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MORE ON REVERSE OF HÖLDER'S INTEGRAL INEQUALITY

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Abstract. In 2012, Sulaiman [7] proved integral inequalities concerning reverse of Holder's. In this paper two results are given. First one is further improvement of the reverse Hölder inequality. We note that many existing inequalities related to the Hölder inequality can be proved via obtained this inequality in here. The second is further generalization of Sulaiman's integral inequalities concerning reverses of Holder's [7].

1. Introduction

Hölder's inequality is one of the most important inequalities of pure and applied mathematics. It was the key for resolving many problems in social science natural science. Hölder's inequality reads :

Theorem 1.1. *Let f, g be non-negative integrable functions, let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_a^b f(x)g(x)dx \leq \left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \left(\int_a^b f^q(x)dx\right)^{\frac{1}{q}}. \tag{1}$$

The reverse Hölder's inequality have been explored by a number of scientists, the famous ones are:

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Lemma 1.2. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *if*

$$0 < m \leq \frac{f(x)}{g(x)} \leq M,$$

then

$$\left(\int_a^b f(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g(x)dx\right)^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \int_a^b f^{\frac{1}{p}}(x) g^{\frac{1}{q}}(x)dx. \tag{2}$$

See ([3], [4] p.126, [6] p.3).

Lemma 1.3. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *if*

$$0 < m \leq \frac{f^p(x)}{g^q(x)} \leq M,$$

then

$$\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \int_a^b f(x)g(x)dx. \tag{3}$$

See ([1] p.212, [2] p.9, [5] p.206).
Sulaiman proved the following important inequality:

Theorem 1.4. [7] *Let* $p > 0$, $q > 0$ *and* f, g *be positive functions satisfying*

$$0 < m \leq \frac{f(x)}{g(x)} \leq M \qquad \text{for all } x \in [a,\, b],$$

then

$$\begin{aligned} &\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}} \\ \leq\ & \frac{M}{m} \left(\int_a^b (f(x)g(x))^{\frac{p}{2}}\, dx\right)^{\frac{1}{p}} \left(\int_a^b (f(x)g(x))^{\frac{q}{2}}\, dx\right)^{\frac{1}{q}}. \end{aligned} \tag{4}$$

In this paper, by using a simple proof method, we will generalize the inequalities above.

2. Main Results

In this section we give our results Theorems 2.1 and 2.6. Let $-\infty < a < b < +\infty$ and $\alpha,\ \beta \in \mathbb{R}$.

THEOREM 2.1. *Let* $\alpha,\ \beta > 0,\ \ p > 1,\ \frac{1}{p}+\frac{1}{q} = 1$ *and* $f, g > 0$ *integrable functions on* $[a, b]$, w *a weight function (measurable and positive) on* $[a,\ b]$. *If*

$$0 < m \le \frac{f^{\alpha}(x)}{g^{\beta}(x)} \le M \quad \text{for all } x \in [a,\ b], \tag{5}$$

then

$$\left(\int_a^b f^{\alpha}(x)w(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^{\beta}(x)w(x)dx\right)^{\frac{1}{q}} \le \left(\frac{M}{m}\right)^{\frac{1}{pq}} \int_a^b f^{\frac{\alpha}{p}}(x)g^{\frac{\beta}{q}}(x)w(x)dx. \tag{6}$$

Proof. From the assumption (5), we get

$$m^{-\frac{1}{q}} \ge f^{-\frac{\alpha}{q}} g^{\frac{\beta}{q}} \ge M^{-\frac{1}{q}}$$

i.e.

$$\left(\frac{1}{m}\right)^{\frac{1}{q}} f^{\alpha} \ge f^{\frac{\alpha}{p}} g^{\frac{\beta}{q}} \ge \left(\frac{1}{M}\right)^{\frac{1}{q}} f^{\alpha}$$

then

$$m^{\frac{1}{q}} f^{\frac{\alpha}{p}} g^{\frac{\beta}{q}} \le f^{\alpha} \le M^{\frac{1}{q}} f^{\frac{\alpha}{p}} g^{\frac{\beta}{q}}. \tag{7}$$

