely, suppose that $Img(\mu) = \{1, \alpha\}$, where $\alpha \in [0, 1)$ and the set $\mu_* = \{x \in A : \mu(x) = 1\}$ is a prime ideal of A. To show that μ is fuzzy prime, Assume on the contrary that there exist fuzzy ideals σ and θ of A such that $\sigma \nsubseteq \mu$ and $\theta \nsubseteq \mu$. Then $\sigma(x) > \mu(x)$ and $\theta(y) > \mu(y)$ for some $x, y \in A$. Since $Img(\mu) = \{1, \alpha\}$, we get $1 > \mu(x) = \mu(y) = \alpha$. So, $x \notin \mu_*$ and $y \notin \mu_*$, which gives $x \wedge y \notin \mu_*$; that is, $\mu(x \wedge y) = \alpha$. Now consider the following;

$$\begin{aligned} (\sigma \cap \theta)(x \wedge y) &= \sigma(x \wedge y) \wedge \theta(x \wedge y) \\ &\geq \sigma(x) \wedge \theta(y) \\ &> \mu(x) \wedge \mu(y) \\ &= \alpha \\ &= \mu(x \wedge y) \end{aligned}$$

which is a contradiction to our assumption $\sigma \cap \theta \subseteq \mu$. Therefore μ is fuzzy prime. □

Definition 3.6. For any subset H of A and each $\alpha \in [0, 1)$ define a fuzzy subset α_H of A by:

$$\alpha_H(x) = \begin{cases} 1 & \text{if } x \in H \\ \alpha & \text{otherwise} \end{cases}$$

for all $x \in A$.

The above theorem confirms that fuzzy prime ideals of A are only of the form α_P for some prime ideal P and some $\alpha \in [0, 1)$. This establishes a one-to-one correspondence between the class of all fuzzy prime ideals of A and the collection of all pairs (P, α); where P is a prime ideal in A and $\alpha \in [0, 1)$.

Corollary 3.7. *Let P be an ideal of A and $\alpha \in [0, 1)$. Then P is a prime ideal if and only if the fuzzy subset α_P of A is a fuzzy prime ideal of A.*

Corollary 3.8. *A proper ideal P of A is prime if and only if its characteristic mapping χ_P is a prime fuzzy ideal of A.*

Lemma 3.9. *If μ is a fuzzy prime ideal of A, then*

$$\mu(a) \vee \mu(b) \geq \mu(a \wedge b)$$

for all $a, b \in A$.

Proof. We use proof by contradiction. Suppose if possible that there exist $a, b \in A$ such that

$$\mu(a) \vee \mu(b) < \mu(a \wedge b)$$

By Theorem 3.5, $Im(\mu) = \{1, \alpha\}$ for some $\alpha \in [0, 1)$. So $\mu(a) \vee \mu(b) = \alpha < 1 = \mu(a \wedge b)$. Then $a \wedge b \in \mu_*$, $a \notin \mu_*$ and $b \notin \mu_*$. Again by Theorem 3.5 this is a contradiction. This completes the proof. □

LEMMA 3.10. *Let $x \in A$. If μ is a prime fuzzy ideal of A, then*

$$\text{either } \mu(x) = 1 \text{ or } \mu(x') = 1$$

LEMMA 3.11. *Let μ be a fuzzy ideal of A, $a \in A$ and $\alpha \in [0, 1)$. If $\mu(a) \leq \alpha$, then there exists a prime fuzzy ideal θ of A such that $\mu \subseteq \theta$ and $\theta(a) \leq \alpha$.*

Proof. Put $\mathcal{P} = \{\theta \in \mathcal{FI}(A) : \mu \subseteq \theta \text{ and } \theta(a) \leq \alpha\}$. Then $\mathcal{P}$ is nonempty and it forms a poset together with the inclusion ordering of fuzzy sets. By applying Zorn's lemma we can choose a maximal element in $\mathcal{P}$ and let θ be maximal in $\mathcal{P}$; that is, θ is a fuzzy ideal of A and it is maximal with properties: $\mu \subseteq \theta$ and $\theta(a) \leq \alpha$. It remains to show that θ is a prime fuzzy ideal. We use contradiction. For; let ν and σ be fuzzy ideals of A such that $\nu \cap \sigma \subseteq \theta$. Suppose if possible that $\nu \nsubseteq \theta$ and $\sigma \nsubseteq \theta$. If we put $\theta_1 = \theta \vee \nu$ and $\theta_2 = \theta \vee \sigma$, then both θ_1 and θ_2 are fuzzy ideals of A properly containing θ. It follows by the maximality of θ in $\mathcal{P}$ that $\theta_1 \notin \mathcal{P}$ and $\theta_2 \notin \mathcal{P}$. So that $\theta_1(a) > \alpha$ and $\theta_2(a) > \alpha$ which implies that $\theta(a) > \alpha$. This is absurd. Therefore θ is a prime fuzzy ideal of A satisfying the desired condition. □

COROLLARY 3.12. *Let μ be a fuzzy ideal of A, $a \in A$. If $\mu(a) = 0$, then there exists a prime fuzzy ideal θ of A such that $\mu \subseteq \theta$ and $\theta(a) = 0$.*

DEFINITION 3.13. A fuzzy subset λ of A is said to be multiplicatively closed if:

$$\lambda(x \wedge y) \geq \lambda(x) \wedge \lambda(y)$$

for all x, y A.

THEOREM 3.14. *Let μ be a fuzzy ideal of A, λ a multiplicatively closed fuzzy subset of A and $\alpha \in [0, 1]$. If $\mu \cap \lambda \leq \alpha$, then there exists a prime fuzzy ideal θ of A such that:*

$$\mu \subseteq \theta \text{ and } \theta \cap \lambda \leq \alpha$$

Proof. Put $\mathcal{P} = \{\theta \in \mathcal{FI}(A) : \mu \subseteq \theta \text{ and } \theta \cap \lambda \leq \alpha\}$. Then $\mathcal{P}$ is nonempty and it forms a poset together with the inclusion ordering of fuzzy sets. By applying Zorn's lemma we can choose a maximal element, say θ in $\mathcal{P}$; that is, θ is a fuzzy ideal of A and it is maximal with properties: $\mu \subseteq \theta$ and $\theta \cap \lambda \leq \alpha$. It remains to show that θ is a fuzzy prime ideal. Let ν and σ be fuzzy ideals of A such that $\nu \cap \sigma \subseteq \theta$. Suppose on contrary that $\nu \nsubseteq \theta$ and $\sigma \nsubseteq \theta$. If we put $\theta_1 = \theta \vee \nu$ and $\theta_2 = \theta \vee \sigma$, then both θ_1 and θ_2 are fuzzy ideals of A properly containing θ and $\theta_1 \cap \theta_2 \subseteq \theta$. By the maximality of θ in $\mathcal{P}$, it follows that $\theta_1 \notin \mathcal{P}$ and $\theta_2 \notin \mathcal{P}$. So that $\theta_1 \cap \lambda \nleq \alpha$ and $\theta_2 \cap \lambda \nleq \alpha$. Then there exist $a, b \in A$ such that $(\theta_1 \cap \lambda)(a) > \alpha$ and $(\theta_2 \cap \lambda)(b) > \alpha$. Put $x = a \wedge b$ and consider the following:

$$\begin{aligned}
(\theta \cap \lambda)(x) &= \theta(x) \wedge \lambda(x) \\
&= \theta(a \wedge b) \wedge \lambda(a \wedge b) \\
&\geq (\theta_1 \cap \theta_2)(a \wedge b) \wedge \lambda(a) \wedge \lambda(b) \\
&= \theta_1(a \wedge b) \wedge \theta_2(a \wedge b) \wedge \lambda(a) \wedge \lambda(b) \\
&\geq \theta_1(a) \wedge \theta_2(b) \wedge \lambda(a) \wedge \lambda(b) \\
&= \theta_1(a) \wedge \lambda(a) \wedge \theta_2(b) \wedge \lambda(b) \\
&= (\theta_1 \cap \lambda)(a) \wedge (\theta_2 \cap \lambda)(b) \\
&> \alpha
\end{aligned}$$

This is a contradiction. Therefore θ is a fuzzy prime ideal of A satisfying the desired condition. □

Corollary 3.15. *For any non-constant fuzzy ideal μ of A:*

$$\mu = \cap\{\theta : \theta \text{ is a fuzzy prime ideal of } A \text{ such that } \mu \subseteq \theta\}$$