Multiplying right hand side of (7) by $w(x)$ and integrating on $[a, b]$, we have

$$\left(\int_a^b f^{\alpha}(x)w(x)dx\right)^{\frac{1}{p}} \le M^{\frac{1}{pq}} \left(\int_a^b f^{\frac{\alpha}{p}}(x)g^{\frac{\beta}{q}}(x)w(x)dx\right)^{\frac{1}{p}}. \tag{8}$$

Similarly, from the assumption (5) we have

$$m^{\frac{1}{p}} \le f^{\frac{\alpha}{p}} g^{-\frac{\beta}{p}} \le M^{\frac{1}{p}}$$

i.e.

$$m^{\frac{1}{p}} g^{\beta} \le f^{\frac{\alpha}{p}} g^{\frac{\beta}{q}} \le M^{\frac{1}{p}} g^{\beta}$$

then

$$\left(\frac{1}{M}\right)^{\frac{1}{p}} f^{\frac{\alpha}{p}} g^{\frac{\beta}{q}} \le g^{\beta} \le \left(\frac{1}{m}\right)^{\frac{1}{p}} f^{\frac{\alpha}{p}} g^{\frac{\beta}{q}}.$$

Finally, we deduce that

$$\left(\int_a^b g^{\beta}(x)w(x)dx\right)^{\frac{1}{q}} \le (\frac{1}{m})^{\frac{1}{pq}} \left(\int_a^b f^{\frac{\alpha}{p}}(x)g^{\frac{\beta}{q}}(x)w(x)dx\right)^{\frac{1}{q}}. \tag{9}$$

By multiplying the inequalities (8) and (9), we have the required inequality (6). □

REMARK 2.2. If we choose $\alpha = 1,\ \beta = 1$ and $w(x) = 1$ in Theorem 2.1, then Theorem 2.1 reduces to Lemma 1.2.

REMARK 2.3. If we take $\alpha = p,\ \beta = q$ and $w(x) = 1$ in Theorem 2.1, then Theorem 2.1 reduces to Lemma 1.3.

If we have

$$m \le \frac{f^p(x)}{g^q(x)} \le M,$$

then

$$m^{\frac{1}{q}} \le \frac{f^{\frac{p}{q}}(x)}{g(x)} \le M^{\frac{1}{q}},$$

$$m^{\frac{1}{q}} \le \frac{f^{p-1}(x)}{g(x)} \le M^{\frac{1}{q}}$$

we deduce that

COROLLARY 2.4. *Let $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1$ and f, g non-negative integrable functions on $[a, b]$ satisfying*

$$0 < m \le \frac{f^{p-1}(x)}{g(x)} \le M,$$

then

$$\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}} \le \left(\frac{M}{m}\right)^{\frac{1}{p}} \int_a^b f(x)g(x)dx. \tag{10}$$

If we have

$$m \le \frac{f^p(x)}{g^q(x)} \le M,$$

then

$$m^{\frac{1}{p}} \le \frac{f(x)}{g^{q-1}(x)} \le M^{\frac{1}{p}}$$

we deduce that

Corollary 2.5. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* f, g *non-negative integrable functions on* $[a, b]$ *satisfying*

$$0 < m \leq \frac{f(x)}{g^{q-1}(x)} \leq M,$$

then

$$\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{q}} \int_a^b f(x)g(x)dx. \tag{11}$$

Theorem 2.6. *Let* α, c, p, q, p', $q' > 0$ *and let* f, g *be two non-negative measurable functions on* $[a, b]$. *If*

$$0 < c < m \leq \frac{\alpha f(x)}{g(x)} \leq M \quad \text{for all } x \in [a,\, b], \tag{12}$$

then

$$\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}}$$

$$\leq \frac{M}{\alpha}\left(\frac{\alpha}{m}\right)^{\frac{2p'}{p'+q'}} (m+c)^{\frac{p'-q'}{p'+q'}} (M+c)^{\frac{q'-p'}{p'+q'}} \tag{13}$$

$$\times \left(\int_a^b \left(f^{p'}(x)g^{q'}(x)\right)^{\frac{p}{p'+q'}} dx\right)^{\frac{1}{p}} \left(\int_a^b \left(f^{p'}(x)g^{q'}(x)\right)^{\frac{q}{p'+q'}} dx\right)^{\frac{1}{q}}.$$