Proof. Let us put $\sigma = \cap\{\theta : \theta$ *is a fuzzy prime ideal of* A *such that* $\mu \subseteq \theta\}$. It is clear that $\mu \subseteq \sigma$. To prove the other inclusion, let $x \in A$. Put $\mu(x) = \alpha$. By Lemma 3.11 there exists a fuzzy prime ideal θ of A such that $\mu \subseteq \theta$ and $\theta(x) \leq \alpha$; that is, $\sigma \subseteq \theta$ and $\theta(x) \leq \alpha$. Thus $\sigma(x) \leq \mu(x)$ and hence $\sigma \subseteq \mu$. Therefore the equality holds. □

Corollary 3.16. *The intersection of all prime fuzzy ideals of A coincides with μ^0; that is,*

$$\mu^0 = \cap\{\theta : \theta \text{ is a fuzzy prime ideal of } A\}$$

4. Fuzzy Prime Spectrum of a C-algebra

In this section, we study the space of fuzzy prime ideals of $C-$algebra equipped with the hull-kernel topology. Consider the following notation

1. $X = \{\mu : \mu$ is a prime fuzzy ideal of $A\}$
2. For any fuzzy subset θ of A, let $V(\theta) = \{\mu \in X : \theta \subseteq \mu\}$ and $X(\theta) = \{\mu \in X : \theta \nsubseteq \mu\}$

THEOREM 4.1. *The collection*

$$\mathcal{T} = \{X(\theta) : \theta \text{ is a fuzzy ideal of } A\}$$

is a topology on X.

Proof. Since $X(\mu_0) = \emptyset$ and $X(1_A) = X$, $\mathcal{T}$ contains both $\emptyset$ and X. Also for any fuzzy ideals θ_1 and θ_2 of A, we have $X(\theta_1) \cap X(\theta_2) = X(\theta_1 \cap \theta_2)$. This shows that $\mathcal{T}$ is closed under finite intersections. Further, let $\{\theta_i : i \in I\}$ be any family of fuzzy ideals of A. We verify that $\cup_{i\in I} X(\theta_i) = X(\langle \cup_{i\in I}\theta_i\rangle)$. Let $\mu \in X(\langle \cup_{i\in I}\theta_i\rangle)$. Then $\langle \cup_{i\in I}\theta_i\rangle \nsubseteq \mu$ which implies that $\theta_i \nsubseteq \mu$ for some $i \in I$. Otherwise if $\theta_i \subseteq \mu$ for each $i \in I$, then it would be true that $\langle \cup_{i\in I}\theta_i\rangle \subseteq \mu$. So that $\mu \in \cup_{i\in I} X(\theta_i)$ whence $X(\langle \cup_{i\in I}\theta_i\rangle) \subseteq \cup_{i\in I} X(\theta_i)$. To prove the other inclusion, let $\mu \in \cup_{i\in I} X(\theta_i)$. Then $\mu \in X(\theta_i)$ for some $i \in I$; that is, $\theta_i \nsubseteq \mu$ for some $i \in I$. Since $\theta_i \subseteq \cup_{i\in I}\theta_i \subseteq \langle \cup_{i\in I}\theta_i\rangle$, we get $\langle \cup_{i\in I}\theta_i\rangle \nsubseteq \mu$. So that $\mu \in X(\langle \cup_{i\in I}\theta_i\rangle)$. Whence $\cup_{i\in I} X(\theta_i) \subseteq X(\langle \cup_{i\in I}\theta_i\rangle)$ and hence the equality holds. Therefore $\mathcal{T}$ is closed under arbitrary union and hence it is a topology on X. □

DEFINITION 4.2. The topological space $(X, \mathcal{T})$ is called the fuzzy prime spectrum of A and it is denoted by $F - spec(A)$.

LEMMA 4.3. *For any fuzzy subset* λ *of* A*:*

$$X(\lambda) = X(\langle \lambda \rangle)$$

Proof. As $\lambda \subseteq \langle \lambda \rangle$, it follows that $X(\lambda) \subseteq X(\langle \lambda \rangle)$. If $\mu \in X(\langle \lambda \rangle)$, then $\langle \lambda \rangle \nsubseteq \mu$ which implies that $\lambda \nsubseteq \mu$. Otherwise, if $\lambda \subseteq \mu$ then $\langle \lambda \rangle \subseteq \mu$ which is impossible. So that $\mu \in X(\lambda)$ and hence $X(\lambda) = X(\langle \lambda \rangle)$. □

LEMMA 4.4. *For any fuzzy subsets* λ *and* ν *of* A

$$X(\lambda) = X(\nu) \Rightarrow \langle \lambda \rangle = \langle \nu \rangle$$

Proof. We use contradiction. Suppose if possible that $\langle\lambda\rangle \neq \langle\nu\rangle$. Then there exists $x \in A$ such that $\langle\lambda\rangle(x) > \langle\nu\rangle(x)$. Let say $\langle\nu\rangle(x) = t$. Then by Lemma 3.11 there exists a prime fuzzy ideal θ of A such that $\langle\nu\rangle \subseteq \theta$ and $\theta(x) = t < \langle\lambda\rangle(x)$. So $\theta \in X(\lambda)$ and $\theta \notin X(\nu)$. Therefore $X(\lambda) \neq X(\nu)$ and this completes the proof. □

LEMMA 4.5. *For any fuzzy points x_α, y_β of A.*

$$X(x_\alpha) \cap X(y_\beta) = X(x_\alpha \wedge y_\beta)$$

LEMMA 4.6. *For any $\alpha \in (0,1]$; $X(x_\alpha) = \emptyset$ if and only if x is a left zero for $\wedge$.*

LEMMA 4.7. *The subfamily $\mathcal{B} = \{X(x_\alpha) : x \in A, \alpha \in (0,1]\}$ of $\mathcal{T}$ is a base for $\mathcal{T}$.*

Proof. Let θ be any fuzzy ideal of A and $\mu \in X(\theta)$. Then μ is a prime fuzzy ideal of A such that $\theta \nsubseteq \mu$. There exists $x \in A$ such that $\theta(x) > \mu(x)$. If we put $\beta = \theta(x)$, then x_β is a fuzzy point of A such that $x_\beta \in \theta$ and $x_\beta \nsubseteq \mu$. So that $\mu \in X(x_\beta) \subseteq X(\theta)$. Thus $\mathcal{B}$ is a base for $\mathcal{T}$. □

LEMMA 4.8. *If A has a meet identity element T, then for each $\alpha \in (0,1]$, the set*

$$A_\alpha = \{\mu \in X : Im(\mu) = \{1, \alpha\}\}$$

is a compact subspace of X.

Proof. Remember that A_α can be made to be subspace of X by the relativized topology $\mathcal{T}_\alpha$ where

$$\mathcal{T}_\alpha = \{X(\theta) \cap A_\alpha : \theta \in \mathcal{FI}(A)\}$$

It is also clear that the family

$$\mathcal{B}_\alpha = \{X(x_t) \cap A_\alpha : x \in A \text{ and } t \in (\alpha, 1]\}$$

constitutes a base for $\mathcal{T}_\alpha$. Suppose that the family

$$\mathcal{C} = \{X((x_i)_t) \cap A_\alpha : i \in \Delta \text{ and } t \in K \subseteq (\alpha, 1]\}$$

is a basic open cover for A_α. If we take $r = Sup\{t : t \in K\}$, then the family $\{X((x_i)_r) \cap A_\alpha : i \in \Delta\}$ covers A_α. That is;

$$\begin{aligned} A_\alpha &= \bigcup_{i\in\Delta}[X((x_i)_r) \cap A_\alpha] \\ &= A_\alpha \cap \bigcup_{i\in\Delta}[X((x_i)_r)] \\ &= A_\alpha \cap X[\cup_{i\in\Delta}(x_i)_r] \\ &= A_\alpha \cap [X - V[\cup_{i\in\Delta}(x_i)_r]] \end{aligned}$$