Proof. By the assumption (12) we have

$$m + c \leq \frac{\alpha f(x) + c\, g(x)}{g(x)} \leq M + c \tag{14}$$

and

$$\frac{M+c}{M} \leq \frac{\alpha f(x) + c\, g(x)}{\alpha f(x)} \leq \frac{m+c}{m}. \tag{15}$$

Integrating the left inequalities of (14) and (15), we get

$$(m+c)\left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}} \leq \left(\int_a^b (\alpha f(x) + c\, g(x))^q dx\right)^{\frac{1}{q}}$$

and

$$\alpha\left(\frac{M+c}{M}\right)\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \leq \left(\int_a^b (\alpha f(x) + c\, g(x))^p dx\right)^{\frac{1}{p}}.$$

Multiplying these inequalities we obtain

$$
\begin{aligned}
&\frac{\alpha}{M}(M+c)(m+c)\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}}\left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}} \\
(16)\qquad &\leq \left(\int_a^b (\alpha f(x)+c\,g(x))^p dx\right)^{\frac{1}{p}}\left(\int_a^b (\alpha f(x)+c\,g(x))^q dx\right)^{\frac{1}{q}}.
\end{aligned}
$$

From the right inequalities of (14) and (15), we get

$$(\alpha f(x)+c\,g(x))^{q'} \leq (M+c)^{q'} g^{q'}(x)$$

and

$$(\alpha f(x)+c\,g(x))^{p'} \leq \left(\frac{\alpha}{m}(m+c)\right)^{p'} f^{p'}(x),$$

then

$$(\alpha f(x)+c\,g(x))^{p'+q'} \leq \left(\frac{\alpha}{m}(m+c)\right)^{p'} (M+c)^{q'} f^{p'}(x)g^{q'}(x).$$

Thus, we get

(17)

$$(\alpha f(x)+c\,g(x)) \leq \left(\frac{\alpha}{m}(m+c)\right)^{\frac{p'}{p'+q'}} (M+c)^{\frac{q'}{p'+q'}} \left(f^{p'}(x)g^{q'}(x)\right)^{\frac{1}{p'+q'}}.$$

From the inequality (17), we deduce that

$$
\begin{aligned}
&\left(\int_a^b (\alpha f(x)+c\,g(x))^p dx\right)^{\frac{1}{p}}\left(\int_a^b (\alpha f(x)+c\,g(x))^q dx\right)^{\frac{1}{q}} \\
(18)\qquad &\leq \left(\frac{\alpha}{m}(m+c)\right)^{\frac{2p'}{p'+q'}} (M+c)^{\frac{2q'}{p'+q'}} \\
&\times \left(\int_a^b \left(f^{p'}(x)g^{q'}(x)\right)^{\frac{p}{p'+q'}}\right)^{\frac{1}{p}} \left(\int_a^b \left(f^{p'}(x)g^{q'}(x)\right)^{\frac{q}{p'+q'}}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Finally, by the inequalities(16) and (18), we obtain desired inequality (13). □

Remark 2.7. If we choose $p'=q'=\alpha=1$ in Theorem 2.6, then we get Theorem 1.4.

References

[1] C. J. Zhao and W. S. Cheung, *Hölder's reverse in equality and its applications,* Publ. Inst. Math. **99** (113)(2016), 211–216.

[2] B. Halim and A. Senouci, *Some generalizations involving open problems of F. Qi.*, Int. J. Open Problems Compt. Math. **12** (1) (2019), 10–21.

[3] L. Xiao-Hua, *On the inverse of Hölder inequality,* Math. Practice and Theory. **1** (1) (1990), 84–88.

[4] D. S. Mitrinovic, J. Pecaric and A. M. Fink, *Classical and new inequalities in analysis,* **61**,126–127, Springer Science and Business Media, 2013.

[5] J. Pecaric and C. E. M. Pearce, *On an inverse to the Hölder inequality,* Int. J. Mathematics and Mathematical Sciences. **20** (1) (1997), 205–207.

[6] S. Saitoh, V. K. Tuan and M. Yamamoto, *Reverse convolution inequalities and applications to inverse heat source problems,* J. Inequal. Pure Appl. Math. **3** (5) (2002), 80.

[7] W. T. Sulaiman, *Reverses of Minkowski's, Hölder's, and Hardy's integral inequalities,* Int. J. Mod. Math. Sci. **1** (1) (2012), 14–24.

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