which implies that $A_\alpha \cap V[\cup_{i\in\Delta}(x_i)_r] = \emptyset$. For any prime fuzzy ideal P of A, consider the fuzzy prime ideal α_P of A given in lemma 3.7. We have $\alpha_P \in A_\alpha$. Since $A_\alpha \cap V[\cup_{i\in\Delta}(x_i)_r] = \emptyset$, it yields that $\alpha_P \notin V[\cup_{i\in\Delta}(x_i)_r]$. So that, $\cup_{i\in\Delta}(x_i)_r \nsubseteq \alpha_P$. If $(x_i)_r \subseteq \alpha_P$ for all $i \in \Delta$, then $\cup_{i\in\Delta}(x_i)_r \subseteq \alpha_P$ which is impossible. Thus, there exists $j \in \Delta$ such that $(x_j)_r \nsubseteq \alpha_P$ implying that $\alpha_P(x_j) < r \leq 1$. Then $\alpha_P(x_j) = \alpha$ and hence $x_i \notin P$. That is; for each prime ideal P of A, there exists $j \in \Delta$ such that $x_j \notin P$. Equivalently saying that every prime ideal P does not contain the ideal $\langle\{x_i : i \in \Delta\}\rangle$. So $\langle\{x_i : i \in \Delta\}\rangle = A$, and hence $T \in \langle\{x_i : i \in \Delta\}\rangle$. Then $T = \vee_{i=1}^n (a_i \wedge x_i)$ for some $a_i \in A, i = 1, 2, ..., n \in \Delta$. We show that $V[\cup_{i=1}^n (x_i)_r] \cap A_\alpha = \emptyset$. Suppose if possible that there is some $\mu \in V[\cup_{i=1}^n (x_i)_r] \cap A_\alpha$ which implies that $\mu(x_i) \geq r > \alpha$ for all $1 \leq i \leq n$. So $\mu(x_i) = 1$ for all $1 \leq i \leq n$. Now consider:

$$\begin{aligned} \mu(T) &= \mu[\bigvee_{i=1}^{n}(a_i \wedge x_i)] \\ &\geq \bigwedge_{i=1}^{n} \mu(x_i) \\ &= 1 \end{aligned}$$

Since $\mu(T) \leq \mu(x)$ for all $x \in A$, it follows that μ is constant which is a contradiction. Therefore $V[\cup_{i=1}^n (x_i)_r] \cap A_\alpha = \emptyset$. Hence the subfamily $\{X((x_i)_r) : 1 \leq i \leq n\}$ covers A_α and therefore A_α is compact. □

Notation. Let us denote the fuzzy point x_α by x_* when $\alpha = 1$.

THEOREM 4.9. *The following are equivalent*

(i) A is a Boolean algebra

(ii) $X - X(x_*) = X((x')_*)$

(iii) For any $\mu \in X$ and each $x \in A$; $\mu(x) \neq \mu(x')$

(iv) $X(x_*) \cup X(y_*) = X((x \vee y)_*)$ *for all* $x, y \in A$.

Proof. (1) $\Rightarrow$ (2). Suppose that A is a Boolean algebra and let $\mu \in X$. Then consider the following:

$$\begin{aligned} \mu \in X - X(x_*) &\Leftrightarrow \mu \notin X(x_*) \\ &\Leftrightarrow x_* \in \mu \\ &\Leftrightarrow \mu(x) = 1 \\ &\Leftrightarrow x \in \mu_* \\ &\Leftrightarrow x' \notin \mu_* \\ &\Leftrightarrow \mu(x') < 1 \\ &\Leftrightarrow (x')_* \notin \mu \\ &\Leftrightarrow \mu \in X((x')_*) \end{aligned}$$

Therefore $X - X(x_*) = X((x')_*)$.
(2) $\Rightarrow$ (3). Suppose that $X - X(x_*) = X((x')_*)$ for all $x \in A$. Let $\mu \in X$. To prove (3) we use contradiction. Suppose if possible that there $x \in A$ such that $\mu(x) = \mu(x')$. By Lemma 3.10 we get $\mu(x) = \mu(x') = 1$, which gives $x_* \in \mu$ and $(x')_* \in \mu$. So, $\mu \notin X(x_*)$ and $\mu \notin X((x')_*) = X - X(x_*)$, which is a contradiction. Therefore $\mu(x) \neq \mu(x')$ for all $x \in A$.
(3) $\Rightarrow$ (4). Suppose that $\mu(x) = \mu(x')$ for all $x \in A$. Let $x, y \in A$. The inclusion $X((x \vee y)_*) \subseteq X(x_*) \cup X(y_*)$ is clear and we proceed to show the other inclusion. Let $\mu \in X(x_*) \cup X(y_*)$. Then either $\mu \in X(x_*)$ or $\mu \in X(y_*)$. So either $x_* \notin \mu$ or $y_* \notin \mu$, which gives that either $\mu(x) < 1$ or $\mu(y) < 1$. It follows from Lemma 3.10 that either $\mu(x') = 1$ or $\mu(y') = 1$, which implies that $\mu(x' \wedge y') = 1$. By (3), we get $\mu((x' \wedge y')') < 1$; that is $\mu(x \vee y) < 1$. This means $(x \vee y)_* \notin \mu$ and that $\mu \in X((x \vee y)_*)$. Thus the equality holds.
(4) $\Rightarrow$ (1). Assume the condition in (4). To prove that A is a Boolean algebra it suffices to show that $\langle x \vee y \rangle = \langle y \vee x \rangle$ for all $x, y \in A$. Let $x, y \in A$. By (4) we get $X((x \vee y)_*) = X((y \vee x)_*)$. It follows from Lemma 4.4 that $\langle (x \vee y)_* \rangle = \langle (y \vee x)_* \rangle$, which gives $\langle x \vee y \rangle = \langle y \vee x \rangle$ and this completes the proof. □

Definition 4.10. A topological space X is called a T_0-space if for each $x \neq y \in X$, there exists an open set H in X such that

$$x \in H \text{ and } y \notin H (\text{or } y \in H \text{ and } x \notin H)$$

THEOREM 4.11. *The space X is a T_0-space.*

Proof. Let μ and θ be fuzzy prime ideals of A such that $\mu \neq \theta$. Then either $\mu \nsubseteq \theta$ or $\theta \nsubseteq \mu$. Without loss of generality we can assume that $\mu \nsubseteq \theta$. Then there exists $x \in A$ such that $\mu(x) > \theta(x)$. Let us put $\alpha = \mu(x)$. Then x_α is a fuzzy point of A such that $x_\alpha \in \mu$ and $x_\alpha \notin \theta$; that is, $\mu \notin X(x_\alpha)$ and $\theta \in X(x_\alpha)$. This means $X(x_\alpha)$ is an open set in X containing θ but not contain μ. Therefore X is a T_0 space. □

THEOREM 4.12. *For any $\mu \neq \nu \in X$, $\nu \in \overline{\{\mu\}}$ if and only if $\mu \subseteq \nu$.*

Proof. Let $\mu \neq \nu \in X$. Suppose that $\nu \in \overline{\{\mu\}}$. Then $\mu \in U$ for each neighborhood U of ν in X. Since neighborhoods of ν in X are of the form $X(\theta)$ for some fuzzy subset θ of A with $\theta \nsubseteq \nu$, it is equivalent to say that $\mu \in X(\theta)$, and hence $\theta \nsubseteq \mu$, for all fuzzy subsets θ of A with $\theta \nsubseteq \nu$. In other words, for any fuzzy subset θ of A the following holds:

$$\theta \subseteq \mu \Rightarrow \theta \subseteq \nu$$

which gives that $\mu \subseteq \nu$.
Conversely, suppose that $\mu \subseteq \nu$ and let U be a neighborhood of ν in X. Then $U = X(\theta)$ for some fuzzy subset θ of A with $\theta \nsubseteq \nu$. Since $\mu \subseteq \nu$, we get $\theta \nsubseteq \mu$, which gives that $\mu \in X(\theta) = U$; that is, $\{\mu\} \cap U \neq \emptyset$. Therefore $\nu \in \overline{\{\mu\}}$. □

COROLLARY 4.13. *For each $\mu \in X$,*

$$V(\mu) = \overline{\{\mu\}}$$

THEOREM 4.14. *Suppose that A is a Boolean algebra, $\alpha \in [0, 1)$. Let $A_\alpha = \{\mu \in X : Im(\mu) = \{1, \alpha\}\}$. For $x, y \in A$ and $\beta \in (0, 1]$ we have the following:*

1. *The set $X(x_\beta) \cap A_\alpha$ is both open and closed in A_α, provided that $\beta > \alpha$.*
2. *$X(x_\beta) \cup X(y_\beta) = X(z_\beta)$ for some $z \in A$.*
3. *The space A_α is Hausdorff.*

